The Solution of Problems in Second Order Elasticity Theory

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1. Introduction. The strain-energy function for an ideal isotropic elastic material which can be subjected to finite deformations may be expressed, in general, in terms of three independent scalar invariants of the strain. If the material is incompressible, then the strain-energy function may be considered to be a function of only two invariants of the strain. This fact, together with the constancy of volume of each element imposed by the incompressibility of the material, makes possible the solution of a number of problems concerning the deformation of highly-elastic incompressible bodies without further assumptions regarding the form of the strain-energy function.

In the case of compressible materials, this simplification is lost and a number of writers have attempted to solve problems by making special assumptions regarding the form of the strain-energy function. These are generally based on apparent simplicity within some mathematical framework and would severely re-

strict, if not invalidate, the application of the conclusions to real compressible materials.

On the other hand, Murnaghan (1937), choosing the strain invariants in such a manner that they vanish for the undeformed material, has expressed the strain-energy function as a triply-infinite power series in terms of these invariants. Now, each of these invariants is a symmetrical function of the principal extensions and, if the latter are sufficiently small, we may approximate the strain-energy function, to any desired order in the principal extensions, by neglecting terms above an appropriate degree in the power-series expansion for the strain-energy function.

For example, by taking only those terms in the power-series expansion for the strain-energy function which are of the third or less order of smallness in the principal extensions, a theory is obtained which enables us to calculate in principle the relation between the applied forces and the displacements in a body up to terms of the second degree in the latter. This provides a first extension of the classical elasticity theory and is referred to as a second order elasticity theory. The expression for the strain-energy in terms of the invariants of the strain involves, in this theory, five elastic constants. Two of these may be related to the Lamé constants of classical elasticity theory, while the remaining three constants do not occur in classical elasticity theory.

Murnaghan (1951) has solved a number of simple problems on the basis of such a second order elasticity theory for compressible materials and Green & Shield (1951) have solved some problems of the torsion of prisms in the incompressible case, for certain particular forms of the cross-section. In the latter case, the appropriate strain-energy function takes the Mooney (1940) form, as has been shown by Rivlin (1949) and Rivlin & Saunders (1951).

Both Murnaghan and Green & Shield take the displacements given by the solution of the problem in classical elasticity theory as a first approximation to the desired result. To these they add small unknown displacements, such that terms of the second degree in their space derivatives can be neglected. The resultant displacements are then substituted in the equations of motion and boundary conditions appropriate to the second order elasticity theory and the resulting equations in the small unknown displacements are linearized, by neglecting terms of higher degree than the first in the space derivatives of these, and then solved.

It is shown in the present paper that boundary value problems within the framework of the second order elasticity theory may be solved systematically by the following process. The corresponding boundary value problem in classical elasticity theory is first solved, so that a system of displacements is obtained which corresponds precisely to the applied force system, provided that terms of the second degree in the space derivatives of the displacement components are neglected. We then calculate, from the second order elasticity theory, the forces additional to those prescribed, which would have to be applied in order to maintain this calculated system of displacements. This additional force system is re-

versed in direction and the displacements which it would then produce in the body under consideration according to classical elasticity theory are calculated. The system of displacements which the prescribed force system would produce in the body according to the second order elasticity theory is then given by summing the two displacement systems calculated in the above manner.*

This method of solution of problems is then applied to the study of second order elastic effects in a circular cylindrical tube of elastic material subjected to simultaneous extension and torsion. The calculation of second order elastic effects in a circular cylindrical rod appears as a special case of this. In solving this problem, the method employed involves the calculation of the differences in the displacements from those predicted by classical elasticity theory, which result from the adoption of a stored-energy function involving third degree terms in the space derivatives of the displacement components. It is seen that the changes in the dimensions of the tube produced by a small torsion are unaltered, as far as terms of the second degree in the torsion, if a constant tensile force is applied to the tube, provided that the extension per unit length produced by the force is also sufficiently small so that terms of the third degree in it can be neglected.

In many problems, we are not primarily interested in the details of the displacement distribution throughout the body, but rather in average effects throughout the body. The method outlined above enables us to solve such problems in certain cases when the more detailed solution would be either intractable or extremely laborious. It is seen that the method reduces the solution of a boundary value problem within the framework of the second order elasticity theory to the solution of two boundary value problems in classical elasticity theory, together with an inverse type of calculation. In the second of these classical boundary value problems, the strain averages may be calculated relatively simply by the Betti method. In the present paper, this procedure is adopted in the calculation of the elongation of a prism of arbitrary cross-section resulting from torsion under conditions of constant tensile force. It is shown that provided the amount of torsion is sufficiently small so that the problem lies within the scope of the second order elasticity theory, the prism undergoes an elongation which is proportional to the square of the amount of torsion and to the length of the prism. It is further shown that the constant of proportionality can be determined in terms of four of the five elastic constants of the material and the cross-sectional dimensions of the prism, provided that the Saint-Venant solution of the classical torsion problem can be obtained for the prism.

In the final section of the paper, it is shown how the method discussed for the solution of problems in the second order elasticity theory may be applied to incompressible materials.

* In the absence of a uniqueness theorem for the displacement distribution produced in a body by a specified system of applied forces, that calculated by the method described here must be regarded, in general, only as a possible displacement distribution which may be maintained by the given force system, as has been pointed out to me by C. TRUESDELL.

2. The Stress-Strain Relations. Let us consider a body of isotropic material, capable of undergoing finite elastic deformation, to be deformed so that a point which is initially at x_i in the rectangular Cartesian co-ordinate system x_i is displaced to $x_i + u_i$ in the same co-ordinate system. The stress components t_{ik} in this co-ordinate system are then given (see, for example, Rivlin (1948)) by

$$t_{ik} = \frac{2}{\tau} \left[g_{ik} \frac{\partial W}{\partial I_1} - G_{ik} \frac{\partial W}{\partial I_2} + \left(I_3 \frac{\partial W}{\partial I_3} + I_2 \frac{\partial W}{\partial I_2} \right) \delta_{ik} \right], \quad (2.1)$$

where W is the strain-energy function for the material,

$$g_{ik} = \left(\delta_{is} + \frac{\partial u_i}{\partial x_s} \right) \left(\delta_{ks} + \frac{\partial u_k}{\partial x_s} \right), \quad (2.2)$$

G_{ik} denotes the co-factor of g_{ik} in $\det g_{ik}$ and I_1 , I_2 and I_3 are scalar invariants of the strain defined by

$$I_1 = g_{ss}, \quad I_2 = G_{ss} \quad \text{and} \quad I_3 = \det g_{ik}. \quad (2.3)$$

τ is defined by $\det (\delta_{ik} + \partial u_i / \partial x_k)$, so that $I_3 = \tau^2$, and δ_{ik} denotes the Kronecker symbol.

For small deformations, such that $\partial u_i / \partial x_k$ are all small compared with unity, $I_1 - 3$, $I_2 - 3$ and $I_3 - 1$ are, in general, of first order of smallness in the quantities $\partial u_i / \partial x_k$. We may, however, construct three other mutually independent scalar invariants of the strain J_1 , J_2 and J_3 which are respectively of first, second and third orders of smallness in quantities $\partial u_i / \partial x_k$. Such scalar invariants may be conveniently defined by the relations

$$J_1 = I_1 - 3, \quad J_2 = I_2 - 2I_1 + 3 \quad \text{and} \quad J_3 = I_3 - I_2 + I_1 - 1, \quad (2.4)$$

which are equivalent to the converse relations

$$I_1 = J_1 + 3, \quad I_2 = J_2 + 2J_1 + 3 \quad \text{and} \quad I_3 = J_3 + J_2 + J_1 + 1. \quad (2.5)$$

Since W is a function of I_1 , I_2 and I_3 , it can be expressed, by means of the relations (2.5), as a function of J_1 , J_2 and J_3 .

If we are concerned with finite deformations of the elastic body which are sufficiently small so that terms of higher degree than the second in the quantities $\partial u_i / \partial x_k$ can be neglected in the expressions for the stress components, then we can, following Murnaghan (1937), express W in the form

$$W = a_0 J_1 + a_1 J_2 + a_2 J_1^2 + a_3 J_1 J_2 + a_4 J_1^3 + a_5 J_3, \quad (2.6)$$

in which $a_0, a_1, \dots, a_5$ are physical constants for the material considered. It has been shown by Murnaghan (1937) that if the material is such that the stress is zero in the undeformed state, $a_0 = 0$, so that

$$W = a_1 J_2 + a_2 J_1^2 + a_3 J_1 J_2 + a_4 J_1^3 + a_5 J_3. \quad (2.7)$$

From (2.4) and (2.7), we obtain

$$\begin{aligned}\frac{\partial W}{\partial I_1} &= \frac{\partial W}{\partial J_1} - 2 \frac{\partial W}{\partial J_2} + \frac{\partial W}{\partial J_3} = (a_5 - 2a_1) + 2(a_2 - a_3)J_1 + a_3J_2 + 3a_4J_1^2, \\ \frac{\partial W}{\partial I_2} &= \frac{\partial W}{\partial J_2} - \frac{\partial W}{\partial J_3} = (a_1 - a_5) + a_3J_1 \quad \text{and} \quad \frac{\partial W}{\partial I_3} = \frac{\partial W}{\partial J_3} = a_5.\end{aligned}\quad (2.8)$$

Neglecting terms of higher degree than the second in $\partial u_i/\partial x_k$ in the expressions for g_{ik} , G_{ik} , I_1 , I_2 , I_3 , J_1 and J_2 , we obtain, with the notation

$$\begin{aligned}e_{ik} &= \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}, \quad \alpha_{ik} = \frac{\partial u_i}{\partial x_s} \frac{\partial u_k}{\partial x_s}, \\ e &= 2\Delta = e_{ss}, \quad \alpha = \alpha_{ss}, \quad E_{ik} = \text{co-factor of } e_{ik} \text{ in } \det e_{ik} \\ &\quad \text{and } E = E_{ss},\end{aligned}\quad (2.9)$$

$$\begin{aligned}g_{ik} &= \delta_{ik} + e_{ik} + \alpha_{ik}, \quad G_{ik} = (1 + 2\Delta + \alpha)\delta_{ik} - e_{ik} - \alpha_{ik} + E_{ik}, \\ I_1 &= 3 + 2\Delta + \alpha, \quad I_2 = 3 + 4\Delta + 2\alpha + E, \quad I_3 = 1 + 2\Delta + \alpha + E, \\ J_1 &= 2\Delta + \alpha \quad \text{and} \quad J_2 = E.\end{aligned}\quad (2.10)$$

Introducing the relations (2.8) and (2.10) into (2.1), and neglecting terms of higher degree than the second in $\partial u_i/\partial x_k$, we obtain

$$\begin{aligned}t_{ik} &= 2\{-a_1 e_{ik} + 2(a_1 + 2a_2)\Delta \delta_{ik}\} \\ &\quad + \{(4a_2 - 2a_3 + a_1)\Delta e_{ik} - a_1 \alpha_{ik} - (a_1 - a_5)E_{ik}\} \\ &\quad + \{(a_1 + 2a_2)\alpha + (a_1 + a_3)E + 2(6a_4 + 2a_3 - a_1 - 2a_2)\Delta^2\}\delta_{ik}.\end{aligned}\quad (2.11)$$

e_{ik} and Δ are homogeneous expressions of the first degree in $\partial u_i/\partial x_k$ and α_{ik} , E_{ik} , α and E are homogeneous expressions of the second degree. If we neglect terms of the second degree in $\partial u_i/\partial x_k$ in the expression for t_{ik} , we obtain the expressions for the stress components of classical elasticity theory

$$t_{ik} = 2[-a_1 e_{ik} + 2(a_1 + 2a_2)\Delta \delta_{ik}]. \quad (2.12)$$

Comparing these equations with the usual stress-strain relations of classical elasticity theory, it is seen that the Lamé constants λ and μ are given by

$$\lambda = 4(a_1 + 2a_2) \quad \text{and} \quad \mu = -2a_1. \quad (2.13)$$

Young's modulus E and Poisson's ratio σ may therefore be expressed in terms of a_1 and a_2 by the formulae

$$E = -8a_1 \frac{a_1 + 3a_2}{a_1 + 4a_2} \quad \text{and} \quad \sigma = \frac{a_1 + 2a_2}{a_1 + 4a_2}. \quad (2.14)$$

Alternatively a_1 and a_2 may be expressed in terms of E and σ by the relations

$$a_1 = -\frac{1}{4} \frac{E}{1 + \sigma} \quad \text{and} \quad a_2 = \frac{1}{8} E \frac{1 - \sigma}{(1 + \sigma)(1 - 2\sigma)}. \quad (2.15)$$

Since, for real materials, $0 < \sigma < \frac{1}{2}$ and $E > 0$, we see that $a_1 < 0$ and $a_2 > 0$. Also, since

$$-\frac{a_1}{a_2} = 2 \left(2 - \frac{1}{1 - \sigma} \right), \quad (2.16)$$

$$a_1 + 2a_2 > 0. \quad (2.17)$$

3. The Equations of Equilibrium and Boundary Conditions. Suppose we consider the deformation of a body of the material by a system of body forces X_i per unit mass of material and surface forces $X_{\nu i}$ per unit area of surface measured in the undeformed state of the body. Then, writing $\xi_i = x_i + u_i$, the equations of motion, applicable at all points of the body, are given by

$$\frac{\partial t_{ik}}{\partial \xi_k} + \rho X_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (3.1)$$

where ρ is the density of the material in the deformed state and t denotes time. The boundary conditions may be written as

$$X_{\nu i} \frac{dS}{dS'} = t_{ik} l'_k. \quad (3.2)$$

where dS and dS' represent elements of area of the surface of the body measured in the undeformed and deformed states respectively, so that $X_{\nu i} dS/dS'$ is the surface traction per unit area of the surface measured in the deformed state of the body, and l'_k are the direction-cosines of the normal to the deformed surface of the body. If ρ_0 is the density of the material of the body in its undeformed state, then

$$\rho = \rho_0 I_3^{-\frac{1}{3}} = \rho_0 / \tau. \quad (3.3)$$

From the relations $\xi_i = x_i + u_i$, we obtain

$$\frac{\partial}{\partial \xi_i} = \frac{1}{\tau} \frac{\partial \tau}{\partial (\partial u_i / \partial x_k)} \frac{\partial}{\partial x_k}. \quad (3.4)$$

Introducing (3.4) and (3.3) into (3.1), we obtain the equations of motion in the form

$$\frac{\partial \tau}{\partial (\partial u_k / \partial x_s)} \frac{\partial t_{ik}}{\partial x_s} + \rho_0 X_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2}. \quad (3.5)$$

Again, if l_i are the direction-cosines of the normal to the surface of the body in its undeformed state, then

$$l'_k = \frac{dS}{dS'} \left[\frac{\partial \tau}{\partial (\partial u_k / \partial x_s)} l_s \right]. \quad (3.6)$$

Introducing (3.6) into (3.2), we obtain the boundary conditions in the form

$$X_{vi} = \frac{\partial \tau}{\partial (\partial u_k / \partial x_s)} l_s t_{ik}. \quad (3.7)$$

Let us write equation (2.11) in the form

$$t_{ik} = t'_{ik} + t''_{ik}, \quad (3.8)$$

where

$$t'_{ik} = 2[-a_1 e_{ik} + 2(a_1 + 2a_2)\Delta \delta_{ik}]$$

and

$$t''_{ik} = 2[\{(4a_2 - 2a_3 + a_1)\Delta e_{ik} - a_1 \alpha_{ik} - (a_1 - a_5)E_{ik}\} \\ + \{(a_1 + 2a_2)\alpha + (a_1 + a_3)E + 2(6a_4 + 2a_3 - a_1 - 2a_2)\Delta^2\}\delta_{ik}]. \quad (3.9)$$

The expression for t'_{ik} contains only terms of the first degree in the space derivatives of u_i and that for t''_{ik} only second degree terms. Introducing (3.8) into (3.5) and neglecting terms of higher degree than the second in the space derivatives of u_i , we obtain

$$\left[(1 + \Delta)\delta_{sk} - \frac{\partial u_s}{\partial x_k} \right] \frac{\partial t'_{ik}}{\partial x_s} + \frac{\partial t''_{ik}}{\partial x_k} + \rho_0 X_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2}. \quad (3.10)$$

Similarly, we obtain from (3.8) and (3.7)

$$X_{vi} = \left[(1 + \Delta)\delta_{sk} - \frac{\partial u_s}{\partial x_k} \right] l_s t'_{ik} + l_k t''_{ik}. \quad (3.11)$$

4. A Method for Solving Boundary Value Problems. It appears from §3 that if the displacements u_i are sufficiently small, so that the strain-energy can be adequately represented by an expression of the form (2.7), then u_i may be determined from equations (3.10), subject to the boundary conditions (3.11), in which t'_{ik} and t''_{ik} are given in terms of u_i by (3.9).

Now, let us suppose that v_i are the displacements in the body predicted by the classical elasticity theory. The stress components τ_{ik} are given by classical elasticity theory as

$$\tau_{ik} = 2[-a_1 e'_{ik} + 2(a_1 + 2a_2)\Delta' \delta_{ik}], \quad (4.1)$$

in which

$$e'_{ik} = \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \quad \text{and} \quad \Delta' = \frac{1}{2} e'_{ss}. \quad (4.2)$$

v_i is then determined from the equations of motion

$$\frac{\partial \tau_{ik}}{\partial x_k} + \rho_0 X_i = \rho_0 \frac{\partial^2 v_i}{\partial t^2}, \quad (4.3)$$

subject to the boundary conditions

$$X_{\nu i} = l_k \tau_{ik}. \quad (4.4)$$

The equations of §3 predict that the body forces X'_i and surface tractions $X'_{\nu i}$ which must, in fact, be applied in order to maintain the deformation v_i in the body on the basis of the second order elasticity theory are given by the equations

$$\left[(1 + \Delta') \delta_{sk} - \frac{\partial v_s}{\partial x_k} \right] \frac{\partial \tau_{ik}}{\partial x_s} + \frac{\partial \tau'_{ik}}{\partial x_k} + \rho_0 X'_i = \rho_0 \frac{\partial^2 v_i}{\partial t^2}$$

and

$$X'_{\nu i} = \left[(1 + \Delta') \delta_{sk} - \frac{\partial v_s}{\partial x_k} \right] l_s \tau_{ik} + l_k \tau'_{ik}, \quad (4.5)$$

in which τ_{ik} is given by (4.1) and τ'_{ik} by

$$\begin{aligned} \tau'_{ik} = & 2[\{(4a_2 - 2a_3 + a_1)\Delta' e'_{ik} - a_1 \alpha'_{ik} - (a_1 - a_3)E'_{ik}\} \\ & + \{(a_1 + 2a_2)\alpha' + (a_1 + a_3)E' + 2(6a_4 + 2a_3 - a_1 - 2a_2)\Delta'^2\} \delta_{ik}], \end{aligned} \quad (4.6)$$

where

$$\alpha'_{ik} = \frac{\partial v_i}{\partial x_s} \frac{\partial v_k}{\partial x_s}, \quad \alpha' = \alpha'_{ss}, \quad E'_{ik} \text{ is the co-factor of } e'_{ik} \text{ in } \det e'_{ik}$$

$$\text{and } E' = E'_{ss}. \quad (4.7)$$

Equations (4.3), (4.4) and (4.5) enable us to find the body forces $X'_i - X_i$ and surface tractions $X'_{\nu i} - X_{\nu i}$ which would have to be applied, in addition to those specified, in order to produce the deformation v_i . These are given by

$$-\rho_0(X'_i - X_i) = \left(\Delta' \delta_{sk} - \frac{\partial v_s}{\partial x_k} \right) \frac{\partial \tau_{ik}}{\partial x_s} + \frac{\partial \tau'_{ik}}{\partial x_k}$$

and

$$X'_{\nu i} - X_{\nu i} = \left(\Delta' \delta_{sk} - \frac{\partial v_s}{\partial x_k} \right) l_s \tau_{ik} + l_k \tau'_{ik}. \quad (4.8)$$

We shall now write down the equations determining the displacements w_i which the system of body forces $X_i - X'_i$ and surface tractions $X_{\nu i} - X'_{\nu i}$ would produce in the body according to classical elasticity theory. These are

$$\frac{\partial \tau''_{ik}}{\partial x_k} + \rho_0(X_i - X'_i) = \rho_0 \frac{\partial^2 w_i}{\partial t^2}$$

and

$$X_{\nu i} - X'_{\nu i} = l_k \tau''_{ik}, \quad (4.9)$$

where τ''_{ik} is given in terms of w_i by

$$\tau''_{ik} = 2[-a_1 e''_{ik} + 2(a_1 + 2a_2) \Delta'' \delta_{ik}]$$

and

$$e''_{ik} = \frac{\partial w_i}{\partial x_k} + \frac{\partial w_k}{\partial x_i}, \quad \Delta'' = \frac{1}{2} e''_{ss}. \quad (4.10)$$

It is seen from equations (4.9) that $\partial w_i / \partial x_k$ is, in general, of the order of magnitude of second degree terms in $\partial v_i / \partial x_k$. Consequently, if we define u_i by

$$u_i = v_i + w_i, \quad (4.11)$$

equations (4.8) may be written, neglecting terms of higher degree than the second in $\partial u_i / \partial x_k$,

$$-\rho_0(X'_i - X_i) = \left(\Delta \delta_{sk} - \frac{\partial u_s}{\partial x_k} \right) \frac{\partial t'_{ik}}{\partial x_s} + \frac{\partial t''_{ik}}{\partial x_k}$$

and

$$X'_{\nu i} - X_{\nu i} = \left(\Delta \delta_{sk} - \frac{\partial u_s}{\partial x_k} \right) l_s t'_{ik} + l_k t''_{ik}, \quad (4.12)$$

where Δ , t'_{ik} and t''_{ik} are defined by equations (2.9) and (3.9).

Again, from equations (4.3), (4.4), (4.9) and (4.12), we see that u_i , defined by (4.11), is determined by equations (3.10) and (3.11). It is thus clear that $v_i + w_i$ represents the displacement resulting from the application to the body of the system of body forces X_i and surface tractions $X_{\nu i}$, if the strain-energy function for the material of the body is taken in the Murnaghan form (2.7).

It thus appears that if the deformation is sufficiently small so that terms of higher order than the second in the space derivatives of the displacement components can be neglected in deriving the relations between the applied forces and the deformation, then the deformation resulting from the application to the body of a given set of surface tractions and body forces can be calculated by the following process.

- (i) The displacements corresponding to the applied force system are first calculated on the basis of the classical elasticity theory.
- (ii) The additional forces which must be applied in order to maintain this deformation according to the finite deformation theory are calculated. These are, of course, of second order of smallness in the space derivatives of the displacement components.
- (iii) The displacements which these additional forces produce in an initially undeformed body according to classical elasticity theory are calculated.
- (iv) Addition of the displacements calculated in steps (i) and (iii) gives the displacements which are produced in the body by the initially applied forces.

Since the equations of classical elasticity theory are linear, it will be quite clear that in carrying out the process indicated in step (iii), the additional force system may be split up in any convenient manner and the displacements corresponding to each of the sub-systems of forces calculated separately, the required displacement configuration being determined by the addition of those so calculated.

It should be noted also that in carrying out the process (i), the displacements v_i derived from equations (4.1) to (4.4) must correspond to the system of body forces X_i and surface tractions $X_{\nu i}$ as far as terms of the second order of smallness. In the usual classical solutions terms of the second order of smallness are, of course, generally neglected. Where such a neglect of second order terms has taken place, the neglected terms can be calculated by introducing the displacements v_i into equations (4.1) to (4.4). Let these neglected body forces and surface tractions be denoted $\mathfrak{X}_i$ and $\mathfrak{X}_{\nu i}$ respectively. Then as far as terms of the second degree in the space derivatives of v_i , v_i satisfy the following equations:

$$\frac{\partial \tau_{ik}}{\partial x_k} + \rho_0(X_i + \mathfrak{X}_i) = \rho_0 \frac{\partial^2 v_i}{\partial t^2}$$

and

$$X_{\nu i} + \mathfrak{X}_{\nu i} = l_k \tau_{ik}, \quad (4.13)$$

where τ_{ik} is defined in terms of v_i by equations (4.2).

From equations (4.13) and (4.5), it is seen that $X'_i - X_i$ and $X'_{\nu i} - X_{\nu i}$ are now given by

$$-\rho_0(X'_i - X_i) = \left(\Delta' \delta_{sk} - \frac{\partial v_s}{\partial x_k} \right) \frac{\partial \tau_{ik}}{\partial x_s} + \frac{\partial \tau'_{ik}}{\partial x_k} - \rho_0 \mathfrak{X}_i$$

and

$$X'_{\nu i} - X_{\nu i} = \left(\Delta' \delta_{sk} - \frac{\partial v_s}{\partial x_k} \right) l_s \tau'_{ik} + l_k \tau'_{ik} + \mathfrak{X}_{\nu i}. \quad (4.14)$$

By an argument identical with that previously employed, in which equations (4.13) and (4.14) replace equations (4.3) and (4.8) respectively, it can be shown that the required deformations of the body can be calculated by the sequence of steps (i) to (iv). Now, however, the force system which must be employed in step (ii) is that given by equations (4.14).

5. The Simultaneous Extension and Torsion of a Cylindrical Tube. We consider a cylindrical tube of external and internal radii a and b respectively to be subjected simultaneously to torsion and simple extension. We shall describe the deformation by means of the displacements v_i undergone by the various points of the body with reference to a rectangular Cartesian co-ordinate system x_i , the x_3 axis of which coincides with the axis of the tube.

This problem can be solved within the framework of classical elasticity theory by superposing the solutions of the simple extension and torsion problems. If ϵ is the extension and ψ is the amount of torsion, then in the neighborhood of $x_3 = 0$ the displacements v_i are given with sufficient accuracy by

$$v_1 = -\sigma\epsilon x_1 - \psi x_2 x_3, \quad v_2 = -\sigma\epsilon x_2 + \psi x_1 x_3 \quad \text{and} \quad v_3 = \epsilon x_3, \quad (5.1)$$

where

$$\sigma = \frac{a_1 + 2a_2}{a_1 + 4a_2}. \quad (5.2)$$

Substituting in (4.2) and (4.1), we obtain in the neighborhood of $x_3 = 0$

$$\begin{aligned} \tau_{11} = \tau_{22} = \tau_{12} = 0, \quad \tau_{33} &= 8\epsilon[a_2 - \sigma(a_1 + 2a_2)], \\ \tau_{23} &= -2a_1\psi x_1 \quad \text{and} \quad \tau_{31} = 2a_1\psi x_2. \end{aligned} \quad (5.3)$$

From (5.3) and (4.3), we obtain

$$X_i = 0. \quad (5.4)$$

On the outer curved surface of the tube,

$$l_1 = x_1/a, \quad l_2 = x_2/a \quad \text{and} \quad l_3 = 0, \quad (5.5)$$

so that (4.4) yields

$$X_{,1} = X_{,2} = X_{,3} = 0. \quad (5.6)$$

A similar result obtains on the inner curved surface. On a plane end of the tube normal to its axis we have

$$l_1 = l_2 = 0 \quad \text{and} \quad l_3 = 1, \quad (5.7)$$

so that (4.4) yields

$$X_{,1} = 2a_1\psi x_2, \quad X_{,2} = -2a_1\psi x_1 \quad \text{and} \quad X_{,3} = 8\epsilon[a_2 - \sigma(a_1 + 2a_2)]. \quad (5.8)$$

It is thus seen that if the displacement components are represented by equation (5.1), the body forces are zero, the curved surfaces of the tube are force-free and the force system acting on the plane ends of the tube normal to its axis consists of a uniformly distributed surface traction $X_{,3}$ acting parallel to the axis of the tube and an azimuthal surface traction proportional to the radial distance from the axis.

In solving the problem on the basis of the second-order elasticity theory, we shall consider the tube to be first subjected to the pure homogeneous deformation defined by

$$v_1 = -\sigma\epsilon x_1, \quad v_2 = -\sigma\epsilon x_2 \quad \text{and} \quad v_3 = \epsilon x_3 \quad (5.9)$$

and shall consider that a torsion of amount ψ is superposed on this deformation. The amount of torsion ψ is defined as the relative rotation in radians of two cross-sections of the tube normal to its axis which are unit distance apart in the state of deformation defined by equations (5.9). The displacement components v_i for the resultant deformation are given by

$$v_1 = -\sigma\epsilon x_1 + (1 - \sigma\epsilon) [x_1 \cos \psi(1 + \epsilon) x_3 - x_2 \sin \psi(1 + \epsilon) x_3 - x_1],$$

$$v_2 = -\sigma\epsilon x_2 + (1 - \sigma\epsilon) [x_1 \sin \psi(1 + \epsilon) x_3 + x_2 \cos \psi(1 + \epsilon) x_3 - x_2]$$

and

$$v_3 = \epsilon x_3. \quad (5.10)$$

Neglecting terms of higher degree than the second in ϵ and ψx_3 , equations (5.10) may be written, in the neighborhood of $x_3 = 0$, as

$$v_1 = -\sigma\epsilon x_1 - \psi x_2 x_3 - \frac{1}{2}\psi^2 x_1 x_3^2 - (1 - \sigma)\epsilon\psi x_2 x_3,$$

$$v_2 = -\sigma\epsilon x_2 + \psi x_1 x_3 - \frac{1}{2}\psi^2 x_2 x_3^2 + (1 - \sigma)\epsilon\psi x_1 x_3$$

and

$$v_3 = \epsilon x_3. \quad (5.11)$$

If in (5.11) terms of second degree in ϵ and ψx_3 are neglected, we obtain the expressions (5.1) for v_i which were used in obtaining the solution of the problem on the basis of classical elasticity theory.

From (5.11) and (4.2), we have, in the neighborhood of $x_3 = 0$,

$$\begin{aligned}
e'_{11} &= -2\sigma\epsilon - \psi^2 x_3^2, & e'_{22} &= -2\sigma\epsilon - \psi^2 x_3^2, & e'_{33} &= 2\epsilon, \\
e'_{23} &= \psi x_1[1 + (1 - \sigma)\epsilon] - \psi^2 x_2 x_3, & e'_{31} &= -\psi x_2[1 + (1 - \sigma)\epsilon] - \psi^2 x_1 x_3, \\
e'_{12} &= 0 \quad \text{and} \quad \Delta' &= \epsilon(1 - 2\sigma) - \psi^2 x_3^2.
\end{aligned} \tag{5.12}$$

Introducing (5.12) and (5.2) into (4.1) and neglecting terms in $\psi^2 x_3^2$, we obtain

$$\begin{aligned}
\tau_{11} &= \tau_{22} = 0, & \tau_{33} &= 8\epsilon[a_2 - \sigma(a_1 + 2a_2)], \\
\tau_{23} &= -2a_1\{\psi x_1[1 + (1 - \sigma)\epsilon] - \psi^2 x_2 x_3\}, \\
\tau_{31} &= 2a_1\{\psi x_2[1 + (1 - \sigma)\epsilon] + \psi^2 x_1 x_3\}
\end{aligned} \tag{5.13}$$

and

$$\tau_{12} = 0.$$

Again, introducing equations (5.11) into (4.7) and neglecting terms of higher degree than the second in ϵ and ψx_i and terms of higher degree than the first in ψx_3 , we obtain

$$\begin{aligned}
\alpha'_{11} &= \sigma^2 \epsilon^2 + \psi^2 x_2^2, & \alpha'_{22} &= \sigma^2 \epsilon^2 + \psi^2 x_1^2, & \alpha'_{33} &= \epsilon^2, \\
\alpha'_{23} &= \psi \epsilon x_1, & \alpha'_{31} &= -\epsilon \psi x_2, & \alpha'_{12} &= -\psi^2 x_1 x_2
\end{aligned}$$

and

$$\begin{aligned}
\alpha &= 2\sigma^2 \epsilon^2 + \epsilon^2 + \psi^2 r^2, \\
E'_{11} &= -4\sigma \epsilon^2 - \psi^2 x_1^2, & E'_{22} &= -4\sigma \epsilon^2 - \psi^2 x_2^2, & E'_{33} &= 4\sigma^2 \epsilon^2, \\
E'_{23} &= 2\sigma \epsilon \psi x_1, & E'_{31} &= -2\sigma \epsilon \psi x_2, & E'_{12} &= -\psi^2 x_1 x_2
\end{aligned} \tag{5.14}$$

and

$$E = -8\sigma \epsilon^2 + 4\sigma^2 \epsilon^2 - \psi^2 r^2, \tag{5.15}$$

where $r^2 = x_1^2 + x_2^2$.

Substituting in equations (4.6) from (5.12), (5.14) and (5.15), we obtain

$$\begin{aligned}
\tau'_{11} &= 2\psi^2\{-(2a_1 - a_6)x_2^2 + (a_1 + 2a_2 - a_3 - a_6)r^2\} + \gamma, \\
\tau'_{22} &= 2\psi^2\{-(2a_1 - a_6)x_1^2 + (a_1 + 2a_2 - a_3 - a_6)r^2\} + \gamma, \\
\tau'_{33} &= 2\psi^2(2a_2 - a_3)r^2 + \gamma', \\
\tau'_{23} &= 2\psi \epsilon x_1\{(4a_2 - 2a_3) - 2\sigma(2a_1 + 4a_2 - 2a_3 - a_6)\}, \\
\tau'_{31} &= -2\psi \epsilon x_2\{(4a_2 - 2a_3) - 2\sigma(2a_1 + 4a_2 - 2a_3 - a_6)\}, \\
\tau'_{12} &= 2(2a_1 - a_6)\psi^2 x_1 x_2,
\end{aligned} \tag{5.16}$$

in which

$$\begin{aligned} \gamma = 2\epsilon^2[\sigma^2\{a_1 + 4a_2 + 12(a_3 + 4a_4)\} - 2\sigma(-a_1 - 4a_2 + 10a_3 + 24a_4 + 2a_5) \\ + \{-a_1 - 2a_2 + 4(a_3 + 3a_4)\}] \end{aligned}$$

and

$$\begin{aligned} \gamma' = 4\epsilon^2[\sigma^2\{-3(a_1 + 2a_2) + 10a_3 + 24a_4 + 2a_5\} \\ - 2\sigma(a_1 + 4a_3 + 12a_4) + 3(a_2 + 2a_4)]. \end{aligned} \quad (5.17)$$

From equations (5.13), (4.13) and (5.4), we obtain, when $x_3 = 0$,

$$\rho_0 \mathfrak{X}_1 = -2a_1\psi^2x_1, \quad \rho_0 \mathfrak{X}_2 = -2a_1\psi^2x_2 \quad \text{and} \quad \rho_0 \mathfrak{X}_3 = 0. \quad (5.18)$$

On the outer curved surface of the tube, we have, from (5.5), (5.13), (4.13) and (5.6), writing $x_3 = 0$,

$$\mathfrak{X}_{r1} = \mathfrak{X}_{r2} = \mathfrak{X}_{r3} = 0. \quad (5.19)$$

A similar result is obtained for the inner curved surface.

On a plane normal to the axis, we have, from (5.7), (5.13), (4.13) and (5.8), taking $x_3 = 0$,

$$\mathfrak{X}_{r1} = 2a_1\psi x_2(1 - \sigma)\epsilon, \quad \mathfrak{X}_{r2} = -2a_1\psi x_1(1 - \sigma)\epsilon \quad \text{and} \quad \mathfrak{X}_{r3} = 0. \quad (5.20)$$

Substituting from (5.11), (5.12), (5.13), (5.16) and (5.18) in the first of equations (4.14), neglecting terms of higher degree than the second in ϵ and ψx_i , and writing $x_3 = 0$ in the results so obtained, we have

$$\begin{aligned} -\rho_0(X'_{r1} - X_1) &= 2\psi^2x_1(4a_1 + 4a_2 - 2a_3 - 3a_5), \\ -\rho_0(X'_{r2} - X_2) &= 2\psi^2x_2(4a_1 + 4a_2 - 2a_3 - 3a_5), \end{aligned}$$

and

$$X'_3 - X_3 = 0. \quad (5.21)$$

The system of body forces described by these equations is equivalent to a distribution of radial body forces, R' per unit mass of the material, given by

$$-\rho_0 R' = 2\psi^2r(4a_1 + 4a_2 - 2a_3 - 3a_5). \quad (5.22)$$

In a similar manner we obtain, from (5.11), (5.12), (5.13), (5.16), (5.5), (5.19) and the second of equations (4.14), for the outer curved surface of the tube

$$\begin{aligned} X'_{r1} - X_{r1} &= [2\psi^2(a_1 + 2a_2 - a_3 - a_5) a^2 + \gamma] x_1/a, \\ X'_{r2} - X_{r2} &= [2\psi^2(a_1 + 2a_2 - a_3 - a_5) a^2 + \gamma] x_2/a \end{aligned}$$

and

$$X'_{r3} - X_{r3} = 0. \quad (5.23)$$

This system of surface tractions is equivalent to a uniform distribution, acting radially outwards on the outer curved surface of the tube, of magnitude R_{r1} , per unit area of surface measured in the undeformed state, given by

$$R_{r1} = 2\psi^2(a_1 + 2a_2 - a_3 - a_5) a^2 + \gamma. \quad (5.24)$$

In a similar manner it can be shown that the system of surface tractions on the inner curved surface of the tube acts radially inwards and has magnitude R_{r2} given by

$$R_{r2} = 2\psi^2(a_1 + 2a_2 - a_3 - a_5) b^2 + \gamma. \quad (5.25)$$

Again, on a plane end of the tube at $x_3 = 0$, we obtain, from (5.11), (5.12), (5.13), (5.16), (5.7), (5.20) and the second of equations (4.14), with $x_3 = 0$,

$$\begin{aligned} X'_{r1} - X_{r1} &= 2\psi\epsilon x_2[(a_1 - 4a_2 + 2a_3) + \sigma(a_1 + 8a_2 - 4a_3 - 2a_5)], \\ X'_{r2} - X_{r2} &= -2\psi\epsilon x_1[(a_1 - 4a_2 + 2a_3) + \sigma(a_1 + 8a_2 - 4a_3 - 2a_5)], \\ X'_{r3} - X_{r3} &= 2\psi^2 r^2(2a_2 - a_3) + \gamma' - 16\sigma\epsilon^2[a_2 - \sigma(a_1 + 2a_2)]. \end{aligned} \quad (5.26)$$

This system of surface tractions is equivalent to the normal surface traction $X'_{r3} - X_{r3}$, together with an azimuthal surface traction of amount Θ given by

$$\Theta_r = -2\psi\epsilon r[(a_1 - 4a_2 + 2a_3) + \sigma(a_1 + 8a_2 - 4a_3 - 2a_5)]. \quad (5.27)$$

Summarizing the results so far obtained, we see that in order to maintain the deformation of the tube described by equations (5.10), the second order elasticity theory predicts that we must apply, in addition to the forces predicted by the classical elasticity theory, the following force system:

- (i) radial body forces R' per unit mass of material, given by (5.22);
- (ii) a radial surface traction R_{r1} per unit area measured in the undeformed state, acting in the outward direction on the outer curved surface of the tube and given by (5.24);
- (iii) a radial surface traction R_{r2} per unit area measured in the undeformed state, acting inwards on the inner curved surface of the tube and given by (5.25);
- (iv) azimuthal surface tractions Θ_r and normal surface tractions $X'_{r3} - X_{r3}$ distributed over the ends of the tube in accordance with equations (5.26) and (5.27).

We may consider the dimensional changes in the tube which arise if one or more of these forces does not act. These are obtained by calculating the displacements which would be produced in the undeformed tube, according to

classical elasticity theory, by forces of equal magnitude but opposite direction. The displacements which would be produced by the various reversed forces are, of course, additive and may therefore be calculated separately. Let us write in equations (5.24), (5.25) and the third of equations (5.26),

$$R_{r1} = R'_{r1} + \gamma, \quad R_{r2} = R'_{r2} + \gamma$$

and

$$X'_{r3} - X_{r3} = Z + \gamma' - 16\sigma\epsilon^2[a_2 - \sigma(a_1 + 2a_2)]. \quad (5.28)$$

We note that R'_{r1} , R'_{r2} and Z are independent of the amount ϵ of the initial extension of the tube, while the remaining terms on the right hand sides of these equations are independent of the amount of torsion ψ . The body force R' is seen, from equation (5.22), to be independent of ϵ .

The displacements which would be produced in the undeformed tube by radial surface tractions $-R'_{r1}$ and $-R'_{r2}$ applied to the outer and inner curved surfaces of the tube, by a body force $-R'$ per unit mass and a longitudinal surface traction $-Z$ acting on the ends of the tube will now be calculated. We shall first consider that the tube is held at constant length and that it is acted upon by the body forces $-R'$ and surface tractions $-R'_{r1}$ and $-R'_{r2}$. Then it is seen from the symmetry of the problem that the displacement is radial at all points. Let U be the radial displacement of a point initially at a distance r from the axis.

Denoting the stress components associated with the deformation in a cylindrical polar co-ordinate system (r, θ, z) by τ''_{rr} , $\tau''_{\theta\theta}$, $\dots$ $\tau''_{r\theta}$, we have

$$\begin{aligned} \tau''_{rr} &= 2 \left[-2a_1 \frac{\partial U}{\partial r} + 2(a_1 + 2a_2) \left(\frac{dU}{dr} + \frac{U}{r} \right) \right], \\ \tau''_{\theta\theta} &= 2 \left[-2a_1 \frac{U}{r} + 2(a_1 + 2a_2) \left(\frac{dU}{dr} + \frac{U}{r} \right) \right], \\ \tau''_{zz} &= 4(a_1 + 2a_2) \left(\frac{dU}{dr} + \frac{U}{r} \right) \end{aligned}$$

and

$$\tau''_{r\theta} = \tau''_{\theta z} = \tau''_{zr} = 0. \quad (5.29)$$

Substituting these expressions for τ''_{rr} , etc. in the equations of motion and boundary conditions, we find that U is determined by the equation

$$8a_2 \left[\frac{d^2 U}{dr^2} + \frac{1}{r} \frac{dU}{dr} - \frac{U}{r^2} \right] = \rho_0 R' = -Kr, \quad (5.30)$$

where K is defined by

$$K = 2\psi^2(4a_1 + 4a_2 - 2a_3 - 3a_6), \quad (5.31)$$

subject to the boundary conditions

$$[\tau''_{rr}]_{r=a} = -R'_{r1} = -2\psi^2 a^2(a_1 + 2a_2 - a_3 - a_6)$$

and

$$[\tau''_{rr}]_{r=b} = -R'_{r2} = -2\psi^2 b^2(a_1 + 2a_2 - a_3 - a_6). \quad (5.32)$$

Equation (5.30) yields

$$U = -\frac{Kr^3}{64a_2} + \frac{1}{2}Ar + \frac{B}{r}, \quad (5.33)$$

where A and B are constants of integration.

From (5.29), (5.32) and (5.33), we obtain the boundary conditions in the form

$$-\frac{Ka^2}{16a_2}(a_1 + 8a_2) + 2A(a_1 + 4a_2) + 4B\frac{a_1}{a^2} = -K'a^2$$

and

$$-\frac{Kb^2}{16a_2}(a_1 + 8a_2) + 2A(a_1 + 4a_2) + 4B\frac{a_1}{b^2} = -K'b^2, \quad (5.34)$$

where

$$K' = 2\psi^2(a_1 + 2a_2 - a_3 - a_6). \quad (5.35)$$

Solving equations (5.34) for A and B , we obtain

$$A = \frac{1}{2} \left[\frac{K(a_1 + 8a_2)}{16a_2} - K' \right] \frac{a^2 + b^2}{a_1 + 4a_2}$$

and

$$B = -\frac{1}{4} \left[\frac{K(a_1 + 8a_2)}{16a_2} - K' \right] \frac{a^2 b^2}{a_1}. \quad (5.36)$$

Introducing (5.36) into (5.33), we find that

$$[U]_{r=b} = \frac{1}{16} \frac{b}{a_1(a_1 + 4a_2)} [K\{b^2 a_1 - a^2(a_1 + 8a_2)\} - 4K'\{b^2 a_1 - a^2 \cdot 4a_2\}]. \quad (5.37)$$

From (5.37), it is seen that if the tube is first extended and then twisted, the

change ΔV of internal volume, per unit length of the tube, on twisting is independent of the initial extension and is given by

$$\begin{aligned} \Delta V &= 2\pi b[U]_{r=b} = \frac{1}{8} \frac{\pi b^2}{a_1(a_1 + 4a_2)} \\ &\quad \cdot [K\{b^2 a_1 - a^2(a_1 + 8a_2)\} - 4K'\{b^2 a_1 - a^2 \cdot 4a_2\}] \\ &= \frac{1}{4} \frac{\pi b^2 \psi^2}{a_1(a_1 + 4a_2)} [(4a_1 + 4a_2 - 2a_3 - 3a_5)\{b^2 a_1 - a^2(a_1 + 8a_2)\} \\ &\quad - 4(a_1 + 2a_2 - a_3 - a_5)\{b^2 a_1 - a^2 \cdot 4a_2\}]. \end{aligned} \quad (5.38)$$

The total change N in the longitudinal force which must be applied to the tube in order to maintain its length constant during torsion is given by

$$N = \int_b^a 2\pi r(\tau''_{zz} + Z) dr, \quad (5.39)$$

where, from (5.28) and (5.26), Z is given by

$$Z = 2\psi^2 r^2(2a_2 - a_3) \quad (5.40)$$

and τ''_{zz} is obtained by substituting from (5.33) in the third of equations (5.29), so that

$$\tau''_{zz} = 4(a_1 + 2a_2) \left(-\frac{1}{16} \frac{Kr^2}{a_2} + A \right). \quad (5.41)$$

Substituting from (5.40) and (5.41) in (5.39) and carrying out the integration, we obtain, after simplification

$$N = \pi\psi^2(a^4 - b^4) \frac{a_5(a_1 + 2a_2) - a_1(2a_2 - a_3)}{a_1 + 4a_2}. \quad (5.42)$$

The fractional extension ϵ' which is produced in the tube on twisting, if it is maintained at constant longitudinal load instead of constant extension is obtained by calculating, on the basis of classical elasticity theory, the extension produced in the tube by a force $-N$ acting longitudinally. We readily obtain

$$\epsilon' = \frac{1}{8} \psi^2(a^2 + b^2) \frac{a_5(a_1 + 2a_2) - a_1(2a_2 - a_3)}{a_1(a_1 + 3a_2)}. \quad (5.43)$$

If we consider a cylindrical rod instead of a tube, then we can obtain the fractional increase in length on twisting under constant longitudinal force by writing $b = 0$ in (5.43). We obtain (*cf.* Murnaghan (1951))

$$\epsilon' = \frac{1}{8} \psi^2 a^2 \frac{a_5(a_1 + 2a_2) - a_1(2a_2 - a_3)}{a_1(a_1 + 3a_2)}. \quad (5.44)$$

6. The Simultaneous Extension and Torsion of a Rod of Arbitrary Cross-Section. As in the previous section, we describe the deformation of the rod by means of the displacement components v_i in a rectangular Cartesian co-ordinate system x_i , the x_3 axis of which is parallel to the length of the rod. If ϵ is the fractional extension and ψ the amount of torsion, then we may write

$$\begin{aligned} v_1 &= -\sigma\epsilon x_1 + (1 - \sigma\epsilon)[x_1 \cos \psi(1 + \epsilon) x_3 - x_2 \sin \psi(1 + \epsilon) x_3 - x_1], \\ v_2 &= -\sigma\epsilon x_2 + (1 - \sigma\epsilon)[x_1 \sin \psi(1 + \epsilon) x_3 + x_2 \cos \psi(1 + \epsilon) x_3 - x_2] \end{aligned}$$

and

$$v_3 = \epsilon x_3 + \psi\phi(x_1, x_2), \quad (6.1)$$

where $\phi(x_1, x_2)$ is the torsion function for the rod given by classical elasticity theory and σ is defined by equation (5.2). Thus, $\phi(x_1, x_2)$ must satisfy the equation

$$\nabla^2 \phi = 0 \quad (6.2)$$

throughout the rod and the relation

$$\left(\frac{\partial \phi}{\partial x_1} - x_2\right) l_1 + \left(\frac{\partial \phi}{\partial x_2} + x_1\right) l_2 = 0 \quad (6.3)$$

on the curved surface of the rod, where $(l_1, l_2, 0)$ denote the direction-cosines of the normal to the surface. Neglecting terms of higher degree than the second in ϵ and ψx_3 , equations (6.1) may be written, in the neighborhood of $x_3 = 0$, as

$$\begin{aligned} v_1 &= -\sigma\epsilon x_1 - \psi x_2 x_3 - \frac{1}{2}\psi^2 x_1 x_3^2 - (1 - \sigma)\epsilon\psi x_2 x_3, \\ v_2 &= -\sigma\epsilon x_2 + \psi x_1 x_3 - \frac{1}{2}\psi^2 x_2 x_3^2 + (1 - \sigma)\epsilon\psi x_1 x_3 \end{aligned}$$

and

$$v_3 = \epsilon x_3 + \psi\phi(x_1, x_2). \quad (6.4)$$

From (6.4) and (4.2), we obtain

$$\begin{aligned} e'_{11} &= e'_{22} = -2\sigma\epsilon - \psi^2 x_3^2, & e'_{33} &= 2\epsilon, \\ e'_{23} &= \psi \left(x_1 + \frac{\partial \phi}{\partial x_2} \right) + (1 - \sigma)\epsilon\psi x_1 - \psi^2 x_2 x_3, \\ e'_{31} &= \psi \left(-x_2 + \frac{\partial \phi}{\partial x_1} \right) - (1 - \sigma)\epsilon\psi x_2 - \psi^2 x_1 x_3, \\ e'_{12} &= 0 \quad \text{and} \quad \Delta' = \epsilon(1 - 2\sigma) - \psi^2 x_3^2. \end{aligned} \quad (6.5)$$

Introducing (6.5) into (4.1) and neglecting terms in $\psi^2 x_3^2$, we obtain

$$\begin{aligned}\tau_{11} &= \tau_{22} = 0, & \tau_{33} &= 8\epsilon[a_2 - \sigma(a_1 + 2a_2)], \\ \tau_{33} &= -2a_1 \left[\psi \left(x_1 + \frac{\partial \phi}{\partial x_2} \right) + (1 - \sigma)\epsilon\psi x_1 - \psi^2 x_2 x_3 \right], \\ \tau_{31} &= -2a_1 \left[\psi \left(-x_2 + \frac{\partial \phi}{\partial x_1} \right) - (1 - \sigma)\epsilon\psi x_2 - \psi^2 x_1 x_3 \right]\end{aligned}$$

and

$$\tau_{12} = 0. \quad (6.6)$$

By introducing (6.6) into the equations of motion and boundary conditions of classical elasticity theory and neglecting terms of higher degree than the first in ψx_i and ϵ , we readily see that, provided that $\phi(x_1, x_2)$ satisfies the relations (6.2) and (6.3), the displacements described by (6.4) may be maintained by a torsional couple alone.

We now calculate the body forces and surface tractions which must be applied according to the second order elasticity theory, in addition to those given by the classical elasticity theory, in order to maintain the deformation described by equations (6.4).

Introducing the relations (6.4) and (6.5) into equations (4.7) and neglecting terms of higher degree than the second in ϵ and ψx_i and terms of higher degree than the first in ψx_3 in the results so obtained, we have

$$\begin{aligned}\alpha'_{11} &= \sigma^2 \epsilon^2 + \psi^2 x_2^2, & \alpha'_{22} &= \sigma^2 \epsilon^2 + \psi^2 x_1^2, \\ \alpha'_{33} &= \epsilon^2 + \psi^2 \left[\left(\frac{\partial \phi}{\partial x_1} \right)^2 + \left(\frac{\partial \phi}{\partial x_2} \right)^2 \right], \\ \alpha'_{23} &= \psi \epsilon \left(x_1 - \sigma \frac{\partial \phi}{\partial x_2} \right) + \psi^2 x_3 \frac{\partial \phi}{\partial x_1}, \\ \alpha'_{13} &= -\psi \epsilon \left(x_2 + \sigma \frac{\partial \phi}{\partial x_1} \right) - \psi^2 x_3 \frac{\partial \phi}{\partial x_2}, \\ \alpha'_{12} &= -\psi^2 x_1 x_2\end{aligned}$$

and

$$\begin{aligned}\alpha' &= 2\epsilon^2 \sigma^2 + \epsilon^2 + \psi^2 (x_1^2 + x_2^2) + \psi^2 \left[\left(\frac{\partial \phi}{\partial x_1} \right)^2 + \left(\frac{\partial \phi}{\partial x_2} \right)^2 \right]; & (6.7) \\ E'_{11} &= -4\sigma \epsilon^2 - \psi^2 \left(x_1 + \frac{\partial \phi}{\partial x_2} \right)^2, & E'_{22} &= -4\sigma \epsilon^2 - \psi^2 \left(x_2 - \frac{\partial \phi}{\partial x_1} \right)^2, \\ & & E'_{33} &= 4\epsilon^2 \sigma^2,\end{aligned}$$

$$E'_{23} = 2\sigma\epsilon\psi\left(x_1 + \frac{\partial\phi}{\partial x_2}\right), \quad E'_{31} = -2\sigma\epsilon\psi\left(x_2 - \frac{\partial\phi}{\partial x_1}\right),$$

$$E'_{12} = -\psi^2\left(x_1 + \frac{\partial\phi}{\partial x_2}\right)\left(x_2 - \frac{\partial\phi}{\partial x_1}\right)$$

and

$$E' = -8\sigma\epsilon^2 + 4\sigma^2\epsilon^2 - \psi^2\left[\left(x_1 + \frac{\partial\phi}{\partial x_2}\right)^2 + \left(x_2 - \frac{\partial\phi}{\partial x_1}\right)^2\right]. \quad (6.8)$$

Introducing (6.7) and (6.8) into (4.6), we obtain

$$\begin{aligned} \tau'_{11} &= 2\psi^2\left[-a_1x_2^2 + (a_1 - a_5)\left(x_1 + \frac{\partial\phi}{\partial x_2}\right)^2 + (a_1 + 2a_2)\left\{\left(\frac{\partial\phi}{\partial x_1}\right)^2 + \left(\frac{\partial\phi}{\partial x_2}\right)^2\right\}\right. \\ &\quad \left.+ (a_1 + 2a_2)(x_1^2 + x_2^2) - (a_1 + a_3)\left\{\left(x_1 + \frac{\partial\phi}{\partial x_2}\right)^2 + \left(x_2 - \frac{\partial\phi}{\partial x_1}\right)^2\right\}\right] + \gamma, \\ \tau'_{22} &= 2\psi^2\left[-a_1x_1^2 + (a_1 - a_5)\left(x_2 - \frac{\partial\phi}{\partial x_1}\right)^2 + (a_1 + 2a_2)\left\{\left(\frac{\partial\phi}{\partial x_1}\right)^2 + \left(\frac{\partial\phi}{\partial x_2}\right)^2\right\}\right. \\ &\quad \left.+ (a_1 + 2a_2)(x_1^2 + x_2^2) - (a_1 + a_3)\left\{\left(x_1 + \frac{\partial\phi}{\partial x_2}\right)^2 + \left(x_2 - \frac{\partial\phi}{\partial x_1}\right)^2\right\}\right] + \gamma, \\ \tau'_{33} &= 2\psi^2\left[2a_2\left\{\left(\frac{\partial\phi}{\partial x_1}\right)^2 + \left(\frac{\partial\phi}{\partial x_2}\right)^2\right\} + (a_1 + 2a_2)(x_1^2 + x_2^2)\right. \\ &\quad \left.- (a_1 + a_3)\left\{\left(x_1 + \frac{\partial\phi}{\partial x_2}\right)^2 + \left(x_2 - \frac{\partial\phi}{\partial x_1}\right)^2\right\}\right] + \gamma', \\ \tau'_{23} &= 2\epsilon\psi\left[x_1\{(4a_2 - 2a_3) - 2\sigma(2a_1 + 4a_2 - 2a_3 - a_5)\}\right. \\ &\quad \left.+ \frac{\partial\phi}{\partial x_2}\{(a_1 + 4a_2 - 2a_3) - \sigma(3a_1 + 8a_2 - 4a_3 - 2a_5)\}\right] - 2a_1\psi^2x_3\frac{\partial\phi}{\partial x_1}, \\ \tau'_{31} &= 2\epsilon\psi\left[-x_2\{(4a_2 - 2a_3) - 2\sigma(2a_1 + 4a_2 - 2a_3 - a_5)\}\right. \\ &\quad \left.+ \frac{\partial\phi}{\partial x_1}\{(a_1 + 4a_2 - 2a_3) - \sigma(3a_1 + 8a_2 - 4a_3 - 2a_5)\}\right] + 2a_1\psi^2x_3\frac{\partial\phi}{\partial x_2}, \end{aligned}$$

and

$$\tau'_{12} = 2\psi^2\left[a_1x_1x_2 + (a_1 - a_5)\left(x_1 + \frac{\partial\phi}{\partial x_2}\right)\left(x_2 - \frac{\partial\phi}{\partial x_1}\right)\right], \quad (6.9)$$

where γ and γ' are given by equations (5.17).

From equations (6.6) and (4.13), we obtain

$$\rho_0 \mathfrak{X}_1 = -2a_1 \psi^2 x_1, \quad \rho_0 \mathfrak{X}_2 = -2a_1 \psi^2 x_2 \text{ and } \rho_0 \mathfrak{X}_3 = 0. \quad (6.10)$$

Substituting from (6.4), (6.6), (6.9) and (6.10) in equations (4.14) and writing $x_3 = 0$ in the results obtained, we find, employing the relation (6.2), that

$$\begin{aligned} -\rho_0 (X'_1 - X_1) &= 2a_1 \psi^2 \left[x_1 \frac{\partial^2 \phi}{\partial x_1 \partial x_2} - x_2 \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial \phi}{\partial x_2} \right] + \frac{\partial \tau'_{11}}{\partial x_1} + \frac{\partial \tau'_{12}}{\partial x_2}, \\ -\rho_0 (X'_2 - X_2) &= 2a_1 \psi^2 \left[-x_2 \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + x_1 \frac{\partial^2 \phi}{\partial x_2^2} - \frac{\partial \phi}{\partial x_1} \right] + \frac{\partial \tau'_{12}}{\partial x_1} + \frac{\partial \tau'_{22}}{\partial x_2} \end{aligned}$$

and

$$-\rho_0 (X'_3 - X_3) = 0. \quad (6.11)$$

The boundary conditions on the curved surface yield, with $x_3 = 0$,

$$X'_{r1} - X_{r1} = l_1 \tau'_{11} + l_2 \tau'_{12} + 2a_1 \psi^2 (l_2 x_1 - l_1 x_2) \left(-x_2 + \frac{\partial \phi}{\partial x_1} \right),$$

$$X'_{r2} - X_{r2} = l_1 \tau'_{12} + l_2 \tau'_{22} + 2a_1 \psi^2 (l_2 x_1 - l_1 x_2) \left(x_1 + \frac{\partial \phi}{\partial x_2} \right),$$

and

$$\begin{aligned} X'_{r3} - X_{r3} &= l_1 \tau'_{13} + l_2 \tau'_{23} + \psi \tau_{33} (x_2 l_1 - x_1 l_2) \\ &= 2\epsilon \psi [(a_1 + 4a_2) - \sigma(3a_1 + 8a_2)] (x_2 l_1 - x_1 l_2). \end{aligned} \quad (6.12)$$

The boundary condition on an initially plane end yields

$$\begin{aligned} X'_{r1} - X_{r1} &= 2\epsilon \psi \left[x_2 \{ (a_1 - 4a_2 + 2a_3) + \sigma(a_1 + 8a_2 - 4a_3 - 2a_5) \} \right. \\ &\quad \left. + \frac{\partial \phi}{\partial x_1} \{ (a_1 + 4a_2 - 2a_3) - \sigma(a_1 + 8a_2 - 4a_3 - 2a_5) \} \right], \\ X'_{r2} - X_{r2} &= 2\epsilon \psi \left[-x_1 \{ a_1 - 4a_2 + 2a_3 \} + \sigma(a_1 + 8a_2 - 4a_3 - 2a_5) \right. \\ &\quad \left. + \frac{\partial \phi}{\partial x_2} \{ (a_1 + 4a_2 - 2a_3) - \sigma(a_1 + 8a_2 - 4a_3 - 2a_5) \} \right] \end{aligned}$$

and

$$\begin{aligned}
X'_{r3} - X_{r3} = 2\psi^2 & \left[(a_1 + 2a_2) \left\{ \left(\frac{\partial \phi}{\partial x_1} \right)^2 + \left(\frac{\partial \phi}{\partial x_2} \right)^2 \right\} + a_1 \left(x_1 \frac{\partial \phi}{\partial x_2} - x_2 \frac{\partial \phi}{\partial x_1} \right) \right. \\
& + (a_1 + 2a_2)(x_1^2 + x_2^2) - (a_1 + a_3) \left\{ \left(x_1 + \frac{\partial \phi}{\partial x_2} \right)^2 + \left(x_2 - \frac{\partial \phi}{\partial x_1} \right)^2 \right\} \Big] \quad (6.13) \\
& + \gamma' - 16\sigma\epsilon^2[a_2 - \sigma(a_1 + 2a_2)].
\end{aligned}$$

We note that on the curved surface $(X'_{r1} - X_{r1} - l_1\gamma)$ and $(X'_{r2} - X_{r2} - l_2\gamma)$ are independent of ϵ , while on the plane ends $\{X'_{r3} - X_{r3} - \gamma' + 16\sigma\epsilon^2[a_2 - \sigma(a_1 + 2a_2)]\}$ is independent of ϵ . The body forces $(X'_1 - X_1)$ and $(X'_2 - X_2)$ are also independent of ϵ . The change in dimensions of the rod due to non-application of these forces can be determined by calculating, on the basis of the classical elasticity theory, the change in dimensions of the undeformed rod if a system of forces of equal magnitude but opposite sign is applied to it. This can be done, either by solving the appropriate problem in classical elasticity theory exactly, obtaining the displacements undergone by each point of the rod, as was done in §5, or by calculating directly the appropriate strain average by the method of Betti.

We can calculate by this method the mean value ϵ' of the fractional elongation parallel to the axis of the rod. We have (see, for example, Love (1927) §123)

$$\begin{aligned}
\epsilon' = & -\frac{1}{8Aa_1} \frac{a_1 + 4a_2}{a_1 + 3a_2} \left[\iint -\sigma\rho_0 \{(X_1 - X'_1)x_1 + (X_2 - X'_2)x_2\} dx_1 dx_2 \right. \\
& - \int \sigma \{(X_{r1} - X'_{r1} + l_1\gamma)x_1 + (X_{r2} - X'_{r2} + l_2\gamma)x_2\} dS \quad (6.14) \\
& \left. + \iint \{X_{r3} - X'_{r3} + \gamma' - 16\sigma\epsilon^2[a_2 - \sigma(a_1 + 2a_2)]\} dx_1 dx_2 \right],
\end{aligned}$$

where A is the cross-sectional area of the rod, the integrations with respect to x_1 and x_2 are carried out over the whole cross-section of the rod, dS denotes an element of length of the periphery of the cross-section of the rod and the integration with respect to S is carried out over the whole of this periphery. $(X_1 - X'_1)$ and $(X_2 - X'_2)$ are given by equations (6.11); $(X_{r1} - X'_{r1})$ and $(X_{r2} - X'_{r2})$ are given by equations (6.12) and $(X_{r3} - X'_{r3})$ by the last of equations (6.13).

It can readily be seen that the force $(X_{r3} - X'_{r3})$, defined by the third of equations (6.12), makes no contribution to ϵ' , since

$$\int (x_2 l_1 - x_1 l_2) dS = 0. \quad (6.15)$$

The forces $(X_{r1} - X'_{r1})$ and $(X_{r2} - X'_{r2})$, defined by the first two of equations (6.13), acting on one end of the rod give a contribution to ϵ' which is equal in magnitude and opposite in sign to that due to the corresponding forces acting

on the other end of the rod. Accordingly, these forces acting on both ends of the rod make no resultant contribution to ϵ' .

Substituting from (6.11) for $(X_1 - X'_1)$ and $(X_2 - X'_2)$, we obtain

$$\begin{aligned}
 \rho_0 \iint \{ (X_1 - X'_1)x_1 + (X_2 - X'_2)x_2 \} dx_1 dx_2 \\
 = 2a_1 \psi^2 \iint \left\{ (x_1^2 - x_2^2) \frac{\partial^2 \phi}{\partial x_1 \partial x_2} - x_1 x_2 \left(\frac{\partial^2 \phi}{\partial x_1^2} - \frac{\partial^2 \phi}{\partial x_2^2} \right) \right. \\
 \left. + \left(x_1 \frac{\partial \phi}{\partial x_2} - x_2 \frac{\partial \phi}{\partial x_1} \right) \right\} dx_1 dx_2 \\
 + \iint \left\{ x_1 \frac{\partial \tau'_{11}}{\partial x_1} + x_1 \frac{\partial \tau'_{12}}{\partial x_2} + x_2 \frac{\partial \tau'_{12}}{\partial x_1} + x_2 \frac{\partial \tau'_{22}}{\partial x_2} \right\} dx_1 dx_2.
 \end{aligned} \tag{6.16}$$

Substituting from (6.12) for $(X_{r1} - X'_{r1})$ and $(X_{r2} - X'_{r2})$, we obtain

$$\begin{aligned}
 \int \{ (X_{r1} - X'_{r1} + l_1 \gamma) x_1 + (X_{r2} - X'_{r2} + l_2 \gamma) x_2 \} dS \\
 = -2a_1 \psi^2 \int (l_2 x_1 - l_1 x_2) \left(x_1 \frac{\partial \phi}{\partial x_1} + x_2 \frac{\partial \phi}{\partial x_2} \right) dS \\
 - \int \{ l_1 [(\tau'_{11} - \gamma)x_1 + \tau'_{12} x_2] + l_2 [\tau'_{12} x_1 + (\tau'_{22} - \gamma)x_2] \} dx_1 dx_2 \\
 = -2a_1 \psi^2 \iint \left\{ -\frac{\partial}{\partial x_1} \left(x_1 x_2 \frac{\partial \phi}{\partial x_1} + x_2^2 \frac{\partial \phi}{\partial x_2} \right) \right. \\
 \left. + \frac{\partial}{\partial x_2} \left(x_1^2 \frac{\partial \phi}{\partial x_1} + x_1 x_2 \frac{\partial \phi}{\partial x_2} \right) \right\} dx_1 dx_2 \\
 - \iint \left\{ \frac{\partial}{\partial x_1} [(\tau'_{11} - \gamma)x_1 + \tau'_{12} x_2] + \frac{\partial}{\partial x_2} [\tau'_{12} x_1 + (\tau'_{22} - \gamma)x_2] \right\} dx_1 dx_2 \\
 = -2a_1 \psi^2 \iint \left\{ (x_1^2 - x_2^2) \frac{\partial^2 \phi}{\partial x_1 \partial x_2} - x_1 x_2 \left(\frac{\partial^2 \phi}{\partial x_1^2} - \frac{\partial^2 \phi}{\partial x_2^2} \right) + \left(x_1 \frac{\partial \phi}{\partial x_2} - x_2 \frac{\partial \phi}{\partial x_1} \right) \right\} \\
 \cdot dx_1 dx_2 - \iint \left\{ x_1 \frac{\partial \tau'_{11}}{\partial x_1} + x_2 \frac{\partial \tau'_{12}}{\partial x_1} + x_1 \frac{\partial \tau'_{12}}{\partial x_2} + x_2 \frac{\partial \tau'_{22}}{\partial x_2} \right\} dx_1 dx_2 \\
 - \iint \{ \tau'_{11} + \tau'_{22} - 2\gamma \} dx_1 dx_2.
 \end{aligned} \tag{6.17}$$

In the above calculation the transformation from a contour to a surface integral is effected by means of Green's theorem.

Again, substituting from (6.13) for $(X_{r3} - X'_{r3})$ we obtain

$$\begin{aligned} & \iint \{X_{r3} - X'_{r3} + \gamma' - 16\sigma\epsilon^2[a_2 - \sigma(a_1 + 2a_2)]\} dx_1 dx_2 \\ &= -2\psi^2 \iint \left\{ (a_1 + 2a_2) \left[\left(\frac{\partial\phi}{\partial x_1} \right)^2 + \left(\frac{\partial\phi}{\partial x_2} \right)^2 \right] + a_1 \left(x_1 \frac{\partial\phi}{\partial x_2} - x_2 \frac{\partial\phi}{\partial x_1} \right) \right. \\ & \quad \left. + (a_1 + 2a_2)(x_1^2 + x_2^2) - (a_1 + a_3) \left[\left(x_1 + \frac{\partial\phi}{\partial x_2} \right)^2 + \left(x_2 - \frac{\partial\phi}{\partial x_1} \right)^2 \right] \right\} dx_1 dx_2. \end{aligned} \quad (6.18)$$

Employing the notation

$$I = \iint (x_1^2 + x_2^2) dx_1 dx_2 \quad \text{and} \quad B = \iint \left[\left(\frac{\partial\phi}{\partial x_1} \right)^2 + \left(\frac{\partial\phi}{\partial x_2} \right)^2 \right] dx_1 dx_2 \quad (6.19)$$

we have, also, (see, for example, Love (1927) §216)

$$\iint \left(x_1 \frac{\partial\phi}{\partial x_2} - x_2 \frac{\partial\phi}{\partial x_1} \right) dx_1 dx_2 = -B$$

and

$$\iint \left\{ \left(x_1 + \frac{\partial\phi}{\partial x_2} \right)^2 + \left(x_2 - \frac{\partial\phi}{\partial x_1} \right)^2 \right\} dx_1 dx_2 = I - B. \quad (6.20)$$

Introducing the results of equations (6.16) to (6.20) into (6.14), we obtain

$$\begin{aligned} \epsilon' = \frac{\psi^2}{4Aa_1(a_1 + 3a_2)} \{ & (I - B)[a_3(a_1 + 2a_2) - a_1(2a_2 - a_3)] \\ & - 2Ba_1(a_1 + 3a_2) \}. \end{aligned} \quad (6.21)$$

This result may be specialized for a wide variety of forms of the cross-section of the rod, for which the classical torsion problem has been solved.

(a) *Elliptical cross-section.* If the cross-section of the rod is elliptical and has semi-axes a and b , then

$$\phi = Kx_1 x_2 \quad \text{where} \quad K = -\frac{a^2 - b^2}{a^2 + b^2}. \quad (6.22)$$

We obtain

$$\begin{aligned} B &= K^2 I = K^2 \iint (x_1^2 + x_2^2) dx_1 dx_2 \\ &= \frac{1}{4} \pi ab(a^2 + b^2) K^2 \end{aligned}$$

and

$$A = \pi ab \quad (6.23)$$

so that

$$\epsilon' = \frac{\psi^2}{8(a^2 + b^2)a_1(a_1 + 3a_2)} \{2a^2b^2[a_5(a_1 + 2a_2) - a_1(2a_2 - a_3)] - (a^2 - b^2)^2 a_1(a_1 + 3a_2)\}. \quad (6.24)$$

For the case when the rod has a circular cross-section of radius a , we may write $a = b$ in equation (6.24), obtaining the result (5.44).

(b) *Rectangular Cross-Section.* If the cross-section of the rod is rectangular and of dimensions $2a \times 2b$, we have

$$I = \frac{4}{3} ab(a^2 + b^2) \quad \text{and} \quad A = 4ab;$$

and

$$B = \frac{4}{3} ab(a^2 - 3b^2) + b^4 \left(\frac{4}{\pi}\right)^5 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5} \tanh \frac{(2n+1)\pi a}{b}. \quad (6.25)$$

Introducing these results into (6.21) we obtain the expression for ϵ' for a rod of rectangular cross-section.

7. The Passage to the Incompressible Case. For a compressible material the strain-energy function W has been expressed as a triply-infinite power series in the strain invariants J_1 , J_2 and J_3 defined by equations (2.3) and (2.4). If the principal extensions are sufficiently small so that terms of higher degree than the third in the principal extensions can be neglected in the expression for W , then W may be expressed by equation (2.7). In this section we shall investigate what further simplifications can be made in the form of W if the elastic material is assumed to be incompressible. For this purpose it is more convenient to express W in terms of three other invariants J'_1 , J'_2 , J'_3 given by

$$J'_1 = I_3 - 1, \quad J'_2 = J_2 \quad \text{and} \quad J'_3 = J_3. \quad (7.1)$$

If the extension ratios are small, then J'_1 , J'_2 and J'_3 are, in general, of first, second and third degrees respectively in the extension ratios and provided that terms of higher degree than the third in the expression for the strain-energy function can be neglected, this can be expressed in the form

$$W = a'_0 J'_1 + a'_1 J'_2 + a'_2 J'^2_1 + a'_3 J'_1 J'_2 + a'_4 J'^3_1 + a'_5 J'_3. \quad (7.2)$$

Comparing equations (7.2) and (2.7), we obtain

$$a'_0 = 0, \quad a'_1 = a_1, \quad a'_2 = a_2, \quad a'_3 + 2a'_2 = a_3, \quad a'_4 = a_4 \quad \text{and} \quad a'_5 = a_5. \quad (7.3)$$

Employing the relations (7.1) and (2.4) in equation (2.1), we obtain

$$t_{ik} = \frac{2}{\tau} \left\{ g_{ik} \left(\frac{\partial W}{\partial J'_3} - 2 \frac{\partial W}{\partial J'_2} \right) - G_{ik} \left(\frac{\partial W}{\partial J'_2} - \frac{\partial W}{\partial J'_3} \right) + \left[(J'_1 + 1) \left(\frac{\partial W}{\partial J'_1} + \frac{\partial W}{\partial J'_3} \right) + (2J'_1 - J'_2 - 2J'_3 + 3) \left(\frac{\partial W}{\partial J'_2} - \frac{\partial W}{\partial J'_3} \right) \right] \delta_{ik} \right\}. \quad (7.4)$$

Introducing the expression (7.2) for W , with $a'_0 = 0$, into (7.4), and writing $g'_{ik} = g_{ik} - \delta_{ik}$ and $G'_{ik} = G_{ik} - \delta_{ik}$, we obtain

$$\begin{aligned} t_{ik} = \frac{2}{\tau} \{ & g'_{ik} (a'_5 - 2a'_1 - 2a'_3 J'_1) - G'_{ik} (a'_1 - a'_5 + a'_3 J'_1) \\ & + [J'_1 (2a'_1 + 2a'_2 - a'_5) + (a'_3 - a'_1 + a'_5) J'_2 \\ & + (2a'_2 + 2a'_3 + 3a'_4) J'^2_1] \delta_{ik} \}. \end{aligned} \quad (7.5)$$

In deriving equation (7.5), we have neglected terms in J'_1 , J'_2 and J'_3 of higher order of smallness than the second in the principal extensions, bearing in mind that J'_1 , J'_2 and J'_3 are, in general, of first, second and third order of smallness respectively in these.

We now consider what relations must be obeyed by the coefficients a'_1 , a'_2 , $\dots$, a'_5 in order that equations (7.2) and (7.5) may represent the strain-energy function and stress-strain relations respectively for an incompressible material.

We must first consider what we mean by the compressibility of a material. Let us consider a uniform dilatation of the material so that the principal extensions are given by

$$e_1 = e_2 = e_3 = e \text{ (say)}. \quad (7.6)$$

Then, from equations (2.2) we obtain

$$g'_{ik} = (1 + e)^2 - 1, \quad G'_{ik} = (1 + e)^4 - 1 \quad (i = k)$$

and

$$g'_{ik} = G'_{ik} = 0 \quad (i \neq k). \quad (7.7)$$

Also, from (7.6), (2.3), and (7.1), we obtain

$$J'_1 = (1 + e)^6 - 1. \quad (7.8)$$

Introducing these results into (7.5) we obtain

$$t_{ik} = T \text{ (say)} = \frac{2}{\tau} \{ 4e(a'_1 + 3a'_2) + O(e^2) \} \quad (i = k)$$

and

$$t_{ik} = 0 \quad (i \neq k), \quad (7.9)$$

where $O(e^2)$ denotes terms of the order of e^2 .

We consider the material to become less and less compressible as t_{ik}/e increases, *i.e.*, as $(a'_1 + 3a'_2)$ increases. An incompressible material is then an ideal limiting case obtained by allowing $(a'_1 + 3a'_2)$ to tend to infinity. This can only occur if a'_2 tends to infinity, for it can readily be shown that as a'_1 tends to infinity the forces necessary to produce deformations in which the volume is unchanged also tend to infinity. Thus, in order that equations (7.2) and (7.5) may represent the strain-energy function and stress-strain relations respectively for an incompressible material we must allow a'_2 to tend to infinity.

We note that if in equations (7.9) we consider the force T to be specified, then as the material tends to incompressibility, $a'_2 \rightarrow \infty$ and $e \rightarrow 0$ in such a way that their product is finite. Since $e = \frac{1}{3}J'_1 + O(e^2)$, we see that as the material tends to incompressibility, $a'_2 J'^2_1 \rightarrow 0$ in the expression (7.2) for W and $a'_2 J'_1$ tends to an undetermined finite quantity in the expression (7.5) for t_{ik} .

Since, for an incompressible material, we have $J'_1 = 0$, we obtain from equation (7.2), with $a'_0 = 0$,

$$W = a'_1 J'_2 + a'_5 J'_3, \quad (7.10)$$

and from equations (7.5),

$$t_{ik} = 2[g'_{ik}(a'_5 - 2a'_1) - G'_{ik}(a'_1 - a'_5)] + p\delta_{ik}, \quad (7.11)$$

where p is an undetermined quantity.

We have seen from an independent argument (Rivlin (1948), Rivlin & Saunders (1951)) that the strain-energy function for an incompressible material may be written, to terms of the third order of smallness in the principal extensions, as

$$W = C_1(I_1 - 3) + C_2(I_2 - 3), \quad (7.12)$$

where C_1 and C_2 are constants, and the stress components may be written, to terms of the second order of smallness in the principal extensions, as

$$t_{ik} = 2[g'_{ik}C_1 - G'_{ik}C_2] + p. \quad (7.13)$$

Comparing equations (7.12) and (7.13) with (7.10) and (7.11) respectively, bearing in the mind the relations (7.1) and (2.4), we obtain

$$C_1 = a'_5 - 2a'_1 \quad \text{and} \quad C_2 = a'_1 - a'_5, \quad (7.14)$$

which with (7.3) yield

$$a_1 = -(C_1 + C_2) \quad \text{and} \quad a_5 = -(C_1 + 2C_2). \quad (7.15)$$

Now, from (7.3), the condition $a'_2 \rightarrow \infty$ is equivalent to the conditions $a_2 \rightarrow \infty$ and $a_3 \rightarrow \infty$, in such a way that $a_3 - 2a_2 (=a'_3)$ may remain finite. By introducing these conditions together with (7.15) into results obtained for a compressible material by the methods given in this paper, we can obtain the results

applicable to an incompressible material for which the strain-energy function is given by (7.12). For example, let us consider the expression (5.44) for the extension undergone by a cylindrical rod, as the result of the application of a torsional couple producing an amount of torsion ψ . We obtain, by allowing a_2 and a_3 to tend to infinity in the manner described and by employing the relations (7.15),

$$\epsilon' = \frac{1}{12} \psi^2 a^2 \frac{C_1 + 2C_2}{C_1 + C_2}. \quad (7.16)$$

In a similar manner, we can specialize the result (6.24) for a rod of elliptical cross-section to the incompressible case, obtaining

$$\epsilon' = \frac{\psi^2}{24(a^2 + b^2)} [(C_1 + C_2)(3a^2 - b^2)(3b^2 - a^2) + 4C_2 a^2 b^2], \quad (7.17)$$

in agreement with the result of Green & Shield (1951). More generally, we can specialize the result (6.21) for a rod of arbitrary cross-section to the incompressible case to obtain

$$\epsilon' = \frac{\psi^2}{6A(C_1 + C_2)} [(I - 4B)(C_1 + C_2) + C_2(I - B)]. \quad (7.18)$$

It is evident that the method adopted, in this paper, for the solution of problems in second order elasticity theory cannot be applied directly to the case of incompressible materials, for which the strain-energy function is given by (7.12), since the additional forces required on the basis of the second order theory to maintain the deformation predicted by the first order theory will, in general, be infinite. However, the solutions appropriate to such materials may be obtained from the corresponding solutions in the compressible case.

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