*The Elementary Basic Principles of the Unified Theory of Relativity. B**

VÁCLAV HLAVATÝ

Graduate Institute for Applied Mathematics, Indiana University, Bloomington, Indiana

CONTENTS

PART I

1.	Introduction.....	2
2.	Field equations.....	4
3.	First information about the solution of (2.1).....	7
4.	The solution of (2.1).....	9

PART II

5.	Illustration.....	14
6.	Application.....	18
7.	The field $S_{\lambda\mu}{}^\nu$	22
8.	The field $U^\nu{}_{\lambda\mu}$	27
9.	Continuation.....	29
10.	Paths of photons.....	32

PART III

11.	Auxiliary results.....	35
12.	The tensor $V_{\nu\lambda\mu}$	37
13.	The tensor $S_{\omega\mu}{}^\nu$	40
14.	The connection ${}^m\Gamma_{\lambda\mu}^\nu$	42
15.	Another method of approximation.....	47
16.	Appendix: Problem B'	50

---

* Prepared under Army Contract DA 33-008 ORD 467.

PART I.

1. Introduction. In this paper we shall deal with a connection $\Gamma_{\lambda\mu}^\nu$ defined by

$$(1.1) \quad \partial_\omega g_{\lambda\mu} = \Gamma_{\lambda\omega}^\alpha g_{\alpha\mu} + \Gamma_{\omega\mu}^\alpha g_{\lambda\alpha},$$

where $g_{\lambda\mu}$ is an arbitrary tensor and

$$\partial_\omega \equiv \frac{\partial}{\partial x^\omega}.$$

As we mentioned in a previous paper* equation (1.1) represents the second principle (the Principle referred to as *B*) of the unified theory of relativity suggested by Einstein†. The first principle (Principle *A*) was dealt with in the aforementioned paper* while the third principle (Principle *C*) dealing with conditions imposed on $\Gamma_{\lambda\mu}^\nu$ will be treated in the next paper.

The tensor $g_{\lambda\mu}$ may be decomposed into its symmetrical part

$$h_{\lambda\mu} = g_{(\lambda\mu)} = h_{\mu\lambda}$$

and its skew-symmetrical part

$$k_{\lambda\mu} = g_{[\lambda\mu]} = -k_{\mu\lambda}.$$

Throughout this paper we confine ourselves to the case where the components $g_{\lambda\mu} = g_{\lambda\mu}(x)$ are finite in a certain part of the space and the determinant

$$(1.2) \quad h = \text{Det} | h_{\lambda\mu} | \neq 0.$$

Then there exists only one tensor $h^{\lambda\mu} = h^{\mu\lambda}$ such that

$$h^{\lambda\nu} h_{\lambda\mu} = \delta_\mu^\nu,$$

where δ_μ^ν is the Kronecker delta. Moreover we shall consider $h_{\lambda\mu}$ as the metric tensor of our space, so that all metric notions referred to in this paper are based on $h_{\lambda\mu}$. Departing from the convention of the first paper*, here we raise and lower indices by means of $h^{\lambda\mu}$ and $h_{\lambda\mu}$ §. The Christoffel symbols $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ are defined by means of $h_{\lambda\mu}$ in the usual way:

$$(1.3) \quad \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} = \frac{1}{2} h^{\nu\alpha} (\partial_\lambda h_{\mu\alpha} + \partial_\mu h_{\alpha\lambda} - \partial_\alpha h_{\lambda\mu}),$$

while ∇_μ will be referred to as the symbol of the covariant derivative with respect to $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$.

* V. HLAVATÝ, *The elementary basic principles of the unified theory of relativity A*, Journal of Rat. Mech. and Analysis **1** (1952), 539–562.

† A. EINSTEIN, *The meaning of relativity*, (Princeton University Press, 1950), Appendix II.

§ One exception to this rule will be stated below.

As long as it is not otherwise stated we assume also

$$(1.4a) \quad g \equiv \text{Det } |g_{\lambda\mu}| \neq 0$$

in the part of space we are considering. Then there is only one tensor $g^{\lambda\mu}$ such that

$$(1.4b) \quad g_{\lambda\mu} g^{\lambda\nu} = g_{\mu\lambda} g^{\mu\nu} = \delta_{\lambda}^{\nu}.$$

The tensor $g^{\lambda\nu}$ is the exception to the rule of raising the indices; *i.e.*, we have in general

$$g^{\lambda\nu} \neq g_{\alpha\beta} h^{\alpha\lambda} h^{\beta\nu}.$$

The problem *B* consists in solving the equations (1.1) for the $\Gamma_{\lambda\mu}^{\nu}$, so that the connection appears as a function of the tensor $g_{\lambda\mu}$ only. Although this problem was proposed by Einstein* for $n = 4$, we do not restrict ourselves to this particular case and assume only $n > 1$.

This paper is divided into three parts. The first and present part deals exclusively with the solution $\Gamma_{\lambda\mu}^{\nu}$ of (1.1). First we split this system in two covariant systems (Section 2). The first of these systems expresses $\Gamma_{\lambda\mu}^{\nu}$ by means of its skew-symmetric part; the second one expresses this skew-symmetric part in terms of $g_{\lambda\mu}$ and its covariant derivatives. This method enables us to find necessary and sufficient conditions for $g_{\lambda\mu}$ in order that the system (1.1) admit exactly one solution $\Gamma_{\lambda\mu}^{\nu}$ (Section 3). Moreover we find necessary and sufficient conditions for $g_{\lambda\mu}$ in order that the system (1.1) admit more than one solution and necessary and sufficient conditions that the system (1.1) admit no solution (illustrations for each of these three cases are exhibited in the second part). Let us stress here only that against all expectation $g \neq 0$ is not a sufficient condition for (1.1) to admit exactly one solution. Finally in the fourth section we exhibit the solution $\Gamma_{\lambda\mu}^{\nu}$ in terms of $g_{\lambda\mu}$ (equation (4.13)).

The second part consists of Sections 5–10. In the first two sections we compute in some special cases the solution as given by (4.13). The method used here may be applied also for more general cases expressed by means of non-holonomic coordinates. Although the problem of solving the system (1.1) has been completely answered from the theoretical point of view, the actual computation to be performed (in order to test the physical theory based on (1.1) by means of experiments) is rather tiresome. Therefore we consider in the remaining sections of this part the cases in which this computation may be avoided. All these considerations are based on a special structure of the tensor $g_{\lambda\mu}$. Some of these cases are exhibited in order to know what to avoid in dealing with the problem *C* (in the next paper). Two of the cases treated here deserve to be mentioned: In Section 9 we deal with a connection $\Gamma_{\lambda\mu}^{\nu}$, a solution of (1.1) all whose autoparallel lines are at the same time the paths “of the old general theory of relativity†,”

---

* Loco cit. sub† on page 2.

† For $n = 4$.

while in Section 10 we consider the case in which minimal autoparallel lines are identified with paths of photons (in the old general theory of relativity).

The third part consists of Sections 11–16. In Sections 11–14 we exhibit a procedure for approximating in a finite number m of steps (in general two are sufficient) the solution $\Gamma_{\lambda\mu}^\nu$ by another one, say by ${}^m\Gamma_{\lambda\mu}^\nu$, such that the difference $\Gamma_{\lambda\mu}^\nu - {}^m\Gamma_{\lambda\mu}^\nu$ is as small as we please. This method is based on the assumption that $k_{\lambda\mu}$ is small with respect to $h_{\lambda\mu}$ (but not negligibly small)[†]. In Section 15 we use these results to find an m -th approximation ($m = 1, 2, 3, \dots$) to $\Gamma_{\lambda\mu}^\nu$, assuming that the product of $r \geq 2m + 1$ components of $k_{\lambda\mu}$ is negligibly small.

The last section of this part is devoted to a global consideration of another connection ${}^*\Gamma_{\lambda\mu}^\nu$ which is given by

$$(1.5) \quad \partial_\omega g_{\lambda\mu} = {}^*\Gamma_{\lambda\omega}^\alpha g_{\alpha\mu} + {}^*\Gamma_{\mu\omega}^\alpha g_{\lambda\alpha}$$

instead of (1.1). It turns out that this system, unlike (1.1), imposes a condition on $g_{\lambda\mu}$ namely $g/h = \text{const.}$

2. Field equations. In this section we derive some simple consequences from the equations

$$(2.1) \quad \partial_\omega g_{\lambda\mu} = \Gamma_{\lambda\omega}^\alpha g_{\alpha\mu} + \Gamma_{\omega\mu}^\alpha g_{\lambda\alpha}.$$

Theorem (2.1a). *Put*

$$(2.2a) \quad S_{\lambda\mu}^\nu = \Gamma_{[\lambda\mu]}^\nu,$$

$$(2.2b) \quad Q_{\omega\mu\lambda} = S_{\omega\mu}^\alpha g_{\lambda\alpha},$$

and denote by D_ω the symbol of the covariant derivative with respect to $\Gamma_{\lambda\mu}^\nu$. Then equation (2.1) is equivalent to

$$(2.3a) \quad D_\omega g_{\lambda\mu} = 2Q_{\omega\mu\lambda}.$$

Proof. We have first

$$D_\omega g_{\lambda\mu} = \partial_\omega g_{\lambda\mu} - \Gamma_{\lambda\omega}^\alpha g_{\alpha\mu} - \Gamma_{\mu\omega}^\alpha g_{\lambda\alpha}.$$

Substituting into this equation $\partial_\omega g_{\lambda\mu}$ taken from (2.1) we obtain (2.3a).

Corollary (2.1). *We have*

$$(2.3b) \quad D_\omega h_{\lambda\mu} = 2Q_{\omega(\mu\lambda)},$$

$$(2.3c) \quad D_\omega k_{\lambda\mu} = 2Q_{\omega[\mu\lambda]}.$$

Proof. This is an evident consequence of (2.3a) and

$$g_{(\lambda\mu)} = h_{\lambda\mu}, \quad g_{[\lambda\mu]} = k_{\lambda\mu}.$$

[†] The exact definition is in Section 11.

Using the tensor $g^{\lambda\nu}$ defined by (1.4b) one finds a formula similar to (2.3a).

Theorem (2.1b). *We have*

$$(2.3d) \quad D_\omega g^{\lambda\nu} = -2S_{\omega\alpha}{}^\nu g^{\lambda\alpha}.$$

Proof. One obtains from (1.4b) and (2.3a)

$$g^{\lambda\nu} D_\omega g_{\lambda\mu} = -g_{\lambda\mu} D_\omega g^{\lambda\nu} = 2g^{\lambda\nu} S_{\omega\mu}{}^\alpha g_{\lambda\alpha} = 2S_{\omega\mu}{}^\nu,$$

and this equation yields (2.3d).

Corollary (2.2). *We have*

$$(2.3e) \quad D_\omega g = 2S_{\omega\alpha}{}^\alpha g.$$

Proof. If we put $\nu = \mu$ in the preceding equation we obtain

$$g^{\lambda\mu} D_\omega g_{\lambda\mu} = 2S_{\omega\alpha}{}^\alpha,$$

and this equation yields at once (2.3e).

The equations (2.3b) and (2.3c) are very important for our purposes. Let us consider first (2.3b).

Theorem (2.2). *The equations (2.3b) admit the solution*

$$(2.4) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + S_{\lambda\mu}{}^\nu + 2h^{\nu\alpha} S_{\alpha(\mu}{}^\beta k_{\lambda)\beta}.$$

Proof. We have from (2.3b)

$$(2.5a) \quad 2Q_{\omega(\mu\lambda)} = \partial_\omega h_{\lambda\mu} - \Gamma_{\lambda\omega}^\alpha h_{\alpha\mu} - \Gamma_{\mu\omega}^\alpha h_{\lambda\alpha},$$

and consequently

$$(2.5b) \quad \begin{aligned} (Q_{\omega(\mu\lambda)} + Q_{\lambda(\omega\mu)} - Q_{\mu(\lambda\omega)})h^{\mu\nu} &= \left\{ \begin{matrix} \nu \\ \omega\lambda \end{matrix} \right\} + \frac{1}{2}h^{\mu\nu}(-\Gamma_{\lambda\omega}^\alpha h_{\alpha\mu} - \Gamma_{\mu\lambda}^\alpha h_{\alpha\omega} \\ &\quad + \Gamma_{\omega\mu}^\alpha h_{\alpha\lambda} - \Gamma_{\mu\omega}^\alpha h_{\alpha\lambda} - \Gamma_{\omega\lambda}^\alpha h_{\alpha\mu} \\ &\quad + \Gamma_{\lambda\mu}^\alpha h_{\alpha\omega}), \\ &= \left\{ \begin{matrix} \nu \\ \omega\lambda \end{matrix} \right\} - \Gamma_{\lambda\omega}^\nu + S_{\lambda\omega}{}^\nu - 2h^{\mu\nu} S_{\mu(\lambda}{}^\alpha h_{\omega)\alpha}. \end{aligned}$$

On the other hand

$$(2.5c) \quad \begin{aligned} (Q_{\omega(\mu\lambda)} + Q_{\lambda(\omega\mu)} - Q_{\mu(\lambda\omega)})h^{\mu\nu} &= h^{\mu\nu}(S_{\omega(\mu}{}^\alpha g_{\lambda)\alpha} + S_{\lambda(\omega}{}^\alpha g_{\mu)\alpha} - S_{\mu(\lambda}{}^\alpha g_{\omega)\alpha}), \\ &= -2S_{\mu(\lambda}{}^\alpha g_{\omega)\alpha} h^{\mu\nu}. \end{aligned}$$

Substituting for the left hand term from (2.5c) into (2.5b) we obtain (2.4).

Note. If we knew the tensor $S_{\lambda\mu}{}^\nu$ then equation (2.4) would solve our problem. In physical applications there are some opportunities to obtain such a tensor*.

---

* Thus for instance the angular momentum tensor in the special theory of relativity.

However as far as the geometrical point of view is concerned we do not know it *a priori*, so that we have to determine it from the given data only (*i.e.* by means of $g_{\lambda\mu}$). The first step toward accomplishing this task is dealt with in the next

Theorem (2.3). *Put*

$$(2.6) \quad K_{\omega\mu\nu} = \nabla_{\omega} g_{\nu\mu} + \nabla_{\mu} g_{\omega\nu} + \nabla_{\nu} g_{\omega\mu}.$$

Then the tensor $S_{\lambda\mu}{}^{\nu}$ involved in (2.4) satisfies the conditions

$$(2.7a) \quad 2S_{\omega\mu\nu} = K_{\omega\mu\nu} + 4S_{\nu\beta\gamma} k_{[\omega}{}^{[\beta} k_{\mu]}{}^{\gamma]} + 4k_{[\omega}{}^{\beta} S_{\mu]\beta\gamma} k_{\nu}{}^{\gamma},$$

equivalent to

$$(2.7b) \quad 2S_{\xi\eta\zeta} X_{\omega\mu\nu}^{\xi\eta\zeta} = K_{\omega\mu\nu},$$

where*

$$(2.8) \quad X_{\omega\mu\nu}^{\xi\eta\zeta} = \delta_{[\omega\mu]}^{\xi\eta]}{}_{\nu}{}^{\zeta} - 2\delta_{\nu}^{[\xi} k_{[\omega}{}^{\eta]} k_{\mu]}{}^{\zeta} - 2\delta_{[\mu}^{[\xi} k_{\omega]}{}^{\eta]} k_{\nu]}{}^{\zeta}.$$

Proof. Put

$$(2.9) \quad U^{\nu}{}_{\lambda\mu} = 2h^{\nu\alpha} S_{\alpha(\lambda}{}^{\beta} k_{\mu)\beta} = 2k_{(\mu}{}^{\beta} S^{\nu}{}_{\lambda)\beta},$$

so that (2.4) reduces to

$$(2.10) \quad \Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu}.$$

Consequently we obtain from (2.3c) by virtue of (2.10)

$$(2.11) \quad \begin{aligned} 2Q_{\omega[\mu\lambda]} &= \nabla_{\omega} k_{\lambda\mu} + (S_{\lambda\omega}{}^{\alpha} + U^{\alpha}{}_{\lambda\omega})k_{\mu\alpha} - (S_{\mu\omega}{}^{\alpha} + U^{\alpha}{}_{\mu\omega})k_{\lambda\alpha} \\ &= \nabla_{\omega} g_{\lambda\mu} + 2S_{\omega[\mu}{}^{\alpha} k_{\lambda]\alpha} - 2U^{\alpha}{}_{\omega[\mu} k_{\lambda]\alpha}{}^{\dagger} \\ &= \nabla_{\omega} g_{\lambda\mu} + 2k_{[\mu}{}^{\alpha} S_{\lambda]\omega\alpha} - 2U_{\alpha\omega[\mu} k_{\lambda]}{}^{\alpha}. \end{aligned}$$

Therefore

$$(2.12) \quad 2(Q_{\omega[\mu\nu]} + Q_{\mu[\nu\omega]} - Q_{\nu[\omega\mu]}) = K_{\omega\mu\nu} + 2k_{\nu}{}^{\alpha} S_{\omega\mu\alpha} - 4U_{\alpha\nu[\mu} k_{\omega]}{}^{\alpha}.$$

On the other hand

$$(2.13) \quad 2(Q_{\omega[\mu\nu]} + Q_{\mu[\nu\omega]} - Q_{\nu[\omega\mu]}) = 2S_{\omega\mu}{}^{\alpha} g_{\nu\alpha}.$$

Substituting for the left hand member of (2.12) from (2.13) we obtain

$$(2.14) \quad 2S_{\omega\mu\nu} = K_{\omega\mu\nu} - 4U_{\alpha\nu[\mu} k_{\omega]}{}^{\alpha},$$

* We are using the abbreviation $\delta_{\omega\mu\nu}^{\xi\eta\zeta} = \delta_{\omega}^{\xi} \delta_{\mu}^{\eta} \delta_{\nu}^{\zeta}$.

† We are using here the formula $\nabla_{\omega} g_{\lambda\mu} = \nabla_{\omega} h_{\lambda\mu} + \nabla_{\omega} k_{\lambda\mu} = \nabla_{\omega} k_{\lambda\mu}$.

and this equation together with (2.9) leads to (2.7a), which also may be written in the form (2.7b).

In the following section the equation (2.7b) will be dealt with.

3. First information about the solution of (2.1). We shall first rewrite equation (2.7b) for later use. We define the quantity $'X_{cd\nu}^{ab\xi}$ in the following way:

$$'X_{cd\nu}^{ab\xi} = 2X_{cd\nu}^{ab\xi} \quad \text{for} \quad a < b, c < d,$$

the remaining $'X_{\omega\mu\nu}^{\xi\eta\zeta} = 0$.

Then we have

$$S_{\xi\eta\zeta} X_{cd\nu}^{\xi\eta\zeta} = S_{ab\xi} 'X_{cd\nu}^{ab\xi},$$

so that we have from (2.7b)

$$(3.1) \quad 2S_{ab\xi} 'X_{cd\nu}^{ab\xi} = K_{cd\nu}, \quad c < d, a < b.$$

In this section we shall investigate equation (3.1). Using the quantities $'X_{\nu cd}^{\xi ab}$ we denote by $((X))$ the square matrix of $n\binom{n}{2}$ columns built up in the following way:

In an arbitrarily selected row there are elements $'X_{cd\nu}^{ab\xi}$ with fixed $cd\nu$, while an arbitrary column contains the elements $'X_{cd\nu}^{ab\xi}$ with fixed $ab\xi$. Moreover we denote by $((X, K))$ the matrix of $n\binom{n}{2}$ rows and $n\binom{n}{2} + 1$ columns, the last one being constituted by the elements $K_{cd\nu}$, while the remaining square matrix in $((X, K))$ is $((X))$.

The following three theorems will be proved simultaneously.

Theorem (3.1a). *A necessary and sufficient condition for the equations (2.1) to admit exactly one solution $\Gamma_{\lambda\mu}^{\nu}$ is*

$$(3.2a) \quad r((X)) = n\binom{n}{2}^*.$$

Theorem (3.1b). *A necessary and sufficient condition for equations (2.1) to admit more than one solution is*

$$(3.2b) \quad r((X)) = r((X, K)) < n\binom{n}{2}.$$

Theorem (3.1c). *A necessary and sufficient condition for equations (2.1) to admit no solution is*

$$(3.2c) \quad n\binom{n}{2} > r((X)) < r((X, K)).$$

---

* $r((X))$ and $r((X, K))$ are the ranks of $((X))$ and $((X, K))$, respectively.

Proof. The

$$n \binom{n+1}{2} + n \binom{n}{2} = n^3$$

equations (2.3b, c) are equivalent to n^3 equations (2.3a), which are equivalent to (2.1). Hence (2.3b, c) and (2.1) are equivalent. On the other hand the equations (2.3b) are equivalent to (2.4), and (2.3c) are equivalent to (3.1). Hence the equations (2.1) are equivalent to (2.4) and (3.1). Thus a necessary and sufficient condition for (2.1) to admit exactly one solution $\Gamma_{\lambda\mu}^\nu$ is that (2.4) and (3.1) admit exactly one solution $\Gamma_{\lambda\mu}^\nu$ given by (2.4). In other words, this necessary and sufficient condition reduces to the requirement that (3.1) admits exactly one solution. This is possible if and only if (3.2a) holds, as follows from elementary algebra. The remaining theorems may be proved in a similar way.

Note. The requirements (3.2) impose some conditions on the tensor $g_{\lambda\mu}$. Hence there arises the question whether these conditions can be satisfied. In the next two theorems which we prove simultaneously we present the answer to this question as far as the conditions (3.2b, c) are concerned.

Theorem (3.2a). *The set of all tensors $g_{\lambda\mu}$ which satisfy the conditions (3.2b) is not an empty set.*

Theorem (3.2b). *The set of all tensors $g_{\lambda\mu}$ which satisfy the conditions (3.2c) is not an empty set.*

Proof. Consider a tensor $g_{\lambda\mu}$ for which

$$(3.3a) \quad r((X)) < n \binom{n}{2}.$$

Such a tensor may always be found: Assume first that in general $r((X)) = m = n \binom{n}{2}$. Then the equation

$$(3.3b) \quad r((X)) = m - 1$$

represents one equation for the components $g_{\lambda\mu}$, and this condition is equivalent to (3.3a). If, however, $m < n \binom{n}{2}$ then (3.3a) is automatically satisfied. Assume now that the components of the tensor $g_{\lambda\mu}$, (which satisfies (3.3a)) are all constant. This is also a requirement which may be satisfied. Under these circumstances, the $h_{\lambda\mu}$ as well as the $k_{\lambda\mu}$ are constant, so that ∇_μ reduces to ∂_μ and in particular $K_{\omega\mu\nu} = 0$. Consequently

$$(3.3c) \quad r((X)) = r((X, K)),$$

and this equation together with (3.3a) proves the theorem (3.2a).

Put $*g_{\lambda\mu} = \rho g_{\lambda\mu}$, where $g_{\lambda\mu}$ is the tensor with constant components from the

previous proof and $\rho = \rho(x)$ is an arbitrary function of position. Then we have also

$$*h_{\lambda\mu} = \rho h_{\lambda\mu}, \quad *h^{\lambda\nu} = \frac{1}{\rho} h^{\lambda\nu}, \quad *k_{\lambda\mu} = \rho k_{\lambda\mu},$$

and in particular

$$*k_{\lambda}{}^{\nu} = *k_{\lambda\mu} *h^{\mu\nu} = k_{\lambda\mu} h^{\mu\nu} = k_{\lambda}{}^{\nu}.$$

Consequently we obtain by virtue of (2.8)

$$*X_{\omega\mu\lambda}^{\xi\eta\zeta} = X_{\omega\mu\lambda}^{\xi\eta\zeta},$$

so that we have from (3.3a)

$$(3.3d) \quad r((X)) = r(*X) < n \binom{n}{2}.$$

On the other hand, ρ being an arbitrary factor, it is not difficult to choose it in such a way that

$$(3.3e) \quad r(*X, *K) > r(*X).$$

The conditions (3.3d, e) are satisfied for the tensor $*g_{\lambda\mu}$, so that theorem (3.2b) is proved.

Corollary (3.1). *There are tensors $g_{\lambda\mu}$ for which the system (2.1) does not admit any solution. There are tensors $g_{\lambda\mu}$ for which the system (2.1) admits more than one solution.*

The proof (based on Theorems (3.1b,c) and Theorems (3.2a,b)) is obvious.

Note. So far we have not proved the existence of a tensor $g_{\lambda\mu}$ which satisfies equation (3.2a). In order to prove the existence of such a tensor we need first some more information based on the following

Agreement (3.1). *If not stated otherwise we will confine ourselves to the cases for which the requirement (3.2c) is not satisfied.*

Note. By virtue of this agreement we are excluding systems (2.1) constituted by incompatible equations. In other words we are considering only such systems (2.1) as admit at least one solution.

4. The solution of (2.1).

Theorem (4.1a). *In order that the system (2.1) admit more than one solution (or that (3.2b) be satisfied) it is sufficient that*

$$(4.1) \quad g = 0.$$

Proof. Assume that (4.1) is satisfied. Then there are at least two vectors A^{λ} and B^{μ} such that

$$(4.2) \quad \text{a) } g_{\alpha\mu} B^{\mu} = 0 \quad \text{b) } A^{\lambda} g_{\lambda\alpha} = 0.$$

Let $C^\omega \neq 0$ be an arbitrary vector and put

$$A^{\omega\lambda\mu} = C^\omega A^\lambda B^\mu.$$

Then we have

$$(4.3) \quad \text{a) } A^{\omega\lambda\mu} g_{\lambda\mu} = 0 \quad \text{b) } Q_{\omega\mu\lambda} A^{\omega\lambda\mu} = 0.$$

Equation (4.3a) yields by virtue of (4.2)

$$(4.3c) \quad \begin{aligned} A^{\omega\lambda\mu} D_\omega g_{\lambda\mu} &= -g_{\lambda\mu} D_\omega A^{\omega\lambda\mu}, \\ &= -g_{\lambda\mu} A^\lambda B^\mu D_\omega C^\omega - g_{\lambda\mu} B^\mu C^\omega D_\omega A^\lambda - g_{\lambda\mu} A^\lambda C^\omega D_\omega B^\mu, \\ &= 0. \end{aligned}$$

From (4.3b, c) we have

$$(4.4) \quad A^{\omega\lambda\mu} (D_\omega g_{\lambda\mu} - Q_{\omega\mu\lambda}) = 0$$

independently of $\Gamma_{\lambda\mu}^\nu$. Hence equations (2.3a) are linearly dependent and, since (2.3a) are equivalent to (2.1), equations (2.1) are linearly dependent.* Thus by virtue of Agreement (3.1) they admit more than one solution.

Note. The condition (4.1) is not a necessary one. Later on we shall exhibit a tensor $g_{\lambda\mu}$ with $g \neq 0$ for which the system admits more than one solution.

In the next theorem we denote by $K^{\lambda\mu}$ the tensor density whose components are cofactors of $k_{\lambda\mu}$ in $k = \text{Det } |k_{\lambda\mu}|$.

Theorem (4.1b). *Assume $K^{\lambda\mu} \neq 0$. In order that the system (2.1) admit more than one solution (or that (3.2b) be satisfied) it is sufficient that*

$$(4.5) \quad g_{\lambda\alpha} K^{v\alpha} \delta_\omega^\mu + g_{\alpha\lambda} K^{\alpha\mu} \delta_\omega^\nu = 0.$$

Proof. Put

$$\Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + T_{\lambda\mu}{}^\nu,$$

and remember that

$$\nabla_\omega g_{\lambda\mu} = \nabla_\omega k_{\lambda\mu}.$$

---

* It is worthwhile to stress here the importance of Agreement (3.1) by the following illustration: Assume that some of the first $n^3 - 1$ equations (2.1) are linearly dependent while the last equation (for $\omega = \omega_0, \mu = \mu_0, \lambda = \lambda_0$) is incompatible with them. Then there is least one tensor $A^{\omega\lambda\mu}$ with

$$A^{\omega_0\lambda_0\mu_0} = 0$$

independent of the $\Gamma_{\lambda\mu}^\nu$, for which (4.4) holds (independently of the $\Gamma_{\lambda\mu}^\nu$). According to our assumption about the last equation (2.1), the system (2.1) does not admit any solution, so that this case is excluded by the Agreement (3.1).

Hence the system (2.1) may be written

$$(4.6) \quad \nabla_{\omega} k_{\lambda\mu} = T_{\lambda\omega}{}^{\alpha} g_{\alpha\mu} + T_{\omega\mu}{}^{\alpha} g_{\lambda\alpha}.$$

Multiplying this equation by $K^{\lambda\mu}$ we obtain

$$(4.7a) \quad K^{\lambda\mu} \nabla_{\omega} k_{\lambda\mu} = T_{\beta\gamma}{}^{\alpha} (g_{\alpha\mu} K^{\beta\mu} \delta^{\gamma}_{\omega} + g_{\lambda\alpha} K^{\lambda\gamma} \delta^{\beta}_{\omega}),$$

where obviously

$$(4.7b) \quad K^{\lambda\mu} \nabla_{\omega} k_{\lambda\mu} = \nabla_{\omega} k,$$

for any k equal to or different from zero. If n is odd then $k = 0$ so that whenever (4.5) holds the equation (4.7a) is satisfied (by virtue of (4.7b)) *independently of the* $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ *and* $T_{\lambda\mu}{}^{\nu}$. Assume now that n is even. Then $K^{\lambda\mu} = K^{[\lambda\mu]}$, so that we obtain from (4.5)

$$n g_{\lambda\alpha} K^{\lambda\alpha} = g_{\alpha\lambda} K^{\lambda\alpha}$$

or

$$(n - 1)h_{\lambda\alpha} K^{\lambda\alpha} = (n + 1)k_{\alpha\lambda} K^{\lambda\alpha},$$

and this equation because of the skew symmetry of $K^{\lambda\mu}$ reduces to

$$k_{\alpha\lambda} K^{\alpha\lambda} = 0.$$

This equation is equivalent to $k = 0$, so that we may apply the same reasoning as before. In both cases equation (4.7b) is satisfied *independently of the* $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ *and* $T_{\lambda\mu}{}^{\nu}$, and the theorem follows easily.

Note I. For $K^{\lambda\mu} = 0$ equation (4.7a) is meaningless for our purpose, so that we had to assume $K^{\lambda\mu} \neq 0$.

Note II. One could try to prove a similar theorem, with $K^{\lambda\mu}$ being now the cofactor of $g_{\lambda\mu}$ in g . But then we have

$$g_{\lambda\mu} K^{\lambda\mu} = g_{\mu\lambda} K^{\mu\lambda},$$

so that if (4.5) is satisfied we obtain

$$g_{\lambda\mu} K^{\lambda\mu} = 0;$$

this equation yields (4.1), and Theorem (4.1a) may be applied.

Theorem (4.2a). *The set of all tensors $g_{\lambda\mu}$ for which the condition (3.2a) holds is not an empty set.*

The proof will be obtained further on by exhibiting such tensors.

Theorem (4.2b). *A necessary condition for the requirement (3.2a) to be satisfied is*

$$(4.8) \quad g \neq 0.$$

Proof. Assume that the conditions (3.2a) are satisfied. Then the system (2.1) admits exactly one solution. By virtue of Theorem (4.1a) we can not have (4.1), and consequently we must have (4.8).

Note. The condition (4.8) is in general not a sufficient one. Later on we shall see that there are tensors $g_{\lambda\mu}$ for which (4.8) and (3.2b) are satisfied. However, for $n = 2$ we obtain for the determinant X of the matrix (X)

$$X = (g/h)^2.$$

Hence for $n = 2$ the condition (4.8) is also a sufficient one.

Agreement (4.1). *Unless it is stated otherwise explicitly, we shall confine ourselves to tensors $g_{\lambda\mu}$ for which the requirement (3.2a) is satisfied.*

Note. By virtue of Theorem (4.2b) the Agreement (4.1) excludes all tensors $g_{\lambda\mu}$ for which (4.1) holds.

The next theorem is almost trivial but a very important one for later purposes.

Theorem (4.3). *The system (2.1) (which admits only one solution) does not impose any condition on the components $g_{\lambda\mu}$ in the form of an equation.*

Proof. The only condition imposed by the Agreement (4.1) is the condition (3.2a), which is not an equation for the components $g_{\lambda\mu}$. The same holds for the condition (4.8) following necessarily from (3.2a). No other conditions are imposed on $g_{\lambda\mu}$, whose existence is assured by Theorem (4.2a).

Note. The property mentioned in Theorem (4.3) distinguishes geometrically the system (2.1) from other similar systems, in particular from (1.4), as we shall see later.

Theorem (4.4). *Put*

$$(4.9) \quad 'Y_{ab\beta}^{cd\alpha} = \frac{\partial \ln X}{\partial 'X_{cd\alpha}^{ab\beta}}, \quad a < b, \quad c < d,$$

and

$$(4.10) \quad \begin{aligned} &\text{a) } 2Y_{ab\beta}^{cd\alpha} = 'Y_{ab\beta}^{cd\alpha}, \\ &\text{b) } Y_{ab\beta}^{cd\alpha} = Y_{ba\beta}^{dc\alpha} = -Y_{ba\beta}^{cd\alpha} = -Y_{ab\beta}^{dc\alpha}, \\ &\text{c) the remaining } Y_{\lambda\mu\nu}^{\xi\eta\tau} = 0. \end{aligned}$$

Then the solution of (2.7b) (and (3.1)) is

$$(4.11) \quad S_{\lambda\mu\nu} = \frac{1}{2} K_{\alpha\beta\gamma} Y_{[\lambda\mu]\nu}^{[\alpha\beta]\gamma},$$

where $Y_{\lambda\mu\nu}^{\alpha\beta\gamma} = Y_{[\lambda\mu]\nu}^{[\alpha\beta]\gamma}$ is a tensor satisfying the condition $X_{\alpha\beta\gamma}^{\xi\eta\tau} Y_{\lambda\mu\nu}^{\alpha\beta\gamma} = \delta_{[\lambda\mu]\nu}^{[\xi\eta]\tau}$.

Proof. First of all we have from (4.9)

$$(4.12) \quad 'X_{cd\alpha}^{ab\xi} 'Y_{ef\lambda}^{cd\alpha} = \delta_{ef\lambda}^{ab\xi}, \quad a < b, \quad c < d, \quad e < f.$$

Moreover, we have according to (4.10)

$$\begin{aligned} K_{cd\gamma} 'Y_{ef\nu}^{cd\gamma} &= K_{\alpha\beta\gamma} Y_{ef\nu}^{[\alpha\beta]\gamma}, \\ K_{cd\gamma} Y_{[\lambda\mu]\nu}^{cd\gamma} &= K_{\alpha\beta\gamma} Y_{[\lambda\mu]\nu}^{[\alpha\beta]\gamma}. \end{aligned}$$

On the other hand we obtain from (3.1)

$$2S_{ef\nu} = K_{cd\alpha} 'Y_{ef\nu}^{cd\alpha} = K_{\alpha\beta\gamma} Y_{ef\nu}^{[\alpha\beta]\gamma},$$

and this equation leads at once to (4.11). The proof of the remaining part of the theorem is almost obvious.

Theorem (4.5). *The solution of (2.1) is*

$$(4.13) \quad \Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \frac{1}{2}(2\nabla_{(\alpha} g_{\gamma)\beta} + \nabla_{\beta} g_{\alpha\gamma}) Y_{\xi\eta}^{\alpha\beta\gamma} [h^{\xi\nu} \delta_{\lambda\mu}^{\xi\eta} + 2h^{\xi\nu} \delta_{(\lambda}^{\eta} k_{\mu)}^{\xi}].$$

The proof follows at once from (4.11) and (2.4).

Note. Formula (4.13) gives the explicit solution of the system (2.1) for the $\Gamma_{\lambda\mu}^{\nu}$. Of course we could have obtained this solution directly from (2.1),* solving this system of n^3 linear equations for n^3 unknowns $\Gamma_{\lambda\mu}^{\nu}$. However, equation (4.13) is more advantageous, for it requires handling a determinant of $n\binom{n}{2}$ rows only (namely the determinant of the matrix $((X))$) instead of a determinant of n^3 rows. This situation is quite similar to the problem of solving the usual $n\binom{n+1}{2}$ equations for the Christoffel symbols. Instead of dealing with the $n\binom{n+1}{2}$ -rowed determinant of this linear system of $n\binom{n+1}{2}$ linear equations for $n\binom{n+1}{2}$ unknowns we have the solution in the form (1.3), which requires only consideration of the n -rowed determinant h .

Although the question of obtaining the $\Gamma_{\lambda\mu}^{\nu}$ from (2.1) may be looked upon as having been solved from the theoretical point of view, the physical application of this formula is at least as tedious as the application of (1.3). Therefore we try in the second part of this paper to treat some special cases where the formula (4.11) may be replaced by a simpler one. This is possible if and only if the tensor $g_{\lambda\mu}$ is subject to some special structural conditions. Some of these cases will serve us later (in dealing with Problem C) mainly as hints of what is to be avoided.

Finally, in the third part of this paper we shall try to replace the solution (4.13) by another one, more suitable for physical application. This will be possible if and only if the tensor $g_{\lambda\mu}$ is subject to some quantitative (numerical) rather than structural restrictions.

---

* Having in mind Theorem (4.3).

PART II.

5. Illustration. The main scope of this section is to show how to apply the method developed in Part I to the case when $n = 4$. We shall confine ourselves to a simple case of a tensor given by the matrix

$$(5.1) \quad ((g_{\lambda\mu})) = \begin{pmatrix} h_{11} & a & 0 & 0 \\ -a & h_{22} & 0 & 0 \\ 0 & 0 & h_{33} & 0 \\ 0 & 0 & 0 & h_{44} \end{pmatrix},$$

where $a \equiv k_{12} \neq 0$. Remembering (2.4), we see that in order to obtain the connection $\Gamma_{\lambda\mu}^\nu$, defined by the system (2.1) for the tensor (5.1), the most important part of the computation is the computation of $S_{\lambda\mu\nu}$ from the given data. We will confine ourselves to this task only.

Using (2.9) we obtain first

$$(5.2) \quad \begin{aligned} \text{a) } 2S_{34\lambda} &= K_{34\lambda}, \\ \text{b) } 2S_{a uv} &= K_{a uv}, \end{aligned} \quad a = 1, 2, \quad v, u = 3, 4,$$

and

$$(5.3) \quad \begin{aligned} \text{a) } 2S_{12a} &= K_{12a} - 2a^2 h^{11} h^{22} S_{12a}, & a = 1, 2, \\ \text{b) } \begin{cases} 2S_{2u2} &= K_{2u2} + 2a^2 (h^{11})^2 S_{u11}, \\ 2S_{1u1} &= K_{1u1} + 2a^2 (h^{22})^2 S_{u22}, \end{cases} \\ \text{c) } \begin{cases} 2S_{12u} &= K_{12u} + 2a^2 h^{11} h^{22} (S_{u12} - S_{u21}), \\ 2S_{2u1} &= K_{2u1} - 2a^2 h^{11} h^{22} S_{u12}, \\ 2S_{u12} &= K_{u12} + 2a^2 h^{11} h^{22} S_{u21}. \end{cases} \end{aligned}$$

The tensor $g_{\lambda\mu}$ being given, we know the Christoffel symbols $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ as well as $k_{\lambda\mu}$, so that we may consider $K_{\mu\lambda\nu}$ as known (*cf.* (2.6)). Hence the only equations to be investigated are equations (5.3). In order to do so we first note that each of the systems (5.3a), (5.3b) and (5.3c) has in its determinant at least one of the factors

$$(5.4) \quad \begin{aligned} \text{a) } 1 + a^2 h^{11} h^{22} & \quad \text{b) } 1 - a^2 h^{11} h^{22}. \end{aligned}$$

On the other hand we have from (5.1)

$$g = (h_{11} h_{22} + a^2) h_{33} h_{44},$$

so that

$$(5.5) \quad 1 + a^2 h^{11} h^{22} = \frac{1}{h_{11} h_{22}} (h_{11} h_{22} + a^2) = G (=g/h).$$

Because of Agreement (4.1) and Theorem (4.2b) we have to confine ourselves to the case

$$(5.6) \quad G \neq 0.$$

Then equations (5.3a) admit only one solution namely

$$(5.7a) \quad 2S_{12a} = K_{12a}/G, \quad a = 1, 2.$$

As far as the system (5.3b) is concerned, we have first by virtue of (5.1)

$$K_{2u2} = K_{1u1} = 0,$$

so that we may write instead of (5.3b)

$$(5.3d) \quad \begin{aligned} S_{2u2} - a^2 (h^{11})^2 S_{u11} &= 0, \\ S_{2u2} a^2 (h^{22})^2 - S_{u11} &= 0. \end{aligned}$$

The determinant of this system is

$$-[1 - (a^2 h^{11} h^{22})^2] = G(G - 2).$$

The system (5.3d) always admits at least one solution. If in particular $G(G - 2) \neq 0$ then we have from (5.3d)

$$(5.7b) \quad S_{2u2} = S_{u11} = 0.$$

The system (5.3c) is very important for our purpose. Its determinant is

$$-8(1 - a^2 h^{11} h^{22})(1 + a^2 h^{11} h^{22}) = +8G(G - 2).$$

Hence if

$$(5.8a) \quad G = 2$$

this system either admits more than one solution or it does not admit any. Consider the matrix of the whole system (5.3c) (the row of the K 's included). One of its 3×3 determinants is

$$-4[K_{2u1} - K_{u12} a^2 h^{11} h^{22}] = 4a\{(2 - G) \partial_u \ln a - \partial_u \ln h_{11} + a^2 h^{11} h^{22} \partial_u \ln h_{22}\}.$$

Consequently, if (5.8a) and

$$(5.8b) \quad \partial_u \ln \frac{h_{11}}{h_{22}} \neq 0$$

hold, the system (5.3c) does not admit any solution. Hence if we use the tensor (5.1), for which the conditions (5.8a, b) hold, the corresponding system (2.1) does not admit any solution. In other words *the tensor (5.1) for which (5.8a, b) hold belongs to the set of tensors $g_{\lambda\mu}$ for which the conditions (3.2c) are satisfied.**

On the other hand we have by virtue of (5.8a)

$$K_{2u1} = -K_{u12} = \frac{a}{2} \partial_u \ln \frac{h_{11}}{h_{22}},$$

so that if

$$(5.8c) \quad \partial_u \ln \frac{h_{11}}{h_{22}} = 0$$

we have $K_{2u1} = K_{u12} = 0$, and consequently the system (5.3c) for which (5.8a) and (5.8c) hold admits more than one solution.† Hence if we use the tensor (5.1) for which the conditions (5.8a) and (5.8c) hold the corresponding system (2.1) admits more than one solution. In other words *the tensor (5.1) for which (5.8a) and (5.8c) hold belongs to the set of tensors $g_{\lambda\mu}$ for which the conditions (3.2b) hold.*‡

Hence if (5.8a) holds either (3.2b) or (3.2c) are satisfied. These cases being excluded by virtue of Agreement (4.1), we have to confine ourselves to the cases

$$(5.9) \quad G \neq 2.$$

If (5.9) holds the determinant of the system (5.3c) is not equal to zero, and this system yields only one solution, namely

* Note that we have $g \neq 0$ in this case. The conditions (5.8a, b) are satisfied for instance for

$$h_{11} = -e^{2x^3}, \quad h_{22} = -1, \quad h_{33} = -1, \quad h_{44} = 1$$

and $a = e^{x^3}$. In particular the signature of $h_{\lambda\mu}$ is $---+$.

† The matrix of the whole system (5.3c) has the same rank $r = 2$ as the matrix of the corresponding homogeneous system.

‡ Note that we have $g \neq 0$ in this case! The conditions (5.8a) and (5.8c) are satisfied for instance by

$$h_{11} = h_{22} = h_{33} = -h_{44} = -1, \quad a = \pm 1.$$

It is not difficult to exhibit other examples for which (5.8a) and (5.8c) hold and $h_{\lambda\mu}$ is of the signature $---+$.

$$\begin{aligned}
(5.7c) \quad 2G(G-2)S_{12u} &= \begin{vmatrix} K_{12u} & G-1 & G-1 \\ K_{u21} & G-1 & 1 \\ K_{1u2} & 1 & G-1 \end{vmatrix}, \\
2G(G-2)S_{u12} &= K_{u21}(G-1) - K_{1u2}, \\
2G(G-2)S_{u21} &= K_{1u2}(G-1) - K_{u21}.
\end{aligned}$$

Hence if we use the tensor (5.1) for which the conditions (5.6) and (5.9) hold, the system (2.7) admits only one solution, namely (5.2a) and (5.7a,b,c). Substituting these values in (2.4) we obtain the solution of (2.1) which is the only one admitted by (2.1). *Hence the tensor (5.1) for which the conditions (5.6) and (5.9) hold belongs to the set of tensors $g_{\lambda\mu}$ for which the condition (3.2a) is satisfied.*

Summarizing these results we see that there are tensors (5.1) with $g \neq 0$ for which

- a) the corresponding system (2.1) admits exactly one solution so that (3.2a) is satisfied.
- b) the corresponding system (2.1) admits more than one solution so that (3.2b) is satisfied.
- c) the corresponding system (2.1) does not admit any solution so that (3.2c) is satisfied.

Note. From the geometrical point of view no harm is done if we exclude the tensors for which (3.2b) or (3.2c) hold (cf. Agreement (4.1)). It means only that we restrict ourselves to the study of (2.1) based on special tensors (for which (3.2a) holds). However, from the physical point of view this restriction may be of paramount importance: Let us assume for instance that for $n = 4$ we will look upon $k_{\lambda\mu}$ as an electromagnetic field. Then for instance in the case of tensors (5.1) the Agreement (4.1) which excludes the aforementioned cases b) and c) eliminates all electromagnetic fields for which

$$k_{12}^2 = \pm h_{11} h_{22}.$$

In other words: *If we look upon $g_{[\lambda\mu]}$ as describing an electromagnetic field then the unified theory of relativity based on (2.1) does not embrace all possible electromagnetic fields.*

There are two possible ways out of this difficulty: the eliminated cases are geometrically characterized by the conditions (3.2b) or (3.2c) respectively. If one finds a similar *physical* characterization for them (in the case of $n = 4$)* then one could look upon the unified theory of relativity based on (2.1) as describing only those electromagnetic fields which have certain physical properties. For the theory of the remaining fields another basis would have to be found.

---

* In the case of the special theory of relativity the excluded cases of electromagnetic fields given by (5.1) may be characterized in three-space as follows: the electric vector field and the magnetic vector fields are perpendicular and the absolute value of the difference of their squared lengths is equal to one.

Another way out of this difficulty is not to regard $k_{\lambda\mu}$ as describing an electromagnetic field. In this case the unified theory of relativity may be looked upon as based on a special purely geometrical tensor $g_{\lambda\mu}$ (which satisfies (3.2a)) and the electromagnetic field has to be identified by means of Principle *C* (to be dealt with in a later paper).

A method similar to the one used for (5.1) holds also for the tensor

$$(5.10) \quad ((g_{\lambda\mu})) = \begin{pmatrix} \begin{pmatrix} h_{11} & a & 0 & 0 \\ -a & h_{22} & 0 & 0 \\ 0 & 0 & h_{33} & b \\ 0 & 0 & -b & h_{44} \end{pmatrix} \end{pmatrix},$$

where at least one of the functions a, b is not equal to zero. The tensors (5.1), (5.10) and

$$(5.11) \quad ((g_{\lambda\mu})) = \begin{pmatrix} \begin{pmatrix} h_{11} & 0 & 0 & 0 \\ 0 & h_{22} & a & \epsilon a \\ -0 & -a & h_{33} & 0 \\ 0 & -\epsilon a & 0 & h_{44} \end{pmatrix} \end{pmatrix} \quad (\epsilon = \pm 1)$$

will play an important role in the next section.

6. Application. 1. In this section we assume again $n = 4$. Moreover we suppose that the signature of $h_{\lambda\mu}$ is $---+$. Under this assumption it is known that there are only three projectively different classes of tensors $g_{\lambda\mu}$ for which

$$(6.1a) \quad h_{\lambda\mu} = -U_{\lambda}^I U_{\mu}^I - U_{\lambda}^{II} U_{\mu}^{II} - U_{\lambda}^{III} U_{\mu}^{III} + U_{\lambda}^{IV} U_{\mu}^{IV},$$

where U_{λ}^x are unit perpendicular vector fields ($x = I, \dots, IV$);

$$(6.1b) \quad -U_{\lambda}^u U_{\mu}^u h^{\lambda\mu} = U_{\lambda}^{IV} U_{\mu}^{IV} h^{\lambda\mu} = 1 \quad (u = I, II, III);$$

$$(6.1c) \quad U_{\lambda}^x U_{\mu}^y h^{\lambda\mu} = 0, \quad x, y = I, \dots, IV, \quad x \neq y.$$

The corresponding skew symmetric part $k_{\lambda\mu}$ is either

$$(6.2) \quad k_{\lambda\mu} = 2aU_{[\lambda}^I U_{\mu]}^{II} + 2bU_{[\lambda}^{III} U_{\mu]}^{IV}$$

or

$$(6.3) \quad \begin{aligned} \text{a) } k_{\lambda\mu} &= 2aU_{[\lambda}^I U_{\mu]}^{II}, \\ \text{b) } k_{\lambda\mu} &= 2bU_{[\lambda}^{III} U_{\mu]}^{IV}, \end{aligned}$$

or

$$(6.4) \quad k_{\lambda\mu} = 2aU_{[\lambda}^{II} U_{\mu]}^{III} + 2\epsilon aU_{[\lambda}^{II} U_{\mu]}^{IV*}$$

---

* Cf. V. HLAVATÝ, *The Elementary Basic Principles of the Unified Theory of Relativity*. A, Journal of Rational Mechanics and Analysis **1** (1952), 539–562, in a slightly different notation.

In the three first cases the fields U_λ are intrinsically and uniquely given (up to Lorentz transformations) by the field $g_{\lambda\mu}$, in the last case the fields U_λ are not intrinsically defined by $g_{\lambda\mu}$, there being a certain degree of arbitrariness in their definition.

Assume now that the indices x, y, z run from I to IV and put

$$U^\lambda_y U_\lambda^x = \delta_y^x.$$

The fields U^ν_x and U_λ^y may be regarded as mixed components of the unit tensor, the Greek indices referring to some system of holonomic parameters, the indices x, y, z referring to another system of parameters which are in general non-holonomic* ones. Then any tensor may be expressed by means of different kinds of components, as for instance

$$T^{\nu\cdots}_{\lambda\cdots}, \quad T^{x\cdots}_{y\cdots} = T^{\nu\cdots}_{\lambda\cdots} U^\nu_x U_\lambda^y \dots, \quad T^{x\cdots}_{\lambda\cdots} = T^{\nu\cdots}_{\lambda\cdots} U_\nu^x \dots$$

In particular we have

$$\begin{aligned} \text{a)} \quad S_{xyz} &= S_{\omega\mu\nu} U^\omega_x U^\mu_y U^\nu_z, \\ \text{b)} \quad S_{\omega\mu\nu} &= S_{xyz} U_\omega^x U_\mu^y U_\nu^z, \\ \text{a)} \quad h_{xy} &= h_{\lambda\mu} U^\lambda_x U^\mu_y = \begin{cases} -1 & \text{for } x = y = \text{I, II, III} \\ 1 & \text{for } x = y = \text{IV} \\ 0 & \text{for } x \neq y \end{cases}, \\ \text{b)} \quad h^{xy} &= h^{\lambda\mu} U_\lambda^x U_\mu^y = \begin{cases} -1 & \text{for } x = y = \text{I, II, III} \\ 1 & \text{for } x = y = \text{IV} \\ 0 & \text{for } x \neq y \end{cases}, \end{aligned} \tag{6.5}$$

and so on. Consequently the field (6.2) has (xy) -components

$$-k_{yx} = k_{xy} = k_{\lambda\mu} U^\lambda_x U^\mu_y = \begin{cases} a & \text{for } x = \text{I} \quad y = \text{II} \\ b & \text{for } x = \text{III} \quad y = \text{IV}; \\ 0 & \text{for the remaining cases} \end{cases} \tag{6.7}$$

while the (xy) -components of (6.3) are

$$\text{a)} \quad k_{\text{I II}} = -k_{\text{II I}} = a, \quad \text{the remaining } k_{xy} = 0,$$

---

* It is worthwhile to stress that these parameters are not necessarily non-holonomic. This is for instance the case if

$$U_\lambda^x = \delta_\lambda^x$$

(which can happen only in the special theory of relativity).

(6.8)

$$\text{b) } k_{\text{III IV}} = -k_{\text{IV III}} = b, \quad \text{the remaining } k_{xy} = 0;$$

and finally the (xy) -components of (6.4) are

$$(6.9) \quad k_{\text{II III}} = k_{\text{II IV}} \epsilon = a.$$

2. We shall now use these results for solving the system (2.1) in the cases mentioned in Section 1.

Take for instance the case of the tensor (6.8a) which has in general six components. On the other hand it has only one (xy) -component, namely $k_{\text{I II}} = a$. The (xy) -components of the corresponding tensor $g_{\lambda\mu}$ are given by the matrix

$$(6.10) \quad ((g_{xy})) = \left(\begin{pmatrix} -1 & a & 0 & 0 \\ -a & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right),$$

which is of the structure (5.1). Hence we may use all covariant equations which appear in Section 5. In particular we obtain from (5.2)

$$\text{a) } 2S_{\text{III IV } s} = K_{\text{III IV } s},$$

(6.11)

$$\text{b) } 2S_{auv} = K_{auv}; \quad a = \text{I, II}, \quad v, u = \text{III, IV},$$

while the equation (5.7a) gives us

$$(6.12) \quad 2S_{\text{I II } a} = K_{\text{I II } a}/G^*,$$

where of course*

$$(6.13) \quad G^* = \frac{\text{Det } |g_{xy}|}{\text{Det } |h_{xy}|} = a^2 + 1 = \frac{g(\text{Det } |U^\lambda_x|)^2}{h(\text{Det } |U^\lambda_x|)^2}.$$

Moreover equations (5.7b) yield for $G^*(G^* - 2) \neq 0$

$$(6.14) \quad S_{\text{II } u \text{ II}} = S_{u \text{ I I}} = 0, \quad u = \text{III, IV}.$$

Instead of using (5.7c) we use equations (5.3c), namely

$$(6.15) \quad \begin{aligned} 2S_{\text{I II } u} + 2a^2 S_{\text{I } u \text{ II}} + 2a^2 S_{u \text{ II I}} &= K_{\text{I II } u}, \\ -2a^2 S_{\text{I } u \text{ II}} - 2S_{u \text{ II I}} &= K_{\text{II } u \text{ I}}, \quad (u = \text{III, IV}) \\ -2S_{\text{I } u \text{ II}} - 2a^2 S_{u \text{ II I}} &= K_{u \text{ I II}}. \end{aligned}$$

---

* We assume as usually that $g_{\lambda\mu}$ is a real tensor so that $k_{\text{I II}} = a$ is a real function. Hence $1 + a^2 \neq 0$.

The determinant of the system (6.15) is

$$8(a^2 - 1)(a^2 + 1) = 8G^*(G^* - 2).$$

If we restrict ourselves to the case $G^* \neq 2$ then we obtain from (6.15)

$$(6.16) \quad \begin{aligned} \text{a)} \quad 2S_{I \ II \ u} &= \frac{\begin{vmatrix} K_{I \ II \ u} & a^2 & a^2 \\ K_{II \ u \ I} & -a^2 & -1 \\ K_{u \ I \ II} & -1 & -a^2 \end{vmatrix}}{G^*(G^* - 2)}, \\ \text{b)} \quad 2S_{I \ u \ II} &= \frac{\begin{vmatrix} K_{II \ u \ I} & -1 \\ K_{u \ I \ II} & -a^2 \end{vmatrix}}{G^*(G^* - 2)}, \\ 2S_{u \ II \ I} &= \frac{\begin{vmatrix} -a^2 & K_{II \ u \ I} \\ -1 & K_{u \ I \ II} \end{vmatrix}}{G^*(G^* - 2)}. \end{aligned}$$

Equations (6.11), (6.12), (6.14) and (6.16) define the components S_{xyz} . Now using (6.5), we obtain $S_{\omega\mu\nu}$, and substituting these values in (2.4) we obtain the solution $\Gamma_{\lambda\mu}^{\nu}$ of the system (2.1) belonging to the tensor $g_{\lambda\mu}$ whose symmetric and skew-symmetric part respectively are given by (6.1) and (6.3a).

Proceeding in the same manner we obtain the solution $\Gamma_{\lambda\mu}^{\nu}$ of the system (2.1) which is based on the tensor $g_{\lambda\mu}$ whose symmetric and skew symmetric part respectively are given by (6.1) and (6.2) or (6.3b) or (6.4). In the first case we have to use the method based on the solution belonging to (5.10) with $ab \neq 0$, for the second case we have to use (5.10) in a similar way with $a = 0$, $b \neq 0$, while the last case may be treated by means of (5.11).

Using this method of non-holonomic coordinates, we have solved the system (2.1) for all tensors $g_{\lambda\mu}$ (which satisfy the condition (3.2a)) which may occur in the unified theory of relativity. Let us stress that this method is based on a (comparatively) easily obtainable solution corresponding to four tensors only, namely (5.1), (5.10), (with $ab \neq 0$, or $b \neq 0$, $a = 0$) and (5.11).

Advantageous as this method might seem for physical application it is still tedious enough when we bear in mind that the solution of (2.1) is only a preparatory step to Problem C, where we shall impose some differential conditions on the $\Gamma_{\lambda\mu}^{\nu}$ (which are expressed by means of $g_{\lambda\mu}$ only) in order to get $g_{\lambda\mu}$ as a solution of the differential equations introduced by the aforementioned conditions. Hence if we want to use the method just established we must not introduce in the solution $\Gamma_{\lambda\mu}^{\nu}$ new elements beside $g_{\lambda\mu}$. It means that we have first to express the fields U_{λ}^x by means of $g_{\lambda\mu}$ only. This is possible in the first three cases (characterized by the equations (6.2), (6.3a, b)), while in the case characterized by

(6.4) the $U_\lambda{}^\alpha$ are not uniquely* defined by $g_{\lambda\mu}$. Hence in the last case our method is not to be recommended for dealing with the Problem C. In the first three cases, the $U_\lambda{}^\alpha$ being uniquely* determined by the tensor $g_{\lambda\mu}$, there is no objection to using the aforementioned method. On the other hand to get the $U_\lambda{}^\alpha$ from the $g_{\lambda\mu}$ requires a tedious computation (described fully in the paper referred to in an earlier footnote).

Still having in mind physical applications, we introduce in the next sections some cases where the tensor $S_{\lambda\mu}{}^\nu$ may be easily obtained without using the $U_\lambda{}^\alpha$ mentioned in this section.

7. The field $S_{\lambda\mu}{}^\nu$. In this and in the following sections we shall resume again the case of an arbitrary $n \geq 2$. Moreover we drop Agreement (4.1) as unnecessary for our purposes. In particular in this section we shall deal with two particular cases of the field $S_{\lambda\mu}{}^\nu$. First we prove the following

Lemma (7.1). *The set of all tensors*

$$(7.1a) \quad A_\lambda{}^\nu = \delta_\lambda^\nu (n - 1 + k^{\alpha\beta} k_{\alpha\beta}) - k_{\lambda\alpha} k^{\nu\alpha}$$

with

$$(7.1b) \quad \text{Det } |A_\lambda{}^\nu| \neq 0$$

is not an empty set.

Proof. Assume that the symmetric part of $g_{\lambda\mu}$ be given by

$$h_{\lambda\lambda} \neq 0, \quad h_{\lambda\mu} = 0 \quad \text{for} \quad \lambda \neq \mu,$$

while the skew-symmetric part satisfies the conditions $a = k_{12} = -k_{21} \neq 0$, the remaining $k_{\lambda\mu} = 0$. By choice of a suitable value for a we obtain for the tensor (7.1) the condition (7.1b).

Note. If (7.1b) holds there is a unique tensor $B_\lambda{}^\nu$ satisfying

$$(7.2) \quad A_\lambda{}^\nu B_\mu{}^\lambda = \delta_\mu^\nu.$$

In the next theorem we will make use of $B_\lambda{}^\nu$ as well as of

$$(7.3) \quad K_\mu{}^\nu = \delta_\mu^\nu + k^{\nu\alpha} k_{\mu\alpha}.$$

Theorem (7.1). *Assume that (7.1b) holds. Then a necessary and sufficient condition that the system (2.7b) admits a unique solution of the form*

$$(7.4) \quad S_{\lambda\mu}{}^\nu = S_{[\lambda} \delta_{\mu]}^\nu$$

is

$$(7.5a) \quad 2K_{\gamma\alpha}{}^\alpha B_{[\omega}{}^\gamma K_{\mu]}{}^\nu = K_{\omega\mu}{}^\nu (= 4(\nabla_\alpha k_\gamma{}^\alpha) B_{[\omega}{}^\gamma K_{\mu]}{}^\nu).$$

---

* Uniquely up to LORENTZ transformation.

If this condition is satisfied then (7.4) holds and

$$(7.5b) \quad S_\lambda = B_\lambda^\alpha K_{\alpha\beta}{}^\beta (= 2B_\lambda^\alpha \nabla_\beta k_\alpha{}^\beta).$$

Proof. Assume that (2.7b) admits a unique solution of the form (7.4). Substituting from (7.4) into (2.7b) we obtain

$$(7.6a) \quad 2S_{[\omega} \delta_{\mu]}^\nu = K_{\omega\mu}{}^\nu + 2S_{[\mu} k_{\omega]\gamma} k^{\nu\gamma}$$

or

$$(7.6b) \quad 2S_{[\omega} K_{\mu]}^\nu = K_{\omega\mu}{}^\nu.$$

Equation (7.6b) yields

$$(7.7) \quad S_\alpha A_\mu{}^\alpha = K_{\mu\alpha}{}^\alpha (= 2\nabla^\alpha k_{\mu\alpha}).$$

Assuming that (7.1b) holds, we obtain from (7.2) and (7.7) equation (7.5b). Substituting from (7.5b) into (7.6b) we obtain (7.5a). Assume now that (7.1b) and (7.5a) hold and define a vector S_λ by (7.5b) and a tensor $S_{\lambda\mu}{}^\nu$ by (7.4). This tensor satisfies (2.7b), for this latter equation reduces by virtue of (7.5b) and (7.4) to (7.5a), which is satisfied according to our assumption. Hence the equation (2.7b) admits at least one solution of the form (7.4). Assume now that there is another solution

$$S^*_{\lambda\mu}{}^\nu \neq S_{\lambda\mu}{}^\nu$$

of (2.7) which has the form

$$S^*_{\lambda\mu}{}^\nu = S^*_{[\lambda} \delta_{\mu]}^\nu.$$

For its existence it is necessary that we have

$$(7.8a) \quad S_\lambda^* \neq S_\lambda.$$

Applying now the method used in the first part of the proof we obtain

$$S_\alpha^* A_\mu{}^\alpha = K_{\mu\alpha}{}^\alpha,$$

and this equation yields by virtue of (7.1b)

$$S_\lambda^* = B_\lambda^\alpha K_{\alpha\beta}{}^\beta.$$

Comparing this result with (7.5b) we obtain

$$(7.8b) \quad S_\lambda^* = S_\lambda,$$

which contradicts (7.8a). Hence if the equation (2.7b) admits at least one solution $S_{\lambda\mu}{}^\nu$ of the form (7.4) it admits exactly one solution of this form.

Corollary (7.1). *If the conditions (7.1b) and (7.5a) are satisfied, then the system (2.1) admits exactly one solution of the form*

$$(7.9) \quad \Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + \frac{1}{2} K_{\alpha\beta}^{\beta} (B_{\lambda}^{\alpha} g_{\gamma\mu} - B_{\mu}^{\alpha} g_{\lambda\gamma}) h^{\gamma\nu}.$$

Proof. If the conditions (7.1b) and (7.5a) are satisfied then Theorem (7.1) holds. If one substitutes from (7.4) and (7.5b) into (2.4) one obtains (7.9).

Corollary (7.2). *In the case mentioned in Corollary (7.1) the requirements*

$$(7.10) \quad \text{a) } S_{\lambda\alpha}^{\alpha} = 0 \quad \text{b) } \nabla^{\alpha} k_{\lambda\alpha} = 0$$

are equivalent and lead to

$$(7.11) \quad \Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}.$$

We leave the simple proof to the reader.

In the next theorem we shall investigate another possibility for $S_{\lambda\mu}^{\nu}$. We need first the following

Lemma (7.2). *The set of all tensors*

$$(7.12a) \quad L_{\lambda\mu} = k_{\lambda\mu} (1 - k^{\alpha\beta} k_{\beta\alpha}) - k_{\lambda\alpha} k_{\mu\beta} k^{\alpha\beta}$$

with

$$(7.12b) \quad \text{Det } |L_{\lambda\mu}| \neq 0$$

is not an empty set.

Proof. For conveniently chosen $ab \neq 0$ the tensor (5.10) leads to (7.12b). Hence the set of all tensors (7.12a) with (7.12b) is not an empty set.

Note. Whenever (7.12b) holds there is only one tensor $M^{\mu\nu}$ satisfying the condition

$$L_{\mu\lambda} M^{\mu\nu} = \delta_{\lambda}^{\nu}.$$

We shall use the tensor $M^{\mu\nu}$ in the next

Theorem (7.2). *Assume that (7.12b) holds. Then a necessary and sufficient condition that the system (2.7b) admit only one solution of the form*

$$(7.13) \quad S_{\lambda\mu}^{\nu} = p^{\nu} k_{\lambda\mu}$$

is

$$(7.14a) \quad K_{\omega\mu\nu} = K_{\rho\alpha}^{\alpha} M^{\rho\gamma} [k_{\omega\mu} h_{\gamma\nu} - 2k_{\nu\beta} k_{[\omega}^{\beta} k_{\mu]\gamma} - 2k_{\nu\gamma} k_{[\omega}^{\beta} k_{\mu]\beta}].$$

If this condition is satisfied then

$$(7.14b) \quad p^{\nu} = \frac{1}{2} K_{\mu\alpha}^{\alpha} M^{\mu\nu} (= M^{\mu\nu} \nabla^{\alpha} k_{\mu\alpha}).$$

Proof. Assume that (2.7b) admits a unique solution of the form (7.13). Substituting from (7.13) into (2.7b) we obtain

$$(7.15a) \quad 2p^\gamma [k_{\omega\mu} h_{\gamma\gamma} - 2k_{\nu\beta} k_{[\omega}{}^\beta k_{\mu]\gamma} - 2k_{[\omega}{}^\beta k_{\mu]\beta} k_{\nu\gamma}] = K_{\omega\mu\nu},$$

and this equation yields

$$(7.15b) \quad 2p^\gamma L_{\mu\gamma} = K_{\mu\alpha}{}^\alpha (= 2\nabla^\alpha k_{\mu\alpha}).$$

Assuming that (7.12b) holds we obtain from (7.15b) the equation (7.14b). Substituting (7.14b) into (7.15a) we obtain (7.14a). Assume now that (7.12b) and (7.14a) hold and define a vector p^ν by (7.14b) and a tensor $S_{\lambda\mu}{}^\nu$ by (7.13). This tensor satisfies (2.7b), for this latter equation reduces by virtue of (7.14b) and (7.13) to (7.14a), which is satisfied according to our assumption. The uniqueness of the solution (7.13) may be proved in a similar manner as used before in the proof of the corresponding part of Theorem (7.1).

Corollary (7.3). *If the conditions (7.12b) and (7.14a) are satisfied then the system (2.1) admits exactly one solution of the form*

$$(7.16) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + M^{\rho\beta} K_{\rho\alpha}{}^\alpha (\tfrac{1}{2}\delta_\beta^\nu k_{\lambda\mu} + k_{(\mu}^\nu k_{\lambda)\beta}).$$

Proof. If the conditions (7.12b) and (7.14a) are satisfied, then Theorem (7.2) holds. Substituting from (7.13) and (7.14b) into (2.4) one obtains (7.16).

Corollary (7.4). *In the case mentioned in Corollary (7.3) the requirements*

$$(7.17) \quad \text{a) } S_{\lambda\alpha}{}^\alpha = 0 \quad \text{b) } k \neq 0^*$$

yield

$$(7.18) \quad \nabla^\alpha k_{\mu\alpha} = 0,$$

so that (7.16) reduces to

$$(7.19) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\}.$$

Proof. The requirement (7.17a) leads by virtue of (7.13) to

$$p^\alpha k_{\lambda\alpha} = 0.$$

If (7.17b) holds, this equation is satisfied only with $p^\nu = 0$. Then we obtain from (7.14b) the equation (7.18), so that (7.16) reduces to (7.19).

In the next theorem, which is not connected with the previous two theorems, we introduce the tensor densities

$$(7.20) \quad g^{\lambda\nu} = \sqrt{g} g^{\lambda\nu},$$

where $g^{\lambda\nu}$ as well as g is introduced by (1.4).

---

* This condition cannot be satisfied for an odd n .

Theorem (7.3). *We have*

$$(7.21) \quad \partial_\lambda g^{[\lambda\nu]} = -S_{\alpha\beta}{}^\beta g^{(\nu\alpha)}.$$

First of all we obtain from (2.3d) and (2.3e)

$$(7.22) \quad D_\omega g^{\lambda\nu} = -2S_{\omega\alpha}{}^\nu g^{\lambda\alpha} + S_{\omega\alpha}{}^\alpha g^{\lambda\nu},$$

and consequently

$$(7.23a) \quad \begin{aligned} D_\lambda g^{[\lambda\nu]} &= -S_{\lambda\alpha}{}^\nu g^{\lambda\alpha} + S_{\lambda\alpha}{}^\lambda g^{\nu\alpha} + S_{\lambda\alpha}{}^\alpha g^{[\lambda\nu]}, \\ &= -S_{\lambda\alpha}{}^\nu g^{[\lambda\alpha]} - S_{\alpha\beta}{}^\beta g^{[\nu\alpha]} - S_{\alpha\beta}{}^\beta g^{(\nu\alpha)} + S_{\alpha\beta}{}^\beta g^{[\alpha\nu]}, \\ &= g^{[\beta\alpha]}(S_{\alpha\beta}{}^\nu - 2\delta_\beta^\nu S_{\alpha\gamma}{}^\gamma) - S_{\alpha\beta}{}^\beta g^{(\nu\alpha)}. \end{aligned}$$

On the other hand if we write $a^{\lambda\nu} = g^{[\lambda\nu]}$ we have

$$(7.23b) \quad \begin{aligned} D_\lambda a^{\lambda\nu} &= \partial_\lambda a^{\lambda\nu} + \Gamma_{\alpha\lambda}^\lambda a^{\alpha\nu} + \Gamma_{\alpha\lambda}^\nu a^{\lambda\alpha} - \Gamma_{\alpha\lambda}^\alpha a^{\lambda\nu} \\ &= \partial_\lambda g^{[\lambda\nu]} + 2g^{[\alpha\nu]} S_{\alpha\beta}{}^\beta + S_{\alpha\beta}{}^\nu g^{[\beta\alpha]} = D_\lambda g^{[\lambda\nu]}. \end{aligned}$$

Comparing (7.23a) and (7.23b) we obtain (7.21).

Corollary (7.5). *The requirement*

$$(7.24a) \quad S_{\alpha\beta}{}^\beta = 0$$

yields

$$(7.24b) \quad \partial_\lambda g^{[\lambda\nu]} = 0.$$

The proof follows at once from (7.21).

Later on we shall see that Theorem (7.3) is closely connected with the following

Theorem (7.4). *We have*

$$(7.25) \quad \partial_{[\omega} k_{\lambda\mu]} = 2S_{[\omega\lambda\mu]}.$$

Proof. First we have from (2.3c)

$$(7.26a) \quad D_{[\omega} k_{\lambda\mu]} = 2Q_{[\omega\mu\lambda]}.$$

On the other hand

$$(7.26b) \quad \begin{aligned} D_{[\omega} k_{\lambda\mu]} &= \partial_{[\omega} k_{\lambda\mu]} + \Gamma_{[\lambda\omega}^\alpha k_{\mu]\alpha} - \Gamma_{[\mu\omega}^\alpha k_{\lambda]\alpha}, \\ &= \partial_{[\omega} k_{\lambda\mu]} + 2S_{[\omega\mu}{}^\alpha (g_{\lambda]\alpha} - h_{\lambda]\alpha}), \\ &= \partial_{[\omega} k_{\lambda\mu]} + 2Q_{[\omega\mu\lambda]} - 2S_{[\omega\mu\lambda]}. \end{aligned}$$

Comparing (7.26a) and (7.26b) we obtain (7.25).

Note: If $n = 4$ then $S_{\omega\lambda\mu}$ is proportional to a vector density v^ν

$$S_{[\omega\lambda\mu]} = e_{\omega\lambda\mu\nu} v^\nu$$

where $e_{\omega\lambda\mu\nu}$ is the four vector density skew symmetric in all indices whose components are 0, +1, -1. Then (7.25) may be written

$$(7.27) \quad \partial_{[\omega} k_{\lambda\mu]} = 2e_{\omega\lambda\mu\nu} v^\nu.$$

8. The field $U^\nu_{\lambda\mu}$. In this section we shall investigate the field $U^\nu_{\lambda\mu}$ given by (2.9):

$$(8.1) \quad U^\nu_{\lambda\mu} = 2h^{\nu\alpha} S_{\alpha(\lambda}{}^\beta k_{\mu)\beta},$$

which reduces (2.4) to (2.10):

$$(8.2a) \quad \Gamma^\nu_{\lambda\mu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + U^\nu_{\lambda\mu} + S_{\lambda\mu}{}^\nu.$$

The symmetric part of $\Gamma^\nu_{\lambda\mu}$ will be denoted by

$$(8.3) \quad \Lambda^\nu_{\lambda\mu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + U^\nu_{\lambda\mu},$$

so that we have

$$(8.2b) \quad \Gamma^\nu_{\lambda\mu} = \Lambda^\nu_{\lambda\mu} + S_{\lambda\mu}{}^\nu.$$

First we need the following

Lemma (8.1). *The tensor $U^\nu_{\lambda\mu}$ satisfies the following identities*

$$(8.4) \quad \text{a) } U^\nu_{[\lambda\mu]} = 0 \quad \text{b) } U_{(\nu\lambda\mu)} = 0.$$

Proof. These identities follow at once from (8.1).

Theorem (8.1). *A necessary and sufficient condition for a tensor $U^\nu_{\lambda\mu}$ to be of the form (8.1) is*

$$(8.6) \quad U_{\alpha\beta\gamma}[\delta_{\nu\lambda\mu}^{\alpha\beta\gamma} - 2k_{(\mu}{}^\beta k_{\lambda)}{}^\alpha \delta_\nu{}^\gamma + 2k_\nu{}^\alpha k_{(\mu}{}^\beta \delta_{\lambda)}{}^\gamma] = -k_{(\mu}{}^\beta K_{\lambda)\nu\beta}.$$

If this condition is satisfied then

$$(8.7) \quad 2U^\alpha_{\alpha\mu} = \partial_\mu \ln (g/h).$$

Proof. Assume that (8.1) is satisfied. Applying it in (2.14), we obtain (8.6). According to Agreement (4.1) we confine ourselves to the tensors $g_{\lambda\mu}$ for which the system (2.1) admits only one solution. Hence there is only one tensor $S_{\lambda\mu}{}^\nu$ and one tensor $U^\nu_{\lambda\mu}$ such that (8.2a) is satisfied. Hence if we look on (8.6) as a system of equations for the unknowns $U_{\nu\lambda\mu}$, we see that the corresponding determinant D (a function of the components $g_{\lambda\mu}$ only) must be different from zero.

Therefore if an *a priori* given tensor $U^\nu{}_{\lambda\mu}$ satisfies (8.6) this tensor must be of the form (8.1). Assume now that $U^\nu{}_{\lambda\mu}$ satisfies (8.6) so that it is of the form (8.1). Substituting it in (8.2a), we obtain $\Gamma^\nu{}_{\lambda\mu}$. Then we have from (2.3e) and (8.2a)

$$\begin{aligned} D_\omega g &= \partial_\omega g - 2 \Gamma^\alpha{}_{\alpha\omega} g = g[\partial_\omega \ln g - 2 \left\{ \begin{smallmatrix} \alpha \\ \alpha\omega \end{smallmatrix} \right\} - 2U^\alpha{}_{\alpha\omega} - 2S_{\alpha\omega}{}^\alpha], \\ &= 2S_{\omega\alpha}{}^\alpha g, \end{aligned}$$

so that

$$\partial_\omega \ln g - \partial_\omega \ln h - 2U^\alpha{}_{\alpha\omega} = 0,$$

and this equation is equivalent to (8.7).

In the next theorem we will investigate the connection (8.3).

Theorem (8.2). *The connection (8.3) is never a Weyl connection.*

Proof. Denote by ${}^\circ D_\mu$ the symbol of the covariant derivative with respect to $\Lambda^\nu{}_{\lambda\mu}$. A necessary and sufficient condition for $\Lambda^\nu{}_{\lambda\mu}$ to be a Weyl connection is

$$(8.8a) \quad {}^\circ D_\omega h_{\lambda\mu} = Q_\omega h_{\lambda\mu},$$

where

$$(8.8b) \quad \partial_{[\omega} Q_{\mu]} \neq 0.$$

On the other hand we obtain from (8.3) and (8.4a, b)

$$\begin{aligned} (8.9) \quad {}^\circ D_\nu h_{\lambda\mu} &= -U^\alpha{}_{\lambda\nu} h_{\alpha\mu} - U^\alpha{}_{\mu\nu} h_{\lambda\alpha} = -U_{\mu\lambda\nu} - U_{\lambda\mu\nu}, \\ &= -(U_{\mu\nu\lambda} + U_{\lambda\mu\nu}) = U_{\nu\lambda\mu}. \end{aligned}$$

Comparing (8.8a) and (8.9) we obtain

$$Q_\nu h_{\lambda\mu} = U_{\nu\lambda\mu},$$

so that the identity (8.4b) leads to

$$(8.10a) \quad Q_\nu = 0$$

and

$$(8.10b) \quad U_{\nu\lambda\mu} = 0.$$

The theorem follows easily from (8.10b).

Corollary (8.1). *If (8.10b) is satisfied, then we have*

$$(8.11) \quad \Gamma^\nu{}_{\lambda\mu} = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + \frac{1}{2} K_{\lambda\mu}{}^\nu.$$

The proof follows at once from (2.14), (8.2a) and (8.10b).

In the next theorem we will investigate a case similar to the one dealt with in Theorem (7.2).

Theorem (8.3). *A necessary and sufficient condition for $U_{\nu\lambda\mu}$ to be of the form*

$$(8.12) \quad U_{\nu\lambda\mu} = k_{\nu(\lambda} p_{\mu)}$$

is

$$(8.13a) \quad p_{\gamma}[k_{\nu(\lambda} \delta_{\mu)}^{\gamma} - 2k_{\alpha(\beta} \delta_{\nu)}^{\gamma} k_{(\mu}^{\beta} k_{\lambda)}^{\alpha} + k_{\nu}^{\alpha}(k_{\alpha\beta} \delta_{(\lambda}^{\gamma} k_{\mu)}^{\beta} + k_{\alpha(\lambda} k_{\mu)}^{\gamma})] \\ = -k_{(\mu}^{\beta} K_{\lambda)\nu\beta}.$$

If this condition is satisfied then

$$(8.13b) \quad k_{\mu\nu} p^{\nu} = \frac{\partial}{\partial x^{\mu}} \ln \frac{h}{g}.$$

Proof. Assume that (8.12) holds*. Substituting from it for $U_{\nu\lambda\mu}$ into (8.6) and (8.7), we obtain (8.13). Assume that (8.13a) holds. If we define $U_{\nu\lambda\mu}$ by (8.12) then (8.6) (and (8.7)) are satisfied by virtue of (8.13a).

Corollary (8.2). *If (8.13) are satisfied then we have*

$$(8.14a) \quad \Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \frac{1}{2} K_{\lambda\mu}^{\nu} + k^{\alpha\nu} p_{[\lambda} k_{\mu]\alpha} + k_{(\lambda}^{\nu} p_{\mu)}.$$

The proof follows from (2.14), (8.12) and (8.2a).

Note. If $k \neq 0$ then there is a unique tensor $l^{\mu\nu} = l^{[\mu\nu]}$ such that

$$k_{\mu\nu} l^{\mu\alpha} = \delta_{\nu}^{\alpha}.$$

Then the equations (8.13a, b) may be condensed into one, which one obtains from (8.13a) by substituting for p^{ν} from

$$p^{\nu} = l^{\mu\nu} \frac{\partial}{\partial x^{\mu}} \ln \frac{h}{g}.$$

Equation (8.14a) becomes after this substitution

$$(8.14b) \quad \Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \frac{1}{2} K_{\lambda\mu}^{\nu} + \left(h_{\gamma\beta} l^{\rho\beta} \frac{\partial}{\partial x^{\rho}} \ln \frac{h}{g} \right) k^{\alpha\nu} \delta_{[\lambda}^{\gamma} k_{\mu]\alpha} + k_{(\mu}^{\nu} \delta_{\nu)}^{\gamma}.$$

9. Continuation. The case investigated in the previous two sections will be very important in dealing with Problem C mentioned in the Introduction. In this section we shall deal with the field $U^{\nu}_{\lambda\mu}$ from the point of view of paths. A path of a connection is by definition an autoparallel line of this connection. Hence in the classical general theory of relativity the curves described by pho-

---

* It is obvious that (8.4a, b) are satisfied for $U_{\nu\lambda\mu}$ defined by (8.12).

tons or by particles of mass in absence of an electromagnetic field are paths of the connection $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$. Therefore it is obvious that the paths will play an important role in dealing with Problem C.

In order to investigate the paths of our connection $\Gamma_{\lambda\mu}^\nu$ we start with the following

Lemma (9.1). *The connection $\Gamma_{\lambda\mu}^\nu$ and $\Lambda_{\lambda\mu}^\nu$ given by (8.2a) and (8.3) have the same paths.*

Proof. If v^ν is an arbitrary vector field we obtain from (8.2a, b) and (8.3)

$$v^\mu D_\mu v^\nu = v^\mu \circ D_\mu v^{\nu*},$$

and the lemma follows easily.

Theorem (9.1). *A necessary and sufficient condition for the connections $\Gamma_{\lambda\mu}^\nu$ and $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ to have the same paths is*

$$(9.1) \quad U^\nu_{\lambda\mu} = 0.$$

Proof. It is sufficient to prove the statement for the connections $\Lambda_{\lambda\mu}^\nu$ and $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$. A necessary and sufficient condition for these two connections to have the same paths follows at once from (8.3), namely

$$(9.2a) \quad U^\nu_{\lambda\mu} = \delta_\lambda^\nu p_\mu + \delta_\mu^\nu p_\lambda,$$

where p_λ is an arbitrary vector. This equation together with (8.4b) leads at once to

$$(9.2b) \quad p_\mu = 0,$$

and the theorem follows.

Note. Theorem (9.1) is of formal character only as long as we do not know the solution of (2.1). In the next theorem we will express the condition (9.1) in terms of given data $g_{\lambda\mu}$ only, which do not require the knowledge of the solution of (2.1).

Theorem (9.2a). *A necessary and sufficient condition for the connections $\Gamma_{\lambda\mu}^\nu$ and $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ to have the same paths is*

$$(9.3a) \quad k_{(\mu}{}^\beta K_{\lambda)\nu\beta} = 0.$$

If this condition is satisfied then

$$(9.3b) \quad g/h = \text{const.} \quad (\neq 0)$$

---

* $\circ D_\mu$ is again the symbolic vector of the covariant derivative with respect to $\Lambda_{\lambda\mu}^\nu$.

Proof. Substituting for $U^\nu_{\lambda\mu}$ from (9.1) into (8.6) one obtains (9.3a). Substituting for $U^\nu_{\lambda\mu}$ from (9.1) into (8.7) one obtains (9.3b)*.

Note. The tensor $K_{\nu\lambda\beta}$ being defined by (2.6), we know it as soon as we know $g_{\lambda\mu}$. Hence the equations (9.3a, b) do not require knowledge of $\Gamma^\nu_{\lambda\mu}$.

The question arises if there are tensors $g_{\lambda\mu}$ for which (9.3a, b) hold. The next theorem answers this question.

Theorem (9.2b). *The set of all tensors $g_{\lambda\mu}$ satisfying (9.3a, b) is not an empty set.*

The proof consists in exhibiting at least one such tensor: Put $n = 4$ and

$$a = e^{x^3+x^4}.$$

Then a straightforward computation shows that the tensor $g_{\lambda\mu}$ given by the matrix

$$(9.4) \quad ((g_{\lambda\mu})) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & a \\ 0 & -a & 1 & 0 \\ 0 & -a & 0 & -1 \end{pmatrix}$$

satisfies (9.3a, b).

Note I. If we multiply (9.3a) by $h^{\lambda\nu}$ we obtain

$$(9.5a) \quad k^{\alpha\beta} k_{\alpha\beta} = \text{const.}$$

On the other hand if $n = 4$ we have

$$(9.6) \quad g = h + \frac{1}{2}h k^{\alpha\beta} k_{\alpha\beta} + k.$$

Comparing (9.3b), (9.5a) and (9.6), we see that in our case

$$(9.7) \quad k/h = \text{const.}$$

Hence if

$$(9.5b) \quad k^{\alpha\beta} k_{\alpha\beta} = 0$$

we see at once that the equation (9.3b) is a consequence of (9.3a) (cf. Theorem (9.2a)). On the other hand even if (9.3a) holds we may have (9.5a) but not (9.5b): assume for instance that the components $k_{\lambda\mu}$ are covariant constants and that (9.5a) but not (9.5b) holds. The equation (9.3a) is identically satisfied in this case while (9.5b) is not satisfied.

---

* As is stated in the Introduction, we assume the validity of (1.4a) unless otherwise stated.

Corollary (9.1). *In the case mentioned in Theorem (9.2a) we have*

$$(9.8) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + \frac{1}{2} K_{\lambda\mu}^\nu.$$

The proof is obvious and follows from Corollary (8.1).

10. Paths of photons. According to the classical general theory of relativity the curves described by photons are minimal paths of the connection $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$. We shall refer to these curves simply as paths of photons. In this section we will find a connection $\Gamma_{\lambda\mu}^\nu$, a solution of (2.1), whose minimal paths are paths of photons. We shall see that this identification of minimal paths of $\Gamma_{\lambda\mu}^\nu$ with the paths of photons leads straight to the tensor $U^\nu_{\lambda\mu}$ and consequently (by virtue of (2.14) and (8.2a)) to the connection $\Gamma_{\lambda\mu}^\nu$, the solution of (2.1).

We will prove first the following

Theorem (10.1). *If and only if*

$$(10.1) \quad \begin{aligned} \text{a)} \quad & U^\nu_{\lambda\mu} = 2p^\nu h_{\lambda\mu} - \delta_\lambda^\nu p_\mu - \delta_\mu^\nu p_\lambda, \\ \text{b)} \quad & 2(n-1)p_\mu = \partial_\mu \ln(h/g) \neq 0, \end{aligned}$$

then

- 1) every minimal path of $\Gamma_{\lambda\mu}^\nu$ is a path of photons and vice versa;
- 2) there are (non-minimal) paths of $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ which are not paths of $\Gamma_{\lambda\mu}^\nu$; and
- 3) there are (non-minimal) paths of $\Gamma_{\lambda\mu}^\nu$ which are not paths of $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$.

Proof. According to Lemma (9.1) it is obvious that it suffices to prove the same statements about the connections $\Lambda_{\lambda\mu}^\nu$ and $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$. A necessary and sufficient condition that $\Lambda_{\lambda\mu}^\nu$ and $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ have the same paths is obviously (9.2a) (which leads to (9.1)). A necessary and sufficient condition for requirements 1), 2), 3) to be fulfilled is obviously

$$(10.2) \quad U^\nu_{\lambda\mu} = X^\nu_{\lambda\mu} - \delta_\lambda^\nu p_\mu - \delta_\mu^\nu p_\lambda, \quad p_\lambda \neq 0,$$

where by virtue of the requirement 1) the tensor $X^\nu_{\lambda\mu} = X^\nu_{(\mu\lambda)}$ satisfies the condition

$$(10.3a) \quad v_\nu X^\nu_{\lambda\mu} = \rho h_{\lambda\mu}$$

for any arbitrary vector v_ν and a suitably chosen $\rho \neq 0$. This condition is satisfied if and only if

$$(10.3b) \quad X^\nu_{\lambda\mu} = 2q^\nu h_{\lambda\mu}$$

where q^ν is a suitably chosen vector. From (10.2) and (10.3b) we obtain

$$(10.4a) \quad U_{\nu\lambda\mu} = 2q_\nu h_{\lambda\mu} - p_\mu h_{\lambda\nu} - p_\lambda h_{\mu\nu}.$$

Hence the condition (8.4a) is satisfied while the condition (8.4b) leads at once to

$$(10.5) \quad q_\nu = p_\nu, \quad p_\nu \neq 0.$$

The proof of the remaining statement is obvious.

Theorem (10.2). Assume that the requirements 1), 2), 3) of Theorem (10.1) are satisfied. Then there is at most one family of non-minimal paths of $\Gamma_{\lambda\mu}^\nu$ which is also a family of non-minimal paths of $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ and vice versa. Necessary and sufficient conditions for its existence are

$$(10.6) \quad \text{a) } h_{\lambda\mu} p^\lambda p^\mu \neq 0, \quad \text{b) } p^\mu (D_\mu p^{[\nu]} p^{\lambda]}) = 0.$$

If these conditions are satisfied then the family is constituted by curves whose tangential vector field is p^ν .

Proof. The requirements 1), 2), 3) of Theorem (10.1) being satisfied, we have (10.1a, b). Denote by v^ν the tangential vector field of a non-minimal paths of $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$:

$$(10.7) \quad \text{a) } v^\lambda v^\mu h_{\lambda\mu} \neq 0, \quad \text{b) } v^\mu (\nabla_\mu v^{[\nu]} v^{\lambda]}) = 0.$$

On the other hand we have

$$(10.7c) \quad v^\mu ({}^\circ D_\mu v^{[\nu]} v^{\lambda]}) = v^\mu (\nabla_\mu v^{[\nu]} v^{\lambda]}) + v^{[\lambda} U^{\nu]}_{\alpha\beta} v^\alpha v^\beta.$$

Hence if v^ν is the tangential vector field of our family we must have by virtue (10.7b, c)

$$v^{[\lambda} U^{\nu]}_{\alpha\beta} v^\alpha v^\beta = 0,$$

or according to (10.1a)

$$p^{[\nu} v^{\lambda]} v^\alpha v^\beta h_{\alpha\beta} - v^\alpha p_\alpha v^{[\nu} v^{\lambda]} = 0,$$

and this equation yields $v^\nu = \rho p^\nu$, $\rho \neq 0$. On the other hand if (10.7b) is satisfied then we have from (10.7c)

$$v^\mu ({}^\circ D_\mu v^{[\nu]} v^{\lambda]}) = v^{[\lambda} U^{\nu]}_{\alpha\beta} v^\alpha v^\beta,$$

or

$$v^\mu ({}^\circ D_\mu v^{[\nu]} v^{\lambda]}) = 2p^{[\nu} v^{\lambda]} v^\alpha v^\beta h_{\alpha\beta}.$$

If (10.7a) holds then the left hand member is equal to zero if and only if $v^\nu = \rho p^\nu$, $\rho \neq 0$. The remaining statements of our theorem may be proved in a similar way.

In the next theorem we will use the tensors A_λ^ν , B_λ^ν and K_λ^ν defined by (7.1a), (7.2) and (7.3).

Theorem (10.3). *A necessary and sufficient condition for the requirements 1), 2), 3) of Theorem (10.1) to be satisfied is*

$$(10.8a) \quad (n-1)k_{(\mu}{}^\beta K_{\lambda)\nu\beta} = (\delta_\nu^\rho K_{\lambda\mu} - \delta_{(\lambda}^\rho K_{\mu)\nu}) \partial_\rho \ln(g/h) \neq 0.$$

If this condition is satisfied then

$$(10.9a) \quad (n-1)\partial_\lambda k^{\alpha\beta} k_{\alpha\beta} = 2A_\lambda^\rho \partial_\rho \ln(g/h).$$

If (7.1b) holds then (10.8b) is equivalent to

$$(10.9b) \quad (n-1)B_\nu^\lambda \partial_\lambda k^{\alpha\beta} k_{\alpha\beta} = 2\partial_\nu \ln(g/h),$$

and (10.8a) is equivalent to

$$(10.8b) \quad 2k_{(\mu}{}^\beta K_{\lambda)\nu\beta} = [\delta_\nu^\rho K_{\lambda\mu} - \delta_{(\lambda}^\rho K_{\mu)\nu}] B_\rho^\gamma \partial_\gamma \ln(g/h) \neq 0.$$

Proof. If we substitute into (8.6) from (10.1a, b) we obtain (10.8a). Multiplying this equation by $h^{\lambda\mu}$, one obtains (10.9a). The proof of the first part of theorem (10.3) follows easily from the equations (10.8a) and (10.9a) and from theorems (8.1) and (10.1). The second part of theorem (10.3) is obvious.

Note. While the conditions (10.1) require the knowledge of $U_{\lambda\mu}^\nu$, the conditions (10.8) require only the knowledge of $g_{\lambda\mu}$ and might be checked as soon as $g_{\lambda\mu}$ is given.

Theorem (10.4). *Whenever the condition (10.8a) is satisfied, the solution $\Gamma_{\lambda\mu}^\nu$ of (2.1) has the form*

$$(10.10) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \frac{1}{2} K_{\lambda\mu}^\nu + \frac{1}{n-1} \cdot$$

$$[2\delta_{[\lambda}^\nu k_{\mu]}{}^\gamma + \delta_{[\mu}^\gamma k_{\lambda]}{}^\nu + h^{\gamma\nu} g_{\lambda\mu} - \delta_{(\mu}^\gamma \delta_{\lambda)}^\nu] \partial_\gamma \ln(h/g).$$

Proof. Whenever (10.8a) holds we have also (10.9a). Substituting from (10.1a) into (2.14), we obtain $S_{\lambda\mu}^\nu$, and substituting this tensor together with $U_{\lambda\mu}^\nu$ in (8.2a) we have (10.10).

PART III.

In the first part of this paper we have solved the system (2.1) for the $\Gamma_{\lambda\mu}^{\nu}$. The solution is given by (4.13). Although this equation may be looked upon as a solution of Problem *B*, its practical significance for checking purposes (physical experiments) is far inferior to its theoretical significance, for involved in the result is the tensor $Y_{\xi\eta}^{\alpha\beta\gamma}$, whose construction requires us to deal with 24-rowed determinants.

In this part we shall seek another device for solving the system (2.1) by a method which does not present any difficulties from the point of view of testing it by experiments. We shall exhibit two different ways, the second one being constructed under the influence of the physical practice of looking for the "first approximation" only. The first one might be described roughly as follows: we shall find a connection ${}^*\Gamma_{\lambda\mu}^{\nu}$ (easily obtained) such that the difference ${}^*\Gamma_{\lambda\mu}^{\nu} - \Gamma_{\lambda\mu}^{\nu}$ is as small as we please.

11. Auxiliary results. Let us change our notation temporarily, writing ${}^*k_{\lambda\mu}$ for the tensor which we denoted earlier by $k_{\lambda\mu}$, and let us consider all possible products of each of the tensors

$$(11.1a) \quad {}^*k_{\lambda\mu}, \quad \nabla_{\omega} {}^*k_{\lambda\mu}$$

with $h^{\omega\nu}$ in which the corresponding tensor (11.1a) enters only linearly, as for instance

$$(11.1b) \quad {}^*k_{\lambda}^{\nu}, \quad {}^*k^{\nu}_{\lambda}, \quad \nabla^{\omega} {}^*k_{\lambda\mu}, \quad \nabla^{\omega} {}^*k_{\lambda}^{\nu}, \dots,$$

and so on. Put

$$(11.2a) \quad N = 2n \binom{n+1}{2},$$

and denote by $C(\chi)$ the largest of the absolute values of the components of the tensors (11.1b) (of the tensor $h^{\lambda\nu}$). Denote by λ an arbitrary positive number satisfying the conditions

$$(11.2b) \quad 1 \leq \lambda \leq \frac{2\chi}{n+1},$$

and introduce the number

$$(11.3) \quad M = \frac{3n}{4N(\lambda^2 N - 1)[(\lambda^2 N)^m - 1]}, \quad m = 1, 2, 3, \dots$$

and the tensor

$$(11.4) \quad k_{\lambda\mu} = \frac{{}^*k_{\lambda\mu}}{2CN\lambda}.$$

Our investigation will be based on the number M and on the tensor

$$(11.5) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},$$

where $k_{\lambda\mu}$ is given by (11.4).

Definition (11.1). Let A, B, D, Q be four (real) numbers related by

$$(11.6a) \quad |A - B| \leq D, \quad \left| \frac{A}{A - B} \right| \geq Q,$$

or

$$(11.6b) \quad |A - B| \leq D, \quad \left| \frac{B}{A - B} \right| \geq Q.$$

If (11.6a) or (11.6b) are satisfied we shall write either

$$(11.7a) \quad A \cong B, \{D, Q\},$$

or

$$(11.7b) \quad A - B \cong 0, \{D, Q\}.$$

In the next theorems the following tensors are involved:

$$(11.8) \quad \begin{aligned} \text{a)} \quad & {}^1X_{\nu\lambda\mu}^{\alpha\beta\gamma} = 2k_{(\mu}{}^\beta k_{\lambda)}{}^\alpha \delta_\nu{}^\gamma - 2k_\nu{}^\alpha k_{(\mu}{}^\beta \delta_{\lambda)}{}^\gamma, \\ \text{b)} \quad & {}^sX_{\nu\lambda\mu}^{\alpha\beta\gamma} = {}^1X_{\xi\xi\eta}^{\alpha\beta\gamma} {}^{s-1}X_{\nu\lambda\mu}^{\xi\xi\eta}, \end{aligned} \quad s = 2, 3, \cdot$$

and

$$(11.8c) \quad Z_{\nu\lambda\mu}^{\alpha\beta\gamma} = \delta_{\nu(\lambda\mu)}^{\alpha\beta\gamma} - {}^1X_{\nu\lambda\mu}^{\alpha\beta\gamma}.$$

Theorem (11.1). The tensor ${}^sX_{\nu\lambda\mu}^{\alpha\beta\gamma}$ satisfies the relation

$$(11.9) \quad |{}^sX_{\nu\lambda\mu}^{\alpha\beta\gamma}| \leq \frac{1}{N(\lambda^2 N)^s}, \quad s = 2, 3, \cdot$$

Proof. Remembering the significance of C , we see from (11.4) that

$$(11.10) \quad |k_{\lambda\mu}| \leq \frac{1}{2\lambda N}.$$

Hence we have by virtue of (11.8a, b) and (11.10)

$$| {}^1 X_{\nu\lambda\mu}^{\alpha\beta\gamma} | \leq | (k_\mu^\beta k_\lambda^\alpha + k_\lambda^\alpha k_\mu^\beta) \delta_\nu^\gamma | + | (k_\mu^\beta \delta_\lambda^\gamma + k_\lambda^\beta \delta_\mu^\gamma) k_\nu^\alpha | \leq \frac{4}{4\lambda^2 N^2} = \frac{1}{(\lambda^2 N)N}$$

$$| {}^2 X_{\nu\lambda\mu}^{\alpha\beta\gamma} | = | {}^1 X_{\xi\xi\eta}^{\alpha\beta\gamma} {}^1 X_{\nu\lambda\mu}^{\xi\xi\eta} | \leq | {}^1 X_{\xi\xi\eta}^{\alpha\beta\gamma} | | {}^1 X_{\nu\lambda\mu}^{\xi\xi\eta} | \leq \frac{N}{(\lambda^2 N)^2 N^2} = \frac{1}{N(\lambda^2 N)^2}$$

Proceeding in this manner we obtain (11.9) for all $s = 1, 2, 3, \dots$.

Theorem (11.2). *The tensor $Z_{\nu\lambda\mu}^{\alpha\beta\gamma}$ defined by (11.8c) satisfies the equation*

$$(11.11a) \quad Z_{\xi\xi\eta}^{\alpha\beta\gamma} {}^m Y_{\nu\lambda\mu}^{\xi\xi\eta} = \delta_{\nu(\lambda\mu)}^{\alpha\beta\gamma} - {}^m X_{\nu\lambda\mu}^{\alpha\beta\gamma},$$

where

$$(11.11b) \quad \begin{aligned} {}^1 Y_{\nu\lambda\mu}^{\xi\xi\eta} &= \delta_{\nu(\lambda\mu)}^{\xi\xi\eta} \\ {}^m Y_{\nu\lambda\mu}^{\xi\xi\eta} &= \delta_{\nu(\lambda\mu)}^{\xi\xi\eta} + \sum_{s=1}^{m-1} {}^s X_{\nu\lambda\mu}^{\xi\xi\eta}, \end{aligned} \quad m = 2, 3, \dots$$

Proof. Substituting from (11.11b) for ${}^m Y_{\nu\lambda\mu}^{\xi\xi\eta}$ into (11.11a) one obtains

$$\begin{aligned} Z_{\xi\xi\eta}^{\alpha\beta\gamma} {}^m Y_{\nu\lambda\mu}^{\xi\xi\eta} &= \delta_{\nu(\lambda\mu)}^{\alpha\beta\gamma} + \sum_{s=1}^{m-1} {}^s X_{\nu\lambda\mu}^{\alpha\beta\gamma} - \sum_{s=1}^m {}^s X_{\nu\lambda\mu}^{\alpha\beta\gamma}, \\ &= \delta_{\nu(\lambda\mu)}^{\alpha\beta\gamma} - {}^m X_{\nu\lambda\mu}^{\alpha\beta\gamma}. \end{aligned}$$

12. The tensor $U_{\nu\lambda\mu}$. In the following consideration we shall deal with the tensors

$$(12.1) \quad H_{\nu\lambda\mu} = -k_{(\mu}^\beta K_{\lambda)\nu\beta}$$

(where $K_{\omega\mu\nu}$ is defined by (2.6)) and

$$(12.2) \quad {}^m U_{\nu\lambda\mu} = H_{\alpha\beta\gamma} {}^m Y_{\nu\lambda\mu}^{\alpha\beta\gamma}.$$

Theorem (12.1). *The tensors $U_{\nu\lambda\mu}^{\nu}$ (defined by (2.9)) and ${}^m U_{\nu\lambda\mu}$ are related by*

$$(12.3) \quad U_{\nu\lambda\mu} - {}^m U_{\nu\lambda\mu} = U_{\alpha\beta\gamma} {}^m X_{\nu\lambda\mu}^{\alpha\beta\gamma}.$$

Proof. The tensor $U_{\nu\lambda\mu}^{\nu}$ defined by (2.9) satisfies the equation (8.6) (cf. Theorem (8.1)). Using this equation as well as (11.8c) we may write

$$(12.4a) \quad U_{\alpha\beta\gamma} Z_{\xi\xi\eta}^{\alpha\beta\gamma} = H_{\xi\xi\eta}$$

and by virtue of (11.11a)

$$\begin{aligned} U_{\alpha\beta\gamma} Z_{\xi\xi\eta}^{\alpha\beta\gamma} {}^m Y_{\nu\lambda\mu}^{\xi\xi\eta} &= U_{\alpha\beta\gamma} (\delta_{\nu(\lambda\mu)}^{\alpha\beta\gamma} - {}^m X_{\nu\lambda\mu}^{\alpha\beta\gamma}), \\ &= H_{\xi\xi\eta} {}^m Y_{\nu\lambda\mu}^{\xi\xi\eta} = {}^m U_{\nu\lambda\mu}, \end{aligned}$$

and this equation is equivalent to (12.3).

In the next lemma we denote by $U_+ = |U_{\nu\lambda\mu}|$ the largest of all absolute values of the components $U_{\alpha\beta\gamma}$.

Lemma (12.1). *Put*

$$(12.5a) \quad \frac{U_+}{(\lambda^2 N)^m} = \rho.$$

Then

$$(12.6) \quad \rho \leq M.$$

Proof. First we have by virtue of (11.9) and (12.5a)

$$(12.7a) \quad |U_{\alpha\beta\gamma} {}^m X_{\nu\lambda\mu}^{\alpha\beta\gamma}| \leq |U_{\alpha\beta\gamma}| |{}^m X_{\nu\lambda\mu}^{\alpha\beta\gamma}| \leq \frac{NU_+}{(\lambda^2 N)^m N} = \rho.$$

On the other hand we obtain from (12.2), (12.3) and (12.7a)

$$(12.7b) \quad \begin{aligned} U_+ &= |{}^m U_{\nu\lambda\mu} + U_{\alpha\beta\gamma} {}^m X_{\nu\lambda\mu}^{\alpha\beta\gamma}| \leq |H_{\alpha\beta\gamma} {}^m Y_{\nu\lambda\mu}^{\alpha\beta\gamma}| + \rho. \\ &= |H_{\nu\lambda\mu} + \sum_{s=1}^{m-1} H_{\alpha\beta\gamma} {}^s X_{\nu\lambda\mu}^{\alpha\beta\gamma}| + \rho, \\ &\leq |H_{\nu\lambda\mu}| + \sum_{s=1}^{m-1} |H_{\alpha\beta\gamma}| |{}^s X_{\nu\lambda\mu}^{\alpha\beta\gamma}| + \rho. \end{aligned}$$

For the first and second term in the last row we obtain by virtue of (11.9), (12.1) and (11.4)

$$(12.7c) \quad \begin{aligned} |H_{\nu\lambda\mu}| &\leq \frac{1}{2} |k_\mu^\beta| |K_{\lambda\nu\beta}| + \frac{1}{2} |K_\lambda^\beta| |K_{\mu\nu\beta}|, \\ &\leq \frac{n}{2\lambda N} \frac{3}{2\lambda N} = \frac{3n}{4(\lambda N)^2}, \end{aligned}$$

$$(12.7d) \quad \begin{aligned} \sum_{s=1}^{m-1} |H_{\alpha\beta\gamma}| |{}^s X_{\nu\lambda\mu}^{\alpha\beta\gamma}| &\leq \frac{3nN}{4(\lambda N)^2} \sum_{s=1}^{m-1} \frac{1}{N(\lambda^2 N)^s}, \\ &= \frac{3n}{4(\lambda N)^2} \frac{(\lambda^2 N)^{m-1} - 1}{(\lambda^2 N)^{m-1}(\lambda^2 N - 1)}. \end{aligned}$$

Hence

$$\begin{aligned} U_+ &= \rho(\lambda^2 N)^m \leq \frac{3n}{4(\lambda N)^2} \left[1 + \frac{(\lambda^2 N)^{m-1}}{(\lambda^2 N)^{m-1}(\lambda^2 N - 1)} \right] + \rho, \\ &= \frac{3n}{4(\lambda N)^2} \frac{(\lambda^2 N)^m - 1}{(\lambda^2 N)^{m-1}(\lambda^2 N - 1)} + \rho \leq \frac{3n}{4(\lambda N)^2} \frac{\lambda^2 N}{\lambda^2 N - 1} + \rho, \end{aligned}$$

or

$$(12.8) \quad \rho[(\lambda^2 N)^m - 1] \leq \frac{3n}{4N(\lambda^2 N - 1)},$$

and this relation together with (11.3) yields (12.6).

Note. The formulae (12.5a) and (12.6) enable us to estimate U_+ without actually computing the tensor $U_{\alpha\beta\gamma}$. If now $U = |U_{\nu\lambda\mu}|$ denotes the absolute value of an arbitrary non-zero component of the tensor $U_{\alpha\beta\gamma}$, we have according to (12.5a)

$$(12.5b) \quad \frac{U}{(\lambda^2 N)^m} = \rho\xi, \quad 0 < \xi \leq 1.$$

To each combination of the indices ν, λ, μ belongs a certain ξ . In the next theorem we denote by ξ_0 the smallest of these numbers.

Theorem (12.2). *Assume*

$$(12.9) \quad U_{\alpha\beta\gamma} {}^m X_{\nu\lambda\mu}^{\alpha\beta\nu} \neq 0^*.$$

Then

$$(12.10) \quad U_{\nu\lambda\mu} - {}^m U_{\nu\lambda\mu} \cong 0, \{M, \xi_0(\lambda^2 N)^m\}.$$

Proof. The relations (12.3), (12.6) and (12.7) give us

$$(12.11) \quad |U_{\nu\lambda\mu} - {}^m U_{\nu\lambda\mu}| = |U_{\alpha\beta\gamma} {}^m X_{\nu\lambda\mu}^{\alpha\beta\gamma}| \leq \rho \leq M.$$

On the other hand we obtain from (12.5b) and (12.11)

$$\left| \frac{U_{\nu\lambda\mu}}{U_{\alpha\beta\gamma} {}^m X_{\nu\lambda\mu}^{\alpha\beta\gamma}} \right| = \frac{\rho\xi(\lambda^2 N)^m}{|U_{\alpha\beta\gamma} {}^m X_{\nu\lambda\mu}^{\alpha\beta\gamma}|} \geq \frac{\rho\xi(\lambda^2 N)^m}{\rho} = \xi(\lambda^2 N)^m > \xi_0(\lambda^2 N)^m.$$

This relation together with (12.11) proves (12.10).

In the next lemma we denote by $U^+ = |U_{\nu\lambda\mu}^{\nu}|$ the largest of the absolute values of the components $U_{\beta\gamma}^{\alpha}$.

Lemma (12.2). *We have*

$$(12.12) \quad U^+ \leq U_+ \chi n.$$

The proof follows at once from

$$U_{\nu\lambda\mu}^{\nu} = U_{\alpha\lambda\mu} h^{\alpha\nu}.$$

Note. If U denotes the absolute value $|U_{\nu\lambda\mu}^{\nu}|$ of an arbitrary non-zero component $U_{\nu\lambda\mu}^{\nu}$, then we have from (12.12)

$$(12.13) \quad U = U_+ \chi n \zeta, \quad 0 < \zeta \leq 1.$$

---

* If (12.9) is not satisfied then (12.10) reduces to $U_{\nu\lambda\mu} = {}^m U_{\nu\lambda\mu}$. Hence we assume (12.9).

To each combination of the indices ν, λ, μ belongs a certain ζ . We denote by ζ_0 the smallest of these values.

Theorem (12.3). *Assume*

$$U_{\alpha\beta\gamma} {}^m X_{\xi\lambda\mu}^{\alpha\beta\gamma} h^{\xi\nu} \neq 0.$$

Then

$$(12.14) \quad U_{\lambda\mu}^{\nu} - {}^m U_{\lambda\mu}^{\nu} \cong 0, \{ \chi n M, \zeta_0 (\lambda^2 N)^m \}.$$

Proof. First we obtain from (12.3)

$$(12.15) \quad U_{\lambda\mu}^{\nu} - {}^m U_{\lambda\mu}^{\nu} = U_{\alpha\beta\gamma} {}^m X_{\xi\lambda\mu}^{\alpha\beta\gamma} h^{\xi\nu}.$$

Moreover, we get from (12.7a)

$$(12.16) \quad | U_{\alpha\beta\gamma} {}^m X_{\xi\lambda\mu}^{\alpha\beta\gamma} h^{\xi\nu} | \leq \frac{U_+}{(\lambda^2 N)^m} \chi n = \rho \chi n < M \chi n.$$

Taking into account (12.5a), (12.13), (12.15) and (12.16), we have

$$(12.17) \quad \left| \frac{U_{\lambda\mu}^{\nu}}{U_{\lambda\mu}^{\nu} - {}^m U_{\lambda\mu}^{\nu}} \right| = \frac{\rho \chi n \zeta (\lambda^2 N)^m}{| U_{\lambda\mu}^{\nu} - {}^m U_{\lambda\mu}^{\nu} |} \geq \frac{\rho \chi n \zeta (\lambda^2 N)^m}{\rho \chi n} \\ = \zeta \chi (\lambda^2 N)^m \geq \zeta_0 (\lambda^2 N)^m.$$

The relations (12.15)–(12.17) lead at once to (12.14).

Later on we shall use these results.

13. The tensor $S_{\omega\mu}{}^{\nu}$. The method of the previous section may be applied to other tensors also and in particular to $S_{\omega\mu}{}^{\nu}$. To this end we apply it first to the tensor

$$(13.1a) \quad V_{\mu}{}^{\nu}{}_{\omega} = 4 U_{\alpha\beta[\mu} k_{\omega]}{}^{\alpha} h^{\beta\nu}.$$

Theorem (13.1). *Put*

$$(13.1b) \quad {}^m V_{\mu}{}^{\nu}{}_{\omega} = 4 {}^m U_{\alpha\beta[\mu} k_{\omega]}{}^{\alpha} h^{\beta\nu}.$$

Then

$$(13.2) \quad V_{\mu}{}^{\nu}{}_{\omega} - {}^m V_{\mu}{}^{\nu}{}_{\omega} = 4 U_{\alpha\beta\gamma} {}^m X_{\xi[\mu}^{\alpha\beta\gamma} k_{\omega]}{}^{\xi} h^{\xi\nu}.$$

The proof follows at once from (13.1a, b) and (12.3).

In the next lemma we denote by $V^+ = | V_{\mu}{}^{\nu}{}_{\omega} |$ the largest of the absolute values of the components $V_{\alpha}{}^{\beta}{}_{\gamma}$.

Lemma (13.1). *We have*

$$(13.3) \quad V^+ \leq U_+ \frac{2n^2 \chi}{\lambda N} = U_+ \frac{2\chi}{\lambda(n+1)}^*.$$

Proof. Taking into account (11.10) we obtain

$$\begin{aligned} V^+ &= 4 | U_{\alpha\beta[\mu} k_{\omega]}^\alpha h^{\beta\nu} | \leq 2 \{ | U_{\alpha\beta\mu} k_\omega^\alpha h^{\beta\nu} | + | U_{\alpha\beta\omega} k_\mu^\alpha h^{\beta\nu} | \}, \\ &\leq 2 \{ | U_{\alpha\beta\mu} | | k_\omega^\alpha | | h^{\beta\nu} | + | U_{\alpha\beta\omega} | | k_\mu^\alpha | | h^{\beta\nu} | \}, \\ &\leq 4U_+ \frac{\chi n^2}{2\lambda N} = 2U_+ \frac{\chi}{\lambda(n+1)}. \end{aligned}$$

Note. Formula (13.3) enables us to estimate V^+ without having computed the components $V_\alpha^\beta{}_\gamma$. If now $V = | V_\mu^\nu{}_\omega |$ is an arbitrary (not necessarily the largest) absolute value of the non-zero components $V_\alpha^\beta{}_\gamma$, we may write by virtue of (13.3)

$$(13.4) \quad V = U_+ \frac{2\chi}{\lambda(n+1)} \eta, \quad 0 < \eta \leq 1,$$

To every combination of the indices μ, ν, ω belongs a certain η . Let us denote by η_0 the smallest of these values.

Theorem (13.2). *Assume*

$$U_{\alpha\beta\gamma} {}^m X_{\xi\xi[\mu}^{\alpha\beta\gamma} k_{\omega]}^\xi h^{\xi\nu} \neq 0.$$

Then

$$(13.5) \quad V_\mu^\nu{}_\omega \cong {}^m V_\mu^\nu{}_\omega, \{M, \eta_0(\lambda^2 N)^m\}.$$

Proof. First we have from (13.2)

$$(13.6a) \quad | V_\mu^\nu{}_\omega - {}^m V_\mu^\nu{}_\omega | = 4 | U_{\alpha\beta\gamma} {}^m X_{\xi\xi[\mu}^{\alpha\beta\gamma} k_{\omega]}^\xi h^{\xi\nu} |.$$

As far as the right hand member is concerned, we may write by virtue of (12.5a), (12.6), (11.9) and (11.2)

$$\begin{aligned} (13.6b) \quad 4 | U_{\alpha\beta\gamma} {}^m X_{\xi\xi[\mu}^{\alpha\beta\gamma} k_{\omega]}^\xi h^{\xi\nu} | &\leq 2 \{ | U_{\alpha\beta\gamma} {}^m X_{\xi\xi\mu}^{\alpha\beta\gamma} k_\omega^\xi h^{\xi\nu} | + | U_{\alpha\beta\gamma} {}^m X_{\xi\xi\omega}^{\alpha\beta\gamma} k_\mu^\xi h^{\xi\nu} | \}, \\ &\leq 4 \frac{U_+ N}{(\lambda^2 N)^m} \frac{n^2 \chi}{2\lambda N} = 2 \frac{U_+}{(\lambda^2 N)^m} \frac{\chi}{\lambda(n+1)} = \rho \frac{2\chi}{\lambda(n+1)} < \rho \leq M. \end{aligned}$$

Hence we have

---

* Cf. the equation (11.2a).

$$(13.7) \quad |V_{\mu}^{\nu\omega} - {}^mV_{\mu}^{\nu\omega}| < M,$$

and this relation proves the first part of (13.5).

In order to prove the second part of (13.5), we take into account (12.5a), (13.2), (13.4) and (13.6b). Then we see that

$$(13.8) \quad \frac{|V_{\mu}^{\nu\omega}|}{|V_{\mu}^{\nu\omega} - {}^mV_{\mu}^{\nu\omega}|} = \frac{\rho\eta(\lambda^2 N)^m \frac{2\chi}{\lambda(n+1)}}{4 |U_{\alpha\beta\gamma} {}^mX_{\xi\xi[\mu}^{\alpha\beta\gamma} k_{\omega]}^{\xi} h^{\xi\nu}|} \\ \geq \frac{\rho\eta(\lambda^2 N)^m \frac{2\chi}{\lambda(n+1)}}{\frac{2\chi}{\lambda(n+1)}} = \eta(\lambda^2 N)^m \geq \eta_0(\lambda^2 N)^m,$$

and this relation proves the second part of (13.5).

Theorem (13.3). Put

$$(13.9) \quad {}^mS_{\omega\mu}^{\nu} = \frac{1}{2}K_{\omega\mu}^{\nu} - 2 {}^mU_{\alpha\beta[\mu} k_{\omega]}^{\alpha} h^{\beta\nu}.$$

Then

$$(13.10) \quad S_{\omega\mu}^{\nu} - \frac{1}{2}K_{\omega\mu}^{\nu} \cong {}^mS_{\omega\mu}^{\nu} - \frac{1}{2}K_{\omega\mu}^{\nu}, \{\frac{1}{2}M, \eta_0(\lambda^2 N)^m\}^*.$$

Proof. Using (2.14) as well as (13.1) and (13.9), we obtain

$$S_{\omega\mu}^{\nu} - \frac{1}{2}K_{\omega\mu}^{\nu} - ({}^mS_{\omega\mu}^{\nu} - \frac{1}{2}K_{\omega\mu}^{\nu}) = \frac{1}{2}({}^mV_{\mu}^{\nu\omega} - V_{\mu}^{\nu\omega}).$$

This equation together with (13.7) proves the first part of (13.9). The second part of (13.10) is a consequence of (2.14), (13.1), (13.9) and (13.8).

14. The connection ${}^m\Gamma_{\lambda\mu}^{\nu}$.

Lemma (14.1). Put

$$(14.1) \quad \begin{aligned} \text{a) } P_{\lambda\mu}^{\nu} &= \frac{1}{2}V_{\lambda}^{\nu\mu} + U^{\nu}{}_{\lambda\mu}, \\ \text{b) } {}^mP_{\lambda\mu}^{\nu} &= \frac{1}{2}{}^mV_{\lambda}^{\nu\mu} + {}^mU^{\nu}{}_{\lambda\mu}. \end{aligned}$$

Then

$$(14.2) \quad |P_{\lambda\mu}^{\nu}| = \alpha \left(\frac{\eta n}{\lambda N} + \zeta \right) \chi n \rho (\lambda^2 N)^m, \quad 0 < \alpha \leq 1.$$

Moreover

---

* If we disregard the quotient $\eta_0(\lambda^2 N)^m$, then the first symbol of (13.10) may be written also $S_{\omega\mu}^{\nu} \cong {}^mS_{\omega\mu}^{\nu}$

$$(14.3) \quad P_{\lambda\mu}^{\nu} - {}^m P_{\lambda\mu}^{\nu} \cong 0, \quad \left\{ \chi n \beta M, \frac{\alpha \zeta}{\beta} (\lambda^2 N)^m \right\} \\ \left(\beta = \frac{n}{\lambda N} + 1 \right).$$

Proof. First we obtain from (12.5a), (12.13), (13.4) and (14.1a)

$$\begin{aligned} |P_{\lambda\mu}^{\nu}| &\leq \frac{1}{2} |V_{\mu}^{\nu\lambda}| + |U_{\lambda\mu}^{\nu}|, \\ &= U_+ \left(\frac{\chi n}{\lambda(n+1)} + \chi n \zeta \right) = \rho (\lambda^2 N)^m \chi n \left(\frac{\eta}{\lambda n(n+1)} + \zeta \right), \\ &= \rho (\lambda^2 N)^m \chi n \left(\frac{\eta n}{\lambda N} + \zeta \right). \end{aligned}$$

The last term is different from zero. Hence there is an α satisfying the conditions $0 < \alpha \leq 1$ for which (14.2) holds. Moreover, we have by virtue of (12.5a), (12.6), (12.15), (12.16), (13.6) and (14.1)

$$(14.4) \quad \begin{aligned} |P_{\lambda\mu}^{\nu} - {}^m P_{\lambda\mu}^{\nu}| &\leq \frac{1}{2} |V_{\lambda}^{\nu\mu} - {}^m V_{\lambda}^{\nu\mu}| + |U_{\lambda\mu}^{\nu} - {}^m U_{\lambda\mu}^{\nu}| \\ &\leq \rho \left[\frac{\chi}{\lambda(n+1)} + \chi n \right] = \rho \chi n \left[\frac{n}{\lambda N} + 1 \right] \leq M \chi n \left[\frac{n}{\lambda N} + 1 \right]. \end{aligned}$$

On the other hand, we obtain from (12.5a), (14.2) and (14.4)

$$(14.5) \quad \begin{aligned} \frac{|P_{\lambda\mu}^{\nu}|}{|P_{\lambda\mu}^{\nu} - {}^m P_{\lambda\mu}^{\nu}|} &= \frac{\alpha \chi n \rho \left(\frac{\eta n}{\lambda N} + \zeta \right)}{|P_{\lambda\mu}^{\nu} - {}^m P_{\lambda\mu}^{\nu}|} (\lambda^2 N)^m \geq \frac{\alpha \chi n \rho \left(\frac{\eta n}{\lambda N} + 1 \right) \zeta}{\chi n \rho \left(\frac{n}{\lambda N} + 1 \right)} (\lambda^2 N)^m \\ &= \alpha (\lambda^2 N)^m \frac{\left(\frac{\eta n}{\lambda N} + 1 \right) \zeta}{\beta} \geq \frac{\alpha}{\beta} \zeta (\lambda^2 N)^m. \end{aligned}$$

The relations (14.4) and (14.5) prove (14.3).

In the next theorem we introduce a new connection.

Theorem (14.1). *Put*

$$(14.6) \quad {}^m \Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + {}^m S_{\lambda\mu}^{\nu} + {}^m U_{\lambda\mu}^{\nu}.$$

Then

$$(14.7) \quad {}^m \Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \frac{1}{2} K_{\lambda\mu}^{\nu} + H_{\alpha\beta\gamma} {}^m Y_{\xi\eta\xi}^{\alpha\beta\gamma} (\delta_{\lambda\mu}^{\eta\xi} h^{\xi\mu} + 2\delta_{[\lambda}^{\xi} k_{\mu]}^{\xi} h^{\eta\nu})$$

where $H_{\alpha\beta\gamma}$ is defined by (12.1) and ${}^m Y_{\xi\eta\xi}^{\alpha\beta\gamma}$ is defined by (11.11b).

Proof. First we have from (13.9) and (12.2)

$$(14.8) \quad \begin{aligned} {}^m S_{\lambda\mu}{}^\nu &= \tfrac{1}{2} K_{\lambda\mu}{}^\nu + 2 H_{\xi\eta\xi} {}^m Y_{\alpha\beta[\lambda}^{\xi\eta\xi} k_{\mu]}{}^\alpha h^{\beta\nu}, \\ &= \tfrac{1}{2} K_{\lambda\mu}{}^\nu + 2 H_{\alpha\beta\gamma} {}^m Y_{\xi\eta\xi}^{\alpha\beta\gamma} \delta_{[\lambda}{}^\xi k_{\mu]}{}^\eta h^{\gamma\nu}. \end{aligned}$$

Substituting for ${}^m S_{\lambda\mu}{}^\nu$ from (14.8) into (14.6) and again remembering (12.2), we obtain (14.7).

Now we are finally ready to prove the main theorem of this part, in which the solution $\Gamma_{\lambda\mu}^\nu$ of (2.1) will be compared to ${}^m \Gamma_{\lambda\mu}^\nu$.

Theorem (14.2). *Put*

$$(14.9) \quad \begin{aligned} T_{\lambda\mu}{}^\nu &= \Gamma_{\lambda\mu}^\nu - \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} - \tfrac{1}{2} K_{\lambda\mu}{}^\nu, \\ {}^m T_{\lambda\mu}{}^\nu &= {}^m \Gamma_{\lambda\mu}^\nu - \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} - \tfrac{1}{2} K_{\lambda\mu}{}^\nu. \end{aligned}$$

Then

$$(14.10) \quad T_{\lambda\mu}{}^\nu = P_{\lambda\mu}{}^\nu, \quad {}^m T_{\lambda\mu}{}^\nu = {}^m P_{\lambda\mu}{}^\nu,$$

so that

$$(14.11) \quad T_{\lambda\mu}{}^\nu - {}^m T_{\lambda\mu}{}^\nu \cong 0, \quad \left\{ \chi n \beta M, \frac{\alpha \xi}{\beta} (\lambda^2 N)^m \right\}.$$

If we disregard the “quotient” $(\alpha \xi)/\beta (\lambda^2 N)^m$ in (14.11) we may write instead

$$(14.12a) \quad \Gamma_{\lambda\mu}^\nu - {}^m \Gamma_{\lambda\mu}^\nu \cong 0, \quad \{ \chi n \beta M \},$$

where ${}^m \Gamma_{\lambda\mu}^\nu$ is given by (14.7).

Proof. Starting with (2.10) and (2.14) we obtain with regard to (13.1a)

$$(14.13a) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + S_{\lambda\mu}{}^\nu + U^\nu{}_{\lambda\mu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \tfrac{1}{2} K_{\lambda\mu}{}^\nu + \tfrac{1}{2} V_{\lambda}{}^\nu{}_\mu + U^\nu{}_{\lambda\mu},$$

and in a similar way

$$(14.13b) \quad {}^m \Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \tfrac{1}{2} K_{\lambda\mu}{}^\nu + \tfrac{1}{2} {}^m V_{\lambda}{}^\nu{}_\mu + {}^m U^\nu{}_{\lambda\mu}.$$

These equations together with (14.1) and (14.9) lead at once to (14.10). Hence (14.11) is an immediate consequence of Lemma (14.1). The statement about (14.12a) follows at once from (14.11) and (14.9).

Note I. It is not difficult to estimate the “quotient” in order to complete (14.12a). We see from (14.13a) that for $|\Gamma_{\lambda\mu}^\nu| \neq 0$ there is a number γ ,

$$0 < \gamma \leq 1,$$

such that

$$|\Gamma_{\lambda\mu}^\nu| = \gamma \left\{ |P_{\lambda\mu}^\nu| + \left| \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + \frac{1}{2} K_{\lambda\mu}^\nu \right| \right\},$$

and this formula together with (14.5) gives us at once

$$(14.12b) \quad \Gamma_{\lambda\mu}^\nu \cong {}^m\Gamma_{\lambda\mu}^\nu, \{ \chi n \beta M, Q \},$$

where

$$(14.12c) \quad Q \geq \frac{\alpha \gamma \zeta}{\beta} (\lambda^2 N)^m.$$

Note II. In the case of the unified theory of relativity we have $n = 4$. Moreover, one assumes that $h_{\lambda\mu}$ differs very little from the Galilean tensor. Therefore it is reasonable to expect in the most unfavorable case that

$$(14.13) \quad \chi = 1.5.$$

On the other hand we have $N = 80$ for $n = 4$ so that

$$(14.13) \quad \beta = 1 + \frac{n}{\lambda N} = 1 + \frac{1}{20\lambda}.$$

Hence we have

$$(14.13) \quad D = \chi n \beta M = 6 \left(1 + \frac{1}{20\lambda} \right) M,$$

and according to (14.12)

$$(14.14) \quad \begin{aligned} |\Gamma_{\lambda\mu}^\nu - {}^m\Gamma_{\lambda\mu}^\nu| &\leq 6 \left(1 + \frac{1}{20\lambda} \right) M = D, \\ D &= 6 \left(1 + \frac{1}{20\lambda} \right) \left(\frac{3}{4 \times 80 (\lambda^2 80 - 1) [(\lambda^2 80)^m - 1]} \right) \end{aligned}$$

The right hand side is a function of λ and m , where $m = 1, 2, 3, \dots$, and λ satisfies the relations (11.2b) or

$$1 \leq \lambda \leq 0.6.$$

In the following table we exhibit the values of D for some m and λ :*

		$m = 1$	$m = 2$	$m = 3$
(14.14)	$\lambda = 1$	9.46×10^{-6}	1.17×10^{-7}	1.46×10^{-9}
	$\lambda = \sqrt{10}$	8.85×10^{-8}	1.12×10^{-10}	1.40×10^{-13}
	$\lambda = 10$	8.84×10^{-10}	1.10×10^{-12}	1.38×10^{-17}
	$\lambda = 10^2$	8.79×10^{-14}	1.10×10^{-19}	1.37×10^{-25}

As far as the quotient (14.12c) is concerned, we can but guess the value of $\alpha\gamma\zeta/\beta < 1$. There are possible cases where both of the numbers α and γ are equal to one, and the same may be true for ζ . Taking a less favorable possibility into account we put tentatively

$$\alpha\gamma\zeta/\beta = 10^{-3}, \quad Q = 10^{-3}(\lambda^2 N)^m = 10^{m-3} 8^m \lambda^{2m}.$$

Then we obtain the following table of values for Q :

		$m = 1$	$m = 2$	$m = 3$
(14.15)	$\lambda = 1$	8×10^{-2}	6.4	5.12×10^2
	$\lambda = \sqrt{10}$	8×10^{-1}	6.4×10^2	5.12×10^6
	$\lambda = 10$	8	6.4×10^4	5.12×10^8
	$\lambda = 10^2$	8×10^2	6.4×10^8	5.12×10^{14}

If we compare the two tables (14.14) and (14.15) we see that even the cases with $m = 1$ are not unsatisfactory. On the other hand if $m = 1$ we obtain for (14.7)

$$(14.16a) \quad {}^1\Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \frac{1}{2}K_{\lambda\mu}{}^\nu + H_{\zeta\eta\xi}(\delta_{\lambda\mu}^{\eta\xi} h^{\zeta\nu} + 2\delta_{[\lambda}^{\xi} k_{\mu]}^{\zeta} h^{\eta\nu}),$$

or

$$(14.16b) \quad {}^1\Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} - k_{[\mu}{}^\alpha K_{\lambda]}{}^\nu{}_\alpha \\ + \frac{1}{2}K_{\lambda\mu}{}^\nu - k_{[\lambda}{}^\alpha k_{\mu]}{}^\beta K_{\beta\alpha}{}^\nu - k^{\nu\beta} k_{[\mu}{}^\alpha K_{\lambda]}{}_\alpha{}_\beta,$$

where $K_{\omega\mu\nu}$ is given by (2.6). For this connection and $\lambda = 10$ we have under our assumptions

$$(14.17) \quad \Gamma_{\lambda\mu}^\nu \cong {}^1\Gamma_{\lambda\mu}^\nu, \{8.84 \times 10^{-10}, 8\},$$

which seems to be a satisfactory approximation.

---

* It is worthwhile to remember that 10^{-7} is the order of magnitude of the ratio of ordinary velocities to the velocity of light.

15. Another method of approximation. The tensor ${}^m Y_{\alpha\beta\gamma}^{\alpha\beta\gamma}$ as defined by (11.11b) for $m > 1$ is a set of homogeneous polynomials of degrees 0, 2, 4, $\dots$ $2(m-1)$ in the k_α^β . We shall use this fact to find a very fast way of obtaining an approximation for the connection $\Gamma_{\lambda\mu}^\nu$. However, this approximation is possible only if we make an appropriate restrictive assumption concerning the determinant Δ of the system (8.6) for the unknowns $U_{\nu\lambda\mu}$. This determinant is of the form

$$\Delta = 1 + F$$

where F is a sum of polynomials in $k_{\lambda\mu}$. Throughout this section *we shall assume that $1/\Delta$ may be expanded in a power series*, which is a restriction concerning the components $k_{\lambda\mu}$ ("weak" field).

First we introduce the following

Definition (15.1). Let A be a homogeneous polynomial in the variables $k_{\lambda\mu}$ and $\partial_\omega k_{\lambda\mu}$ of order p . Then we write (if necessary) ${}^{(p)}A$ for A . Moreover, if

$$(15.1a) \quad P - Q = \sum_{s=q}^p {}^{(s)}A, \quad q < p,$$

then we write for (15.1a)

$$(15.1b) \quad P \equiv Q \pmod{\exp q'}$$

or

$$(15.1c) \quad P - Q \equiv 0 \pmod{\exp q'}, \quad \text{where } q' \leq q.$$

Lemma (15.1). We have

$$(15.2) \quad \begin{aligned} \text{a)} \quad K_{\omega\mu\nu} &= {}^{(1)}K_{\omega\mu\nu}, \\ \text{b)} \quad H_{\omega\mu\nu} &= {}^{(2)}H_{\omega\mu\nu}. \end{aligned}$$

The proof follows at once from (2.6) and (12.1).

Lemma (15.2). We have

$$(15.3) \quad {}^m X_{\nu\lambda\mu}^{\alpha\beta\gamma} \equiv 0 \pmod{\exp 2m},$$

and

$$(15.4) \quad U_{\lambda\mu}^\nu \equiv 0 \pmod{\exp 2}.$$

Proof. The relation (15.3) follows at once from (11.8). The relation (15.4) is an immediate consequence of (15.2b) and of the system (8.6) as well as of the assumption about its determinant.

Theorem (15.1). The tensors $U_{\lambda\mu}^\nu$ and ${}^m U_{\lambda\mu}^\nu$ are related by

$$(15.5) \quad U_{\lambda\mu}^\nu \equiv {}^m U_{\lambda\mu}^\nu \pmod{\exp 2m + 2},$$

while the tensors $S_{\omega\mu}{}^\nu$ and ${}^m S_{\omega\mu}{}^\nu$ are related by

$$(15.6a) \quad S_{\omega\mu}{}^\nu \equiv {}^m S_{\omega\mu}{}^\nu \pmod{\exp 2m + 3}.$$

Proof. The formula (15.5) follows at once from (15.3), (15.4) and (12.3). The formula (15.6a) may be derived on basis of (13.9), (2.14) and (15.5).

These results enable us to prove the main theorem of this section.

Theorem (15.2). *The connection $\Gamma_{\lambda\mu}^\nu$ (a solution of (2.1)) and the connection ${}^m \Gamma_{\lambda\mu}^\nu$ given by (14.6) (or (14.7)) are related by*

$$(15.6b) \quad \Gamma_{\lambda\mu}^\nu \equiv {}^m \Gamma_{\lambda\mu}^\nu \pmod{\exp 2m + 2}.$$

The proof follows at once from (14.6), (2.10), (15.5) and (15.6a).

Corollary (15.1). *We have*

$$(15.7) \quad \Gamma_{\lambda\mu}^\nu \equiv \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} - k_{(\mu}{}^\alpha K_{\lambda)}{}^\nu{}_\alpha + \frac{1}{2} K_{\lambda\mu}{}^\nu - k_{[\lambda}{}^\alpha k_{\mu]}{}^\beta K^\nu{}_{\beta\alpha} - k^{\nu\beta} k_{[\mu}{}^\alpha K_{\lambda]\alpha\beta} \pmod{\exp 4}.$$

Proof. Substituting $m = 1$ into (15.6b) and remembering (14.16b) one obtains (15.7).

Corollary (15.2). *We have*

$$(15.8) \quad \Gamma_{\lambda\mu}^\nu \equiv \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + \frac{1}{2} K_{\lambda\mu}{}^\nu \pmod{\exp 2}.$$

This relation follows at once from (15.7).

Note. Assume now that the components $k_{\lambda\mu}$ and $\partial_\omega k_{\lambda\mu}$ are so small that it is permissible to neglect every homogeneous polynomial of degree $\geq 2m + 2$ in the variables $k_{\lambda\mu}$ and $\partial_\omega k_{\lambda\mu}$, but no such polynomial of degree $< 2m + 2$. Then we may use by virtue of (15.6) the connection ${}^m \Gamma_{\lambda\mu}^\nu$ instead of the connection $\Gamma_{\lambda\mu}^\nu$ which we were looking for. If in particular $m = 1$ we may use instead of $\Gamma_{\lambda\mu}^\nu$ the connection given by the right hand side of (15.7). Moreover, if we agree to neglect all polynomials of degree ≥ 1 in $k_{\lambda\mu}$ and $\partial_\omega k_{\lambda\mu}$, then we may use the connection given by the right hand side of (15.8). This formula was published in a short paper of mine*. If one assumes a special structure for $h_{\lambda\mu}$, one obtains from (15.8) the formulae of Udeschini and Schrödinger†.

---

* V. HLAVATÝ, *The elementary basic principles of the unified theory of relativity*, Proc. Nat. Acad. Sci. **38** (1952), 243–247.

† P. UDESCHINI, *Le equazioni di prima approssimazione nella nuova teoria relativistica unitaria di Einstein*, Rend. Lincei **IX** (1950), 256–261. E. SCHRÖDINGER, *Studies in the non-symmetric generalization of the theory of gravitation I*, Comm. Dublin Institute for Adv. Studies, Series A, No. 6, 1951. These authors assume

$$h_{\lambda\mu} = {}^0 h_{\lambda\mu} + {}^1 h_{\lambda\mu},$$

Note. It is interesting to consider some of the theorems of Part II with regard to (15.8). Thus for instance the condition (7.5a) in Theorem (7.1) reduces to

$$(15.9a) \quad 2\delta_{[\mu}^{\nu} K_{\omega]\alpha}^{\alpha} - (n-1)K_{\omega\mu}^{\nu} \equiv 0 \pmod{\exp 2}$$

and (7.5b) reduces to

$$(15.9b) \quad (n-1)S_{\lambda} - K_{\lambda\beta}^{\beta} \equiv 0 \pmod{\exp 2}.$$

Theorem (7.2) cannot be applied, for we have for n even

$$k \equiv 0 \pmod{\exp n},$$

and according to (7.12a)

$$L_{\lambda\mu} \equiv k_{\lambda\mu} \pmod{\exp 3}.$$

Hence for $n > 2$

$$\text{Det } |L_{\lambda\mu}| \equiv 0 \pmod{\exp 3},$$

and for $n = 2$

$$\text{Det } |L_{\lambda\mu}| \equiv 0 \pmod{\exp 2},$$

so that (7.12b) is not satisfied “mod exp 2”.

The condition (8.6) of Theorem (8.1) reduces to

$$(15.10) \quad U_{\nu\lambda\mu} \equiv 0 \pmod{\exp 2},$$

and this condition is always satisfied if (15.8) holds. Moreover, we have instead of (8.7) the relation

$$(15.11) \quad g/h \equiv \text{const.} \pmod{\exp 2}.$$

This condition is always satisfied for $n = 4$ for in this case

$$(15.12) \quad g \equiv h \pmod{\exp 2}.$$

Corollary (8.1) is equivalent to the statement expressed by (15.8).

where ${}^0h_{\lambda\mu}$ are constant and ${}^1h_{\lambda\mu}$ is very small with regard to ${}^0h_{\lambda\mu}$. In two other papers, “*Le equazione di seconda approssimazione nella nuova teoria relativistica unitaria di Einstein I, II,*” Rend. Lincei X (1950), 21–24, 121–123, UDESCHINI assumes the tensor $g_{\lambda\mu}$ to be of the form

$$g_{\lambda\mu} = {}^0h_{\lambda\mu} + {}^1g_{\lambda\mu} + {}^2g_{\lambda\mu},$$

where the ${}^2g_{\lambda\mu}$ are small with respect to ${}^1g_{\lambda\mu}$, which are small with respect to ${}^0h_{\lambda\mu}$. A more detailed discussion of some selected papers dealing with the unified theory of relativity will be presented in the next paper of this series.

The requirement of Theorems (9.1) and (9.2a) are satisfied “mod exp 2”, as it is seen also directly from (15.8) (*cf.* also Corollary (9.1))* . The requirement (10.8a) of Theorem (10.3) is not satisfied, for

$$k_{(\mu}{}^{\beta} K_{\lambda)\nu\beta} \equiv 0 \pmod{\exp 2},$$

so that this expression is not $\equiv 0 \pmod{\exp 2}$ as (10.8) requires.

16. Appendix: Problem B'. In this section we return again to the tensor $g_{\lambda\mu}$ used in the first part of this paper and denote by ${}^*\Gamma_{\lambda\mu}^{\nu}$ a connection defined by

$$(16.1a) \quad \partial_{\omega} g_{\lambda\mu} = {}^*\Gamma_{\lambda\omega}^{\alpha} g_{\alpha\mu} + {}^*\Gamma_{\mu\omega}^{\alpha} g_{\lambda\alpha}^{\dagger}$$

Theorem (16.1). *The connection ${}^*\Gamma_{\lambda\mu}^{\nu}$ satisfies the relation*

$$(16.2a) \quad {}^*\Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + {}^*S_{\lambda\mu}{}^{\nu} + {}^*U^{\nu}{}_{\lambda\mu},$$

where

$$(16.2b) \quad {}^*U^{\nu}{}_{\lambda\mu} = -2 {}^*S^{\nu}{}_{(\lambda\mu)}$$

and the tensor ${}^*S_{\lambda\mu}{}^{\nu}$ is defined by

$$(16.2c) \quad 2 {}^*S_{\omega\mu}{}^{\alpha} k_{\nu\alpha} = -K_{\omega\mu\nu} - 4U^{\alpha}{}_{\nu[\omega} k_{\mu]\alpha}.$$

Proof. Denoting by ${}^*D_{\omega}$ the symbol of the covariant derivative with respect to ${}^*\Gamma_{\lambda\mu}^{\nu}$ one may write (16.1a) in the form

$$(16.1b) \quad {}^*D_{\omega} g_{\lambda\mu} = 0,$$

so that

$$(16.4) \quad \text{a) } {}^*D_{\omega} h_{\lambda\mu} = 0, \quad \text{b) } {}^*D_{\omega} k_{\lambda\mu} = 0.$$

Based on these equations the proof follows the pattern of the proofs of Theorems (2.2) and (2.3).

---

* If we use (15.7) then the condition (9.3a) of Theorem (9.2a) reduces to

$$k_{(\mu}{}^{\beta} K_{\lambda)\nu\beta} \equiv 0 \pmod{\exp 4},$$

as it was to be expected from the structure of ${}^*\Gamma_{\lambda\mu}^{\nu}$.

† Compare the order of the indices ω and μ in the last terms in (16.1) and (2.1). The unusual order of these indices in (2.1) was probably inspired by an earlier attempt of EINSTEIN to base the unified theory of relativity on a tensor with Hermitian symmetry

$$g_{\lambda\mu} = h_{\lambda\mu} + i k_{\lambda\mu} \quad (i = \sqrt{-1}).$$

(*Cf.* for instance A. EINSTEIN, A generalized theory of gravitation, *Rev. Modern Physics* **20**, 35–39, 1948). However, equation (2.1) follows also from a variational principle.

Next arises the question of under what circumstances the connection $\Gamma_{\lambda\mu}^\nu$ and $*\Gamma_{\lambda\mu}^\nu$ are equal.

Theorem (16.2). *A necessary and sufficient condition for*

$$(16.5) \quad \Gamma_{\lambda\mu}^\nu = *\Gamma_{\lambda\mu}^\nu$$

is

$$(16.6) \quad \nabla_\omega g_{\lambda\mu} = 0.$$

If this condition is satisfied then

$$(16.7) \quad \Gamma_{\lambda\mu}^\nu = *\Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\}.$$

Proof. Assume that (16.6) is satisfied. Then (16.2c) reduces to

$$(16.8a) \quad *S_{\omega\mu}{}^\alpha k_{\nu\alpha} - 2*S_{\alpha(\nu\omega)} k_{\mu\alpha} + 2*S_{\alpha(\nu\mu)} k_{\omega\alpha} = 0,$$

while (2.7a) reduces to

$$(16.8b) \quad 2S_{\omega\mu\nu} - 4S_{\nu\beta\gamma} k_{[\omega}{}^{[\beta} k_{\mu]}{}^{\gamma]} - 4k_{[\omega}{}^\beta S_{\mu]\beta\gamma} k_{\nu}{}^\gamma = 0.$$

The system (16.8b), homogeneous in the variables $S_{\omega\mu\nu}$, admits only one solution by virtue of Agreement (4.1). Hence we must have

$$(16.9) \quad S_{\omega\mu}{}^\nu = 0,$$

and in a similar manner from (16.8a)

$$(16.10) \quad *S_{\omega\mu}{}^\nu = 0.$$

The equations (16.2a), (2.4), (16.9) and (16.10) yield (16.7). Assume now that (16.5) is satisfied. Then we obtain from (16.1b) and (2.3a) the condition

$$S_{\omega\mu}{}^\alpha g_{\lambda\alpha} = 0,$$

which gives us by virtue of $g \neq 0$ equation (16.9), and this equation together with (16.5) leads to (16.10). Hence the equation (2.3c) reduces to

$$(16.11) \quad \nabla_\omega k_{\lambda\mu} = 0,$$

equivalent to (16.6). The same statement holds for equation (16.4b). From (16.9) and (16.10) we obtain easily (16.7).

Note. The theory of the connection $*\Gamma_{\lambda\mu}^\nu$ could be dealt with along the same line as the theory of $\Gamma_{\lambda\mu}^\nu$. However, there is a fact which ought to be stressed: in the case of the system (2.1) no condition is imposed by this system on $g_{\lambda\mu}$

---

* We confine ourselves to the case when (16.1a) admits only one solution $*\Gamma_{\lambda\mu}^\nu$. The uniqueness of $\Gamma_{\lambda\mu}^\nu$ is secured by the Agreement (4.1).

(cf. Theorem (4.3)), while the system (16.1a) acts in quite another way, as the following theorem shows.

Theorem (16.3). *The system (16.1a) imposes on $g_{\lambda\mu}$ the condition*

$$(16.12) \quad g/h = \text{const.}$$

Proof. We have first from (16.2a)

$$(16.13) \quad \begin{aligned} {}^*\Gamma_{\alpha\mu}^{\alpha} &= \left\{ \begin{matrix} \alpha \\ \alpha\mu \end{matrix} \right\} + {}^*S_{\alpha\mu}^{\alpha} - {}^*S_{\alpha\mu}^{\alpha} - {}^*S_{\mu\alpha}^{\alpha}, \\ &= \left\{ \begin{matrix} \alpha \\ \alpha\mu \end{matrix} \right\} + {}^*S_{\alpha\mu}^{\alpha} - {}^*S_{\mu\alpha}^{\alpha} = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} = \frac{1}{2}\partial_{\mu} \ln h. \end{aligned}$$

On the other hand we obtain from (16.1b)

$${}^*D_{\omega} g = 0,$$

or

$$\partial_{\omega} g - 2 {}^*\Gamma_{\alpha\omega}^{\alpha} g = g(\partial_{\omega} \ln g - 2 {}^*\Gamma_{\alpha\omega}^{\alpha}) = 0,$$

and this equation reduces by virtue of (16.13) to the requirement

$$g\partial_{\omega} \ln \frac{g}{h} = 0,$$

which is equivalent to (16.12).

In the next paper we shall use the results of this paper in order to deal with the problem of imposing some conditions (in the form of differential equations) on the $\Gamma_{\lambda\mu}^{\nu}$ defined by (2.1) (Principle C).