Uniqueness Theorems for the Tricomi Problem

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1. Introduction. In this paper certain uniqueness theorems are established for boundary value problems involving equations of mixed elliptic-hyperbolic type. We consider an equation of the form

$$Lu \equiv K(y) u_{xx} + u_{yy} = 0 \quad (1)$$

where $K(y)$ is a monotone increasing function of y with a continuous second derivative, and $K(0) = 0$. Equation (1) is elliptic for $y > 0$ and hyperbolic for $y < 0$. Considerable interest has been shown recently in equations of this type since certain problems occurring in transonic flow lead to elliptic-hyperbolic partial differential equations. Particular cases of (1) have been considered by many authors since the initial works of Tricomi for the case $K(y) = y$ [8]¹ and of Gellerstedt for $K(y) = y^m$ [4]. Equation (1) itself has been the subject of investigations by Bers [1, 2], Frankl [3], Germain & Bader [5, 6], the author [7], and others.

Let Γ_1 be a rectifiable Jordan arc lying in the upper half-plane with endpoints P and Q lying on the positive x -axis ($P > Q$), and denote by y_m the maximum of the ordinates of points on Γ_1 . Equation (1) has real characteristics for negative values of y given by the two families of curves

$$\frac{dy}{dx} = \frac{1}{\sqrt{-K(y)}}, \quad (2a)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{-K(y)}}. \quad (2b)$$

Thus from each point of the x -axis two characteristics extend into the half-plane $y < 0$. Denote by Γ_2 the characteristic of the family (2a) emanating from

¹ Numbers in brackets refer to the bibliography at the end of the paper.

P and by Γ_3 the characteristic of (2b) emanating from Q . Then Γ_2 and Γ_3 will intersect at some point S in the lower half-plane and the curves Γ_1 , Γ_2 , and Γ_3 will enclose a domain which we designate D . Let A denote the portion of D above the x -axis, B the portion below it. The problem of Tricomi, which we call *problem (T)*, is the following: to find a solution of (1) in D assuming prescribed values on Γ_1 and Γ_2 . Under certain restrictions on Γ_1 this problem was solved by Tricomi for the case $K(y) = y$ and by Gellerstedt for the case $K(y) = y^m$.

Problems similar to problem (T) may be considered for more general domains. Let Γ_1 , P , Q be as above and suppose R is a point on the x -axis lying between P and Q . Suppose γ_1 and γ_3 are the characteristics of (2a) starting from P and R

respectively, and γ_2 and γ_4 are the characteristics of (2b) starting from R and Q respectively. Denote by E the domain bounded by Γ_1 , γ_1 , γ_2 , γ_3 , and γ_4 ; by A that part of E above the x -axis, by B_1 the part bounded by γ_1 , γ_2 and the x -axis, and by B_2 the part bounded by γ_3 , γ_4 and the x -axis. In this case, *problem (T)* consists in finding a unique solution of (1) assuming prescribed values along Γ_1 , γ_1 and γ_3 or along Γ_1 , γ_2 and γ_3 .

A further generalization, *problem (T)*₂, is the following: let R_1 and R_2 be two points on the x -axis lying between P and Q with characteristics γ_2 of (2a) starting from R_1 and γ_3 of (2b) starting from R_2 . Suppose γ_1 of (2a) and γ_4 of (2b) are the characteristics starting from P and Q respectively (see figure). Then γ_2 and γ_3 will intersect at a point S , γ_1 and γ_3 at a point U , and γ_2 and γ_4 at a

point V . The problem consists in finding a solution of (1) in the domain bounded by the arcs Γ_1 , PU , US , SV , and VQ , having prescribed values along Γ_1 and the characteristic arcs PU and SV . We note that this reduces to problem (T_1) if R_1 and R_2 coincide.

In a paper dealing with transonic jet flows Frankl [3] considered problem (T') for equation (1). He established the uniqueness of the solution in the case where the function

$$F(y) \equiv 2 \left(\frac{K}{\overline{K}'} \right)' + 1 \quad (3)$$

is positive for all negative values of y . In the present paper this uniqueness theorem is generalized to include cases where $F(y)$ may be negative. Uniqueness of the solution is established for two types of domains D . In the first theorem the arc Γ_1 in the elliptic half-plane is left arbitrary; but then the hyperbolic domain is restricted so that $F(y)$ may not fall below a fixed negative value. In the second theorem the hyperbolic part of the domain may be selected arbitrarily and $F(y)$ may assume any values for $y < 0$; but then the curve Γ_1 may not extend arbitrarily far into the elliptic portion of the plane.

A *quasi-regular* solution of problem (T) is a function $u(x, y)$ satisfying (1) in D and in addition the following conditions:

$$(1) \quad \int_Q^P u(x, 0) u_y(x, 0) \, dx \quad \text{exists,}$$

$$(2) \quad \iint_A (K u_x^2 + u_y^2) \, dx \, dy \quad \text{exists,}$$

(3) Green's theorem may be applied to the integrals

$$\iint_D u \, Lu \, dx \, dy, \quad \iint_D u_x \, Lu \, dx \, dy, \quad \iint_D u_y \, Lu \, dx \, dy,$$

and the boundary integrals which arise exist in the sense that the limits taken over corresponding interior curves exist as these interior curves approach the boundary.

Theorem 1: Let $K(y)$ be a monotone increasing function with a continuous second derivative, $K(0) = 0$, $F(0) > 0$, $K'(y) \neq 0$ for $y < 0$. Let D be the domain described in the Introduction. There exists a constant $d_0 < 0$ such that if $F(y) \geq d_0$ in D , and $u(x, y)$ is a quasi-regular solution of (1) which vanishes on Γ_1 and Γ_2 , then $u \equiv 0$ in D .

Theorem 2: Let $K(y)$ and D have the same meaning as in Theorem 1. There exists a constant $d_1 > 0$ such that if $y_m < d_1$, and $u(x, y)$ is a quasi-regular solution of D which vanishes on Γ_1 and Γ_2 , then $u \equiv 0$ in D .

Corresponding theorems hold for problems (T_1) and (T_2) .

2. Proof of Theorems 1 and 2. Theorems 1 and 2 are proved simultaneously. We investigate the expression²

$$\iint_D [a(x, y)u + b(x, y)u_x + c(x, y)u_y][Ku_{xx} + u_{yy}] dx dy \quad (4)$$

where $a(x, y)$, $b(x, y)$ and $c(x, y)$ are functions with a continuous first derivative and are yet to be determined. If $u(x, y)$ is a solution of (1) then (4) will vanish. We note the following identities:

$$\begin{aligned} aKuu_{xx} &= (aKuu_x)_x - aKu_x^2 - \frac{1}{2}(a_x Ku^2)_x + \frac{1}{2}a_{xx} Ku^2, \\ auu_{yy} &= (auu_y)_y - au_y^2 - \frac{1}{2}(a_y u^2)_y + \frac{1}{2}a_{yy} u^2, \\ bKu_x u_{xx} &= \frac{1}{2}(bKu_x^2)_x - \frac{1}{2}b_x Ku_x^2, \\ bu_x u_{yy} &= (bu_x u_y)_y - b_y u_x u_y - \frac{1}{2}(bu_y^2)_x + \frac{1}{2}b_x u_y^2, \\ cKu_y u_{xx} &= (cKu_x u_y)_x - \frac{1}{2}(cKu_x^2)_y + \frac{1}{2}(cK)_y u_x^2 - c_x Ku_x u_y, \\ cu_y u_{yy} &= \frac{1}{2}(cu_y^2)_y - \frac{1}{2}c_y u_y^2. \end{aligned}$$

Substitution of these identities into (4) and an application of Green's theorem yield

$$\begin{aligned} 0 &= \iint_D [-aKu_x^2 + \frac{1}{2}a_{xx}Ku^2 - au_y^2 + \frac{1}{2}a_{yy}u^2 - \frac{1}{2}Kb_x u_x^2 - b_y u_x u_y \\ &\quad + \frac{1}{2}b_x u_y^2 + \frac{1}{2}(cK)_y u_x^2 - c_x Ku_x u_y - \frac{1}{2}c_y u_y^2] dx dy \\ &\quad + \oint_{\Gamma_1 + \Gamma_2 + \Gamma_3} [aKuu_x dy - \frac{1}{2}a_x Ku^2 dy - auu_y dx + \frac{1}{2}a_y u^2 dx + \frac{1}{2}bKu_x^2 dy \\ &\quad - bu_x u_y dx - \frac{1}{2}bu_y^2 dy + cKu_x u_y dy + \frac{1}{2}cKu_x^2 dx - \frac{1}{2}cu_y^2 dx] = J_1 + J_2. \end{aligned}$$

If we suppose $u(x, y)$ vanishes on Γ_1 and Γ_2 and if we select b and c to vanish identically in A the integral J_2 becomes

$$\begin{aligned} J_2 &= \int_{\Gamma_3} [aKuu_x dy - auu_y dx - \frac{1}{2}a_x Ku^2 dy + \frac{1}{2}a_y u^2 dx + \frac{1}{2}bKu_x^2 dy \\ &\quad - bu_x u_y dx - \frac{1}{2}bu_y^2 dy + cKu_x u_y dy + \frac{1}{2}cKu_x^2 dx - \frac{1}{2}cu_y^2 dx] \\ &\quad + \int_{\Gamma_2} [\frac{1}{2}bKu_x^2 dy - bu_x u_y dx - \frac{1}{2}bu_y^2 dy + cKu_x u_y dy + \frac{1}{2}cKu_x^2 dx - \frac{1}{2}cu_y^2 dx]. \end{aligned}$$

² This idea is due to K. O. FRIEDRICHS, and I am indebted to him for suggesting it.

Since Γ_2 and Γ_3 are characteristics we have

$$dy = \frac{1}{\sqrt{-K}} dx \quad \text{along } \Gamma_2, \quad dy = -\frac{1}{\sqrt{-K}} dx \quad \text{along } \Gamma_3.$$

Thus we find

$$\begin{aligned} J_2 &= -\frac{1}{2} \int_{\Gamma_3} (c\sqrt{-K} - b) \left(\sqrt{-K} u_x^2 - 2u_x u_y + \frac{1}{\sqrt{-K}} u_y^2 \right) dx \\ &\quad + \int_{\Gamma_3} [a\sqrt{-K} u du - \frac{1}{2}\sqrt{-K} u^2 (a_x dx + a_y dy)] \\ &\quad - \frac{1}{2} \int_{\Gamma_2} (c\sqrt{-K} + b) (\sqrt{-K} u_x + u_y) (u_x dx + u_y dy) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We break up J_1 into three integrals as follows:

$$\begin{aligned} J_1 &= - \iint_A a(Ku_x^2 + u_y^2) dx dy \\ &\quad - \frac{1}{2} \iint_B \left\{ [2aK + Kb_x - (cK)_y] u_x^2 + 2(Kc_x + b_y) u_x u_y \right. \\ &\quad \left. + (c_y + 2a - b_x) u_y^2 \right\} dx dy + \frac{1}{2} \iint_D (Ka_{xx} + a_{yy}) u^2 dx dy \\ &= I_4 + I_5 + I_6. \end{aligned}$$

We shall now show that by a proper choice of a , b , and c each of the integrals $I_1, I_2, \dots, I_6$ must vanish. From this it follows that $u(x, y)$ vanishes identically throughout D . For, the vanishing of I_4 (if $a > 0$) implies that u vanishes in A . The quantities a , b , and c will be such that the integrand of I_2 will be definite in u . Hence u will vanish along Γ_3 . By a well known theorem for hyperbolic equations if u vanishes along Γ_2 and Γ_3 it vanishes throughout B .

For negative values of y we select

$$c = \frac{4aK}{K'}, \quad b = c\sqrt{-K}. \quad (5)$$

With this choice we see that b and c are continuous throughout D except if $\lim_{y \rightarrow 0} K/K' < 0$. If this is the case the application of Green's theorem would introduce the additional boundary integral

$$I_7 = \frac{1}{2} \int_q^P cu_y^2(x, 0) dx$$

which must also be taken into consideration. Since $u = 0$ on Γ_2 we have $du = u_x dx + u_y dy = 0$ along this characteristic and therefore $I_3 = 0$. Obviously I_1 vanishes because of the choice for b . For I_2 we find after integration by parts

$$I_2 = - \int_{\Gamma_3} \left(a_x \sqrt{-K} - a_y + \frac{aK'}{-4K} \right) u^2 dx$$

This integral will be non-positive if

$$a_x \sqrt{-K} - a_y + \frac{aK'}{-4K} \geq 0 \quad \text{for } y < 0;$$

or if the two conditions

$$a(x, y) \geq 0 \quad \text{for } y \leq 0 \quad (6a)$$

$$\frac{a_x \sqrt{-K} - a_y}{a} \geq 0 \quad \text{for } y \leq 0 \quad (6b)$$

hold. The choice of a will be such that (6a) and (6b) are satisfied. I_5 will be non-positive if

$$(Kc_x + b_y)^2 \leq (c_y + 2a - b_x) [Kb_x + 2aK - (cK)_y] \quad \text{for } y \leq 0 \quad (7a)$$

and

$$c_y + 2a - b_x \geq 0 \quad \text{for } y \leq 0. \quad (7b)$$

If (5) is taken into account (7a) becomes an equality for arbitrary a , and (7b) becomes

$$\sqrt{-K} a_x - a_y + a \frac{K'}{-2K} F(y) \geq 0. \quad (8)$$

We now distinguish two cases.

Case 1. $F(y) > 0$ for all $y < 0$. Then we select a to be any positive constant for all x, y in D . (6a) and (6b) are satisfied and I_2 is non-positive. I_4 is clearly non-positive, and I_6 vanishes identically. $c(x, y)$ is negative for $y \leq 0$, and therefore I_7 is also non-positive. Since $I_1 + I_2 + \dots + I_7 = 0$, we conclude that each integral and hence u vanishes throughout D .

Case 2. $F(y)$ is negative for some negative values of y . We select

$$a(x, y) = \begin{cases} e^{\beta x}, & y \leq 0 \\ e^{\beta x} \cos \gamma y, & y \geq 0 \end{cases} \quad (9)$$

where β and γ are constants yet to be determined.

To prove Theorem 1 we select $\gamma = \frac{\pi}{2y_m}$ and $\beta = \frac{\gamma}{\sqrt{K(y_m)}}$. Then (8) is satisfied if

$$F(y) \geq \frac{2K\sqrt{-K(y)}\beta}{K'}.$$

Hence a selection of $d_0 = \min_{y < 0} \frac{2K\sqrt{-K(y)}\beta}{K'}$ yields the condition that I be negative. Since all the remaining integrals are clearly non-positive the result follows.

For the proof of Theorem 2 we choose

$$\beta = \max_{y < 0} \frac{K'}{2K\sqrt{-K}} F(y).$$

Then if $y_m \leq \frac{\pi}{2\beta\sqrt{K(y_m)}} (= d_1)$ we select $\gamma = \beta K(y_m)$ and a simple computation shows that all the integrals are non-positive.

3. Proof of Problems (T₁) and (T₂). For problem (T₁) suppose $u(x, y)$ vanishes on Γ_1, γ_1 and γ_3 . We define a, b and c as in Section 2:

$$a = e^{\beta x}, \quad b = c \sqrt{-K}, \quad c = 4aK/K', \quad \text{for } y \leq 0.$$

It is clear that the integrals in the elliptic portion of the plane are unaffected by the change in the hyperbolic domain, and the result follows. If $u(x, y)$ vanishes on Γ_1, γ_2 and γ_3 then for B_2 we define

$$a = e^{\beta x}, \quad b = c \sqrt{-K}, \quad c = 4aK/K',$$

while for B_1 we have

$$a = me^{-\beta x}, \quad b = -c \sqrt{-K}, \quad c = 4aK/K',$$

where m is a constant selected so that a is continuous for $y > 0$. Since a_x may not be continuous the additional integral

$$-m\beta e^{\beta x} \int [\cos(\gamma y)] K(y) u^2 dy$$

taken in the positive direction along the vertical line through R must be considered. However this integral is obviously non-positive and the result follows.

For problem (T₂) the selection of a, b and c as in Section 2 yields the result without any change.

4. An Application. Consider a two-dimensional adiabatic potential flow of a perfect gas. In the hodograph plane the equation for the stream function ψ is

$$K(\sigma) \psi_{\theta\theta} + \psi_{\sigma\sigma} = 0,$$

where (τ, θ) are the velocity components in polar coordinates and $\sigma = \sigma(\tau)$ is defined by the relation

$$\sigma = - \int_{1/(2\lambda+1)}^{\tau} \frac{(1-t)^{\lambda}}{2t} dt,$$

λ being a constant. The function

$$K(\sigma) = \frac{1 - (2\lambda + 1)\tau}{(1 - \tau)^{2\lambda+1}}$$

is monotone increasing and satisfies the condition $K(0) = 0$. Positive values of K correspond to subsonic velocities, negative values to supersonic velocities, while $\sigma = 0$ corresponds to the speed of sound.

We consider the jet flow out of a symmetrical vessel bounded by two straight infinite walls making an angle $2\theta_0$. Frankl [3] investigated the case of such a flow and showed that for velocities in the transonic range the problem reduces to that of Tricomi. He solved the uniqueness problem under the hypothesis

$$F(y) \geq 0, \sigma \leq 0.$$

For the case under consideration this corresponds to limiting the angle θ_0 of the jet to 54 degrees. Frankl conjectured that transonic jet flows for an angle $\theta > 54^\circ$ are not possible without shock waves. Theorem 2 shows that as far as uniqueness is concerned the angle 54° is of no physical significance and that uniqueness always holds for angles greater than 54° .

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