*On the Existence of Subsonic Flows of a Compressible Fluid**

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The stationary, irrotational flow of a compressible fluid is characterized by a quasi-linear second order partial differential equation with coefficients which are functions of the first derivatives. The equation is sometimes of elliptic type and sometimes of hyperbolic type, depending on the magnitude of the first derivatives. These two types occur where the flow is subsonic, or supersonic, respectively. The question of the existence of such flows satisfying prescribed conditions is uncertain, and even when the flow exists it is not known beforehand whether the flow is always subsonic or partly subsonic and partly supersonic. We shall show here how to establish existence theorems for purely subsonic flows.

In the case of the flow about an obstacle, we shall establish the existence of subsonic flows for a range of prescribed Mach numbers at infinity up to a certain critical value, when the corresponding flow would become sonic somewhere. In previous literature on the subject, Frankl & Keldysh have established the existence of flows for sufficiently small Mach numbers at infinity, while Bers has proved existence theorems for boundary value problems concerning minimal surfaces analogous to problems for flows.¹

The flow of a compressible fluid can be examined either from the partial differential equation itself, or by referring to a variational formulation. It is well-known that the flow is one which makes a certain integral stationary, i.e., the partial differential equation for the flow is the Euler-Lagrange equation of a certain variational integral.² But this fact offers two main difficulties before it

* A summary appears in the May, 1952, issue of *Proc. Nat. Acad. Sci.*

¹ F. FRANKL & M. KELDYSH, *Bull. Acad. Sci. URSS* (1934), 561-601; L. BERS, *Trans. Amer. Math. Soc.* **70** (1951), 465-491.

² A variational formulation is given in a paper by H. BATEMAN, *Proc. Roy. Soc., London*, **A125** (1929), 598-618, and a numerical procedure used in C. WANG, *Jour. of Aeronaut. Sci.* **15** (1948), 675-685. In the latter paper, the variational problem is corrected by an additional

can be made useful, practically or theoretically, in the study of compressible flows. On the one hand, for the flow about an obstacle, the domain of integration is the region exterior to the obstacle, and the integrals involved diverge. We shall show how to subtract from the integrand suitable terms which do not affect the Euler-Lagrange equation and which at the same time lead to convergent integrals. It is in fact in these subtractive terms that the cases of non-zero or zero circulation are distinguished, and we shall obtain existence theorems thereby with or without circulation. The second principal question with the variational approach is due to the fact that the Euler-Lagrange equation is not necessarily always of elliptic type. For this reason, a solution of the Euler-Lagrange equation does not necessarily minimize or maximize the variational integral, even when the flow itself is subsonic (i.e., the solution is of elliptic type). We shall nevertheless show how to modify the variational integral so that every subsonic flow does indeed minimize the new variational integral.

With this final new variational integral, we apply the direct methods of the calculus of variations to establish the existence of subsonic flows satisfying a range of prescribed conditions. It becomes then necessary to discuss the behavior of the solution and of its first derivatives on the boundary of the region and at infinity. Our principal technical effort will be directed to establishing the desired behavior, and this will finally establish the validity of the variational approach. We shall also obtain bounds on integrals of the second derivatives of a solution. These bounds and boundary behavior are obtained by using methods connected with the works of Morrey and of the author in the subject of the calculus of variations.³ Such bounds and boundary behavior are of considerable importance in the general theory of non-linear partial differential equations.⁴ The procedures and theorems obtained here are applicable to general partial differential equations as well as to the specific equation determining the flow of a compressible fluid.

The variational approach is not only useful theoretically, but is also a possible practical device of considerable importance for obtaining numerical procedures and for estimating changes in the flow when prescribed conditions change. All of the advantages of variational procedures, such as inequalities, comparison of two situations, symmetrization, etc., become applicable to the flow of a compressible fluid.⁵

The flow of a compressible fluid may also be examined from the standpoint

term due to the behavior at infinity, related to the additional terms used in the present paper.

³ C. B. MORREY, Multiple Integral Problems in the Calculus of Variations and Related Topics, *Univ. of Calif. Publ. in Math.*, **1** (1943), 1-130; M. SHIFFMAN, *Annals of Math.*, **48** (1947), 274-284.

⁴ See S. BERNSTEIN, *Math. Ann.* **69** (1910), 82-136; J. SCHAUDER & J. LERAY, *Annales de l'École Norm. Sup.*, **61** (1934), 45-78.

⁵ See COURANT-HILBERT, *Methoden der Mathematischen Physik*, Vol. I, and POLYA-SZEGÖ, Isoperimetric Inequalities in Mathematical Physics, *Annals of Math. Study No. 27* (1951). In the latter, pp. 153-157 are of special relevance.

of the partial differential equation itself. From this general standpoint, the idea of our method is to replace the equation by a completely elliptic equation which coincides with the original equation over a subset of its elliptic portion. Existence of a solution to the new equation, its dependence on parameters and in particular its behavior on the boundary of the region are studied. One can then discuss the range of values of the parameters in which the solution everywhere falls within the subset where the new equation coincides with the original equation. These solutions of the new equation are then likewise solutions, everywhere elliptic, of the original partial differential equation.

The principal theorem concerning the flow of a compressible fluid, and the main ideas of our method, are formulated in Part I, while Part VI considers the case of circulation. The remaining parts prove the requisite mathematical theorems concerning behavior on the boundary and at infinity. The various parts are as follows:

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I. FORMULATION AS A MINIMUM PROBLEM

1. Consider a stationary, irrotational two-dimensional flow of a compressible fluid. Such a flow is characterized by a velocity potential $\phi(x, y)$ satisfying the partial differential equation

$$(1.1) \quad \frac{\partial}{\partial x} (\rho \phi_x) + \frac{\partial}{\partial y} (\rho \phi_y) = 0$$

or

$$(1.2) \quad (c^2 - u^2) \frac{\partial^2 \phi}{\partial x^2} - 2uv \frac{\partial^2 \phi}{\partial x \partial y} + (c^2 - v^2) \frac{\partial^2 \phi}{\partial y^2} = 0$$

where $u = \partial \phi / \partial x$, $v = \partial \phi / \partial y$ are the velocity components of the flow, ρ is the density of the fluid, and c is the local velocity of sound. The quantities ρ , c are connected with the square of the velocity $u^2 + v^2$ through the Bernoulli law and

the equation of state of the fluid. If the fluid obeys the polytropic law $p\rho^{-\gamma} = \text{constant}$, where p is the pressure and γ is a constant > 1 , these relations are

$$(1.3) \quad \begin{aligned} \rho &= \rho_0 (c/c_0)^{2/\gamma}, \\ c^2 &= c_0^2 + \frac{\gamma-1}{2} (u_0^2 + v_0^2) - \frac{\gamma-1}{2} (u^2 + v^2), \end{aligned}$$

c_0, u_0, v_0 being the local velocity of sound and the velocity components respectively at a prescribed point. The partial differential equation (1.2) is of elliptic type whenever the discriminant $(c^2 - u^2)(c^2 - v^2) - u^2v^2 = c^2(c^2 - u^2 - v^2)$ is positive, i.e., whenever the flow is subsonic, $u^2 + v^2 < c^2$, and is of hyperbolic type whenever the flow is supersonic, $u^2 + v^2 > c^2$. For the polytropic law, the conditions $u^2 + v^2 \leq c^2$ becomes $u^2 + v^2 \leq \frac{2}{\gamma+1} c_0^2 + \frac{\gamma-1}{\gamma+1} (u_0^2 + v_0^2)$.

We shall establish existence and uniqueness of flows of a compressible fluid which are everywhere subsonic and satisfy prescribed boundary conditions. For the sake of definiteness, we shall consider the flow about a given obstacle and satisfying prescribed conditions at the point at infinity. The obstacle will be supposed to have a boundary B^* which is a smooth curve with continuously turning tangent line (no corners) and with bounded curvature. Also, to be specific, we shall suppose that the fluid obeys the polytropic law, although the exact forms (1.3) of the equations for ρ and c are never used and the procedure is applicable to an arbitrary equation of state.

Let $c_0, u_0 \geq 0, v_0 = 0$ be given at the point at infinity, and let $M_0 = u_0/c_0$ be the Mach number of the flow at infinity. We shall prove that *there is a positive number $\bar{M}_0$ such that for any $M_0 < \bar{M}_0$ there exists a uniquely determined completely subsonic flow with Mach number M_0 at infinity and with no circulation about the obstacle. In this flow, denote the maximum of the local Mach number $\sqrt{u^2 + v^2}/c$ throughout the fluid by M_1 , where $M_1 < 1$, so that the flow and in particular M_1 depend on M_0 . Then, in allowing M_0 to vary in the range $0 \leq M_0 < \bar{M}_0$, both the flow and M_1 depend continuously on M_0 , with $M_1 \rightarrow 1$ as $M_0 \rightarrow \bar{M}_0$ (and $M_1 = 0$ for $M_0 = 0$).*

A similar result holds in the case of prescribed circulation about the obstacle, where $\bar{M}_0$ depends on the circulation as well as on the boundary of the obstacle. The theorem is stated and proved in Part VI. Similar theorems hold for an arbitrary equation of state of the fluid, for other boundary value problems, and for other partial differential equations.

In connection with the uniqueness as stated in the main theorem, we should like to make the following additional comment. We prove below that there is a unique flow for which $\iint [(u - u_0)^2 + v^2] dx dy$ is convergent, the integration being over the unbounded region exterior to the boundary B^* . (The convergence of this integral will be shown to imply zero circulation and that $u \rightarrow u_0, v \rightarrow 0$ as the point at infinity is approached.) The interesting related question arises to show that a flow with zero circulation and with $u \rightarrow u_0, v \rightarrow 0$ at the point at

infinity must have a convergent $\iint [(u - u_0)^2 + v^2] dx dy$. This remains an open question.

2. The partial differential equation (1.1) can be obtained as the Euler equation of a variational expression, namely

$$(2.1) \quad \delta \iint G(\phi_x^2 + \phi_y^2) dx dy = 0,$$

where, in the case when the fluid obeys the polytropic law,

$$(2.2) \quad G(\phi_x^2 + \phi_y^2) = - \left[\frac{2}{\gamma - 1} c_0^2 + u_0^2 - (\phi_x^2 + \phi_y^2) \right]^{\gamma/(\gamma-1)}$$

The Euler equation corresponding to (2.1) is

$$(2.3) \quad \frac{\partial}{\partial x} (G'(\phi_x^2 + \phi_y^2) \cdot \phi_x) + \frac{\partial}{\partial y} (G'(\phi_x^2 + \phi_y^2) \cdot \phi_y) = 0,$$

and this is identical to (1.1) when $G'(\phi_x^2 + \phi_y^2)$ is taken to be proportional to ρ given as a function of $\phi_x^2 + \phi_y^2$ by (1.3). The expression (2.2) is the function $G(\phi_x^2 + \phi_y^2)$. For an equation of state other than the polytropic law, the function $G(\phi_x^2 + \phi_y^2)$ is the expression for the negative of the pressure as a function of $u^2 + v^2$, as given by the Bernoulli law and the equation of state.

For an arbitrary variational problem of the form

$$(2.4) \quad \delta \iint F(\phi_x, \phi_y) dx dy = 0,$$

the Euler equation is

$$F_{pp}(\phi_x, \phi_y)\phi_{xx} + 2F_{pq}(\phi_x, \phi_y)\phi_{xy} + F_{qq}(\phi_x, \phi_y)\phi_{yy} = 0,$$

where p, q are the arguments in $F(p, q)$. The equation is of elliptic type whenever

$$(2.5) \quad F_{pp}F_{qq} - F_{pq}^2 > 0.$$

If the condition (2.5) holds for all value of p, q , the variational problem (2.4) is called a *regular* variational problem. For a regular variational problem, we may suppose that in addition $F_{pp} > 0$ (by multiplying F by -1 if necessary), and this condition together with (2.5) may be expressed in other terms by the statement that $F(p, q)$ as a function of its two arguments p, q is a *convex* function of p, q .

For an integrand of the form (2.1), the condition (2.5) of regularity can be written in a simpler form. Setting $p^2 + q^2 = \Lambda$, so that the integrand $F(p, q)$ is $G(\Lambda)$, we have

$$F_{pp}F_{qq} - F_{pq}^2 = 4(\Lambda[G'(\Lambda)]^2)'$$

where the prime means differentiation with respect to Λ , and the condition (2.5) for the elliptic character of the Euler equation is

$$(2.6) \quad \frac{d}{d\Lambda} (\Lambda G'(\Lambda)^2) > 0, \quad \text{where } \Lambda = p^2 + q^2.$$

The condition $F_{pp} > 0$, together with (2.6), gives

$$(2.7) \quad G'(\Lambda) > 0.$$

These conditions on $G(\Lambda)$ may be expressed more geometrically when G is considered as a function of $\sqrt{\Lambda}$. They are, then, that the integrand G be an *increasing, convex* function of $\sqrt{p^2 + q^2}$.

For the integrand (2.2), the conditions (2.6), (2.7) are satisfied only for $p^2 + q^2 < \frac{2}{\gamma+1} c_0^2 + \frac{\gamma-1}{\gamma+1} u_0^2$. Furthermore, the integrand is only defined for $p^2 + q^2 \leq \frac{2}{\gamma-1} c_0^2 + u_0^2$. We shall show how to overcome these principal difficulties by modifying the problem in such a way that the direct methods of the calculus of variations become applicable.

Before modifying the variational problem, we shall normalize by factoring $\left(\frac{2}{\gamma-1} c_0^2 + u_0^2\right)^{\frac{1}{2}}$ from the velocity components and a constant from the integrand. Then the variational integrand in (2.1) is given by

$$(2.8) \quad G(\phi_x^2 + \phi_y^2) = -[1 - (\phi_x^2 + \phi_y^2)]^{\gamma/(\gamma-1)}$$

with the condition

$$(2.9) \quad \begin{aligned} \phi_x &= \frac{u_0}{\left(\frac{2}{\gamma-1} c_0^2 + u_0^2\right)^{\frac{1}{2}}} = \frac{M_0}{\left(\frac{2}{\gamma-1} + M_0^2\right)^{\frac{1}{2}}} \\ \phi_y &= 0 \end{aligned}$$

at the point at infinity, with no circulation. The condition (2.6) of regularity of the integrand, corresponding to subsonic flow, is given by

$$(2.10) \quad \phi_x^2 + \phi_y^2 < \frac{\gamma-1}{\gamma+1},$$

while the integrand (2.8) is not regular for $\phi_x^2 + \phi_y^2 \geq \frac{\gamma-1}{\gamma+1}$.

The boundary condition on the boundary of the obstacle is $\partial\phi/\partial n = 0$. A variational problem can be formulated in terms of ϕ yielding this condition as a natural

boundary condition, but we prefer to introduce instead the conjugate function $\psi(x, y)$ (stream function of the flow) defined by

$$(2.11) \quad \begin{aligned} \phi_x \cdot G'(\phi_x^2 + \phi_y^2) &= \psi_y, \\ \phi_y \cdot G'(\phi_x^2 + \phi_y^2) &= -\psi_x. \end{aligned}$$

The boundary condition $\partial\phi/\partial n = 0$ then becomes $\partial\psi/\partial s = 0$, or $\psi = \text{constant} = 0$ on the boundary B^* of the obstacle. At infinity, we have

$$(2.12) \quad \psi_x = 0, \quad \psi_y = \frac{M_0}{\left(\frac{2}{\gamma-1} + M_0^2\right)^{1/2}} \cdot G'\left(\frac{M_0^2}{\frac{2}{\gamma-1} + M_0^2}\right) = q_0.$$

The stream function ψ of the flow also has a variational formulation. Indeed from (2.4) we obtain

$$(2.13) \quad (\phi_x^2 + \phi_y^2)G'(\phi_x^2 + \phi_y^2)^2 = \psi_x^2 + \psi_y^2.$$

For $\phi_x^2 + \phi_y^2 \leq (\gamma-1)/(\gamma+1)$, the left-hand side is an increasing function of $\phi_x^2 + \phi_y^2$, and in that range the equation can be uniquely solved for $\phi_x^2 + \phi_y^2$ in terms of $\psi_x^2 + \psi_y^2$. Define the function $g(\psi_x^2 + \psi_y^2)$ by

$$(2.14) \quad g'(\psi_x^2 + \psi_y^2) = \frac{1}{G'(\phi_x^2 + \phi_y^2)^2}.$$

The equation (2.11) can be written

$$(2.15) \quad \begin{aligned} \psi_x \cdot g'(\psi_x^2 + \psi_y^2) &= -\phi_y \\ \psi_y \cdot g'(\psi_x^2 + \psi_y^2) &= \phi_x. \end{aligned}$$

This shows that the stream function $\psi(x, y)$ can be obtained from the variational formulation

$$(2.16) \quad \delta \iint g(\psi_x^2 + \psi_y^2) dx dy = 0.$$

The integrand $g(\psi_x^2 + \psi_y^2)$ is defined, by (2.13) and (2.14), for

$$(2.17) \quad \psi_x^2 + \psi_y^2 \leq \frac{\gamma-1}{\gamma+1} G'\left(\frac{\gamma-1}{\gamma+1}\right)^2 = b_0^2.$$

Furthermore, the integrand $g(\psi_x^2 + \psi_y^2)$ is regular throughout this range excluding the end-point, i.e., for $\psi_x^2 + \psi_y^2 < b_0^2$. For, by (2.14) and (2.13),

$$(\psi_x^2 + \psi_y^2)g'(\psi_x^2 + \psi_y^2)^2 = \frac{\psi_x^2 + \psi_y^2}{G'(\phi_x^2 + \phi_y^2)^2} = \phi_x^2 + \phi_y^2.$$

* Integration of (2.14) and use of (2.13) will yield

$$g(\psi_x^2 + \psi_y^2) = 2(\phi_x^2 + \phi_y^2)G'(\phi_x^2 + \phi_y^2) - G(\phi_x^2 + \phi_y^2).$$

Denote $\phi_x^2 + \phi_y^2$ by Λ , and $\psi_x^2 + \psi_y^2$ by λ . By (2.14),

$$\frac{d}{d\lambda} [\lambda g'(\lambda)^2] = \frac{d\Lambda}{d\lambda} = \frac{1}{\frac{d}{d\Lambda} [\Lambda G'(\Lambda)^2]}.$$

For $\lambda < b_0^2$, $\frac{d}{d\lambda} [\lambda g'(\lambda)^2] > 0$ while $\frac{d}{d\lambda} [\lambda g'(\lambda)^2] \rightarrow +\infty$ as $\lambda \rightarrow b_0^2$. The integrand $g(\psi_x^2 + \psi_y^2)$ is therefore regular throughout the range $\psi_x^2 + \psi_y^2 < b_0^2$, and the variational problem (2.16) can only yield solutions of an elliptic partial differential equation which correspond to everywhere subsonic flows. (Equation (2.13) shows that $\psi_x^2 + \psi_y^2 \leq b_0^2$ for all values of $\phi_x^2 + \phi_y^2$, in the supersonic range $\phi_x^2 + \phi_y^2 > (\gamma - 1)/(\gamma + 1)$ included; if the supersonic range is included, $\phi_x^2 + \phi_y^2$ is a multivalued function of $\psi_x^2 + \psi_y^2$, and the integrand $g(\psi_x^2 + \psi_y^2)$ defined in (2.14) likewise a multivalued function of $\psi_x^2 + \psi_y^2$.)

3. Consider a variational problem (2.16) in which the integrand $g(\psi_x^2 + \psi_y^2)$ is regular for

$$\psi_x^2 + \psi_y^2 < b_0^2,$$

while for $\psi_x^2 + \psi_y^2 \geq b_0^2$ the integrand is either undefined, or not regular, or behaves in an arbitrary manner. Select a value b_1 slightly smaller than b_0 , for example $b_1 = (1 - \epsilon)b_0$ where ϵ is a small positive quantity, and define a new function $h(\psi_x^2 + \psi_y^2)$ which is equal to $g(\psi_x^2 + \psi_y^2)$ for $\psi_x^2 + \psi_y^2 \leq b_1^2$, while in $b_1^2 \leq \psi_x^2 + \psi_y^2 < \infty$ the function $h(\psi_x^2 + \psi_y^2)$ remains an integrand of a *regular* variational problem, and is equal to $e + f \cdot (\psi_x^2 + \psi_y^2)$ for sufficiently large $\psi_x^2 + \psi_y^2$, where e, f are suitable constants. Furthermore, $h(\psi_x^2 + \psi_y^2)$ should be as often differentiable as desired, say with continuous derivatives up to the third order. It is clearly possible to construct the function h since the condition of regularity merely requires h to be an increasing, convex function of $\sqrt{\psi_x^2 + \psi_y^2}$. More explicitly, replacing $\psi_x^2 + \psi_y^2$ by λ , select $\lambda h'(\lambda)^2$ to be an increasing function of λ , defined for all λ , equal to $\lambda g'(\lambda)^2$ for $\lambda \leq b_1^2$, and equal to $f^2 \lambda$, where f is a constant, for all λ larger than say $2b_1^2$; $h(\lambda)$ is then obtained from $h(\lambda) = \int_0^\lambda \sqrt{[\lambda h'(\lambda)^2]/\lambda} d\lambda$. Note that the function $h(\psi_x^2 + \psi_y^2)$ depends on the choice of $b_1 < b_0$.

Consider the region B exterior to the curve B^* , and consider the new variational problem of minimizing the integral

$$(3.1) \quad \iint_B [h(\psi_x^2 + \psi_y^2) - h(q_0^2) - 2q_0 h'(q_0^2)(\psi_y - q_0)] dx dy = \text{minimum}$$

among all functions $\psi(x, y)$ defined over B and vanishing on the boundary B^* . The constant q_0 is defined in (2.12) in terms of M_0 and will specify the behavior of the derivatives of ψ at infinity. The terms $h'(q_0^2) + 2q_0 h'(q_0^2)(\psi_y - q_0)$ are

subtracted from $h(\psi_x^2 + \psi_y^2)$ to obtain convergence of the integral at infinity, and these terms do not affect the Euler equation or the conditions of regularity of the integrand. Denote the variational problem (3.1) by $P(q_0; b_1)$, the parameter b_1 being involved in the integrand $h(\psi_x^2 + \psi_y^2)$.

As will be proved, the variational problem $P(q_0, b_1)$ has a *unique* solution $\psi(x, y)$ which has continuous first and second derivatives, satisfies the Euler equation corresponding to (3.1), vanishes on the boundary, and has a behavior at infinity given by $\psi_x \rightarrow 0, \psi_y \rightarrow q_0$ while $\iint_B [\psi_x^2 + (\psi_y - q_0)^2] dx dy$ converges.

Denote the solution by $\psi(x, y; q_0; b_1)$ to indicate its dependence on the parameter q_0 and on the quantity b_1 occurring in the term $h(\psi_x^2 + \psi_y^2)$ in the integrand. Set

$$V(q_0; b_1) = \max |\text{grad } \psi(x, y; q_0; b_1)|.$$

We shall prove in Parts II, III, IV, that both $\psi(x, y; q_0; b_1)$ and $V(q_0; b_1)$ depend continuously on q_0 . See Theorems 13, 14, 15. This is a vital step in the argument which follows.

In the range $\psi_x^2 + \psi_y^2 \leq b_1^2$, the partial differential equation satisfied by ψ is the desired equation for a subsonic flow of a compressible fluid, since $h \equiv g$ there, but it is not the desired flow outside this range. If we could assert, therefore, that

$$(3.2) \quad V(q_0; b_1) \leq b_1,$$

then the function $\psi(x, y; q_0; b_1)$ would provide the desired subsonic flow.

For $q_0 = 0$, the function $\psi(x, y; 0; b_1)$ is the function which is identically zero since this indeed minimizes (3.1). Thus, $V(0; b_1) = 0$. On the other hand, clearly $V(q_0; b_1) \geq q_0$, so that $V(b_0; b_1) \geq b_0 > b_1$. By the continuity of $V(q_0; b_1)$ as a function of q_0 , there is a value $Q_0(b_1)$ of q_0 such that

$$(3.3) \quad V(q_0; b_1) < b_1 \quad \text{for all } q_0 \text{ in } 0 \leq q_0 < Q_0(b_1) \quad \text{and} \quad V(Q_0(b_1); b_1) = b_1.$$

For all q_0 in $0 \leq q_0 \leq Q_0(b_1)$, the condition (3.2) is satisfied, and $\psi(x, y; q_0; b_1)$ provides the desired subsonic flow of the fluid.

These considerations apply to the variational problem $P(q_0, b_1)$. We now select a second value b_1^1 such that $b_1 < b_1^1 < b_0$, and consider the variational problem $P(q_0, b_1^1)$. The integrand for problem $P(q_0, b_1^1)$ is identical to the integrand for the variational problem $P(q_0, b_1)$ in the range $\psi_x^2 + \psi_y^2 \leq b_1^2$ (since $h \equiv g$ in the range for both cases). For any q_0 in the interval $0 \leq q_0 \leq Q_0(b_1)$, the condition (3.2) for the function $\psi(x, y; q_0; b_1)$ is satisfied, so that $\psi(x, y; q_0; b_1)$ satisfies the variational condition for the problem $P(q_0, b_1^1)$ as well. By the uniqueness of the solution we can assert that

$$(3.4) \quad \psi(x, y; q_0; b_1) \equiv \psi(x, y; q_0; b_1^1) \quad \text{for } 0 \leq q_0 \leq Q_0(b_1),$$

and therefore that

$$(3.5) \quad V(q_0; b_1) = V(q_0; b_1^1) \quad \text{for} \quad 0 \leq q_0 \leq Q_0(b_1).$$

Consequently the interval in which $V(q_0, b_1^1) \leq b_1^1$ is larger than the interval $0 \leq q_0 \leq Q_0(b_1)$, and we have

$$(3.6) \quad Q_0(b_1^1) > Q_0(b_1).$$

Let

$$(3.7) \quad Q_0 = \lim_{b_1 \rightarrow b_0} Q_0(b_1),$$

and let q_0 be any value $< Q_0$. The functions $\psi(x, y; q_0; b_1)$, for all b_1 with $Q_0(b_1) \geq q_0$, coincide and define a subsonic flow $\psi(x, y; q_0)$ of the fluid, with the maximum $V(q_0)$ of its gradient satisfying $V(q_0) < b_0$. For each q_0 in the interval $0 \leq q_0 < Q_0$, we therefore have obtained the stream function $\psi(x, y; q_0)$ of a subsonic flow.

We now show that

$$(3.8) \quad V(q_0) \rightarrow b_0 \quad \text{as} \quad q_0 \rightarrow Q_0.$$

If (3.8) were not true, there would be a positive ϵ and a sequence $q_0^{(i)}$, $i = 1, 2, \dots$, $\rightarrow Q_0$ such that

$$V(q_0^{(i)}) < b_0 - \epsilon, \quad i = 1, 2, \dots$$

Select a value b_1 in $b_0 - \epsilon < b_1 < b_0$. The function $\psi(x, y; q_0^{(i)})$ is a solution of problem $P(q_0^{(i)}, b_1)$, $i = 1, 2, \dots$. Therefore, in accordance with the continuity Theorems 13, 14, 15 in §15,

$$\psi(x, y; q_0^{(i)}) \rightarrow \psi(x, y; Q_0, b_1) \quad \text{and} \quad V(q_0^{(i)}) \rightarrow V(Q_0; b_1),$$

where $\psi(x, y; Q_0, b_1)$ with maximum gradient $V(Q_0; b_1)$, is the solution of problem $P(Q_0, b_1)$. We have $V(Q_0; b_1) \leq b_0 - \epsilon < b_1$, and there is therefore an interval

$$(3.9) \quad |q_0 - Q_0| < \delta$$

about Q_0 such that $\psi(x, y; q_0, b_1)$ all satisfy

$$(3.10) \quad V(q_0; b_1) \leq b_1.$$

All these $\psi(x, y; q_0, b_1)$ are therefore subsonic flows, and provide the solutions for problems $P(q_0, b_1^1)$ for any b_1^1 in $b_1 < b_1^1 < b_0$. But (3.10) and (3.9) then contradict $V(Q_0(b_1^1)) = b_1^1 \rightarrow b_0$ and $Q_0(b_1^1) \rightarrow Q_0$ as $b_1^1 \rightarrow b_0$. The limiting relation (3.8) is established.

The main theorem is obtained when we translate from the stream function $\psi(x, y)$ to the velocity potential $\phi(x, y)$. The condition $\psi_x^2 + \psi_y^2 \rightarrow b_0^2$ means that $\phi_x^2 + \phi_y^2 \rightarrow (\gamma - 1)/\gamma + 1$, or that the local Mach number $\rightarrow 1$. The interval

$0 \leq q_0 < Q_0$ can be replaced, by (2.12), by an interval $0 \leq M_0 < \bar{M}_0$ for the Mach number M_0 at the point at infinity. The main theorem follows.

Actually, we have proved somewhat more than is stated in the main theorem. Let $\Omega(b_1)$ be the set of values of q_0 in the interval $0 \leq q_0 \leq b_0$ for which $V(q_0; b_1) < b_1$. This set $\Omega(b_1)$ is open in $0 \leq q_0 \leq b_0$, and includes $q_0 = 0$, but excludes $q_0 = b_0$. As in (3.5), $\Omega(b_1) \subset \Omega(b_1')$ for $b_1 < b_1' < b_0$. Let $\Omega = \bigcup_{b_1} \Omega(b_1)$, the sum varying over all $b_1 < b_0$. For any q_0 in the open set Ω , there exists a unique stream function $\psi(x, y; q_0)$ providing a subsonic flow with a maximum gradient $V(q_0) < b_0$. Consider the set Q consisting of all the limit points of Ω not belonging to Ω . For each value $\bar{q}_0$ in Q , we have $V(q_0) \rightarrow b_0$ as $q_0 \rightarrow \bar{q}_0$ through values q_0 in Ω .

When we translate to the velocity potential ϕ in place of the stream function ψ , and to the Mach number M_0 in place of q_0 , the following is obtained: *there is an open set $\mathfrak{M}_0$ of values of M_0 in the interval $0 \leq M_0 \leq 1$, $\mathfrak{M}_0$ including 0 but excluding 1, such that a subsonic flow with Mach number M_0 at infinity exists and is unique for M_0 in $\mathfrak{M}_0$. Denote the maximum local Mach number in this flow by M_1 , where $M_1 < 1$. Both the flow and M_1 depend continuously on M_0 , with $M_1 = 0$ when $M_0 = 0$. Let $\bar{\mathfrak{M}}_0$ be the set of limiting values of $\mathfrak{M}_0$ not contained in $\mathfrak{M}_0$. For any $\bar{M}_0$ in $\bar{\mathfrak{M}}_0$, $M_1 \rightarrow 1$ if $M_0 \rightarrow \bar{M}_0$ through values in $\mathfrak{M}_0$. Each such subsonic flow is the unique solution of a variational problem. For any M_0 not in $\mathfrak{M}_0$, there does not exist any subsonic flow with Mach number M_0 at the point at infinity and with maximum local Mach number < 1 .*

It would be interesting to see whether there are any obstacles B^* for which the set $\mathfrak{M}_0$ consists of more than an interval $0 \leq M_0 < \bar{M}_0$.

We should perhaps repeat that we have assumed that the boundary B^* of the obstacle has a continuously turning tangent line and bounded curvature. The conditions on B^* can be weakened but we shall not pursue this question.

Similar theorems hold when there is circulation about the obstacle as well. The variational formulation differs from (3.1) in that a different expression is subtracted from $h(\psi_x^2 + \psi_y^2)$. The variational problem and the resulting theorems are discussed in Part VI.

4. Consider an arbitrary regular variational problem

$$(4.1) \quad \delta \iint_B F(p, q) \, dx \, dy = 0$$

for the determination of $\psi = \psi(x, y)$, where $p = \partial\psi/\partial x$, $q = \partial\psi/\partial y$, and where the domain B is an infinite domain which includes the point at infinity. Concerning the integrand $F(p, q)$, we assume that

$$(4.2) \quad K(\alpha^2 + \beta^2) \geq F_{pp}\alpha^2 + 2F_{pq}\alpha\beta + F_{qq}\beta^2 \geq k(\alpha^2 + \beta^2)$$

for arbitrary α, β, p, q , where k and K are positive constants (*strict* regularity of F). This condition is satisfied by the integrand $h(p^2 + q^2)$ defined in Section 3. The Euler equation for ψ is the elliptic equation

$$(4.3) \quad F_{pp}(\psi_x, \psi_y)\psi_{xx} + 2F_{pq}(\psi_x, \psi_y)\psi_{xy} + F_{qq}(\psi_x, \psi_y)\psi_{yy} = 0.$$

At the point at infinity, we shall suppose that $\psi(x, y)$ is required to behave in such a way that $\psi_x \rightarrow p_0$, $\psi_y \rightarrow q_0$, while

$$(4.4) \quad \iint [(\psi_x - p_0)^2 + (\psi_y - q_0)^2] dx dy$$

converges. (This will be shown to imply zero "circulation".) It is required to find, throughout the domain B , a solution $\psi(x, y)$ of (4.3) which has the above described behavior at the point at infinity and which has the boundary values 0 on the boundary of B .

Consider the variational problem

$$(4.5) \quad I[\psi] = \iint_B [F(\psi_x, \psi_y) - F(p_0, q_0) - F_p(p_0, q_0)(\psi_x - p_0) - F_q(p_0, q_0)(\psi_y - q_0)] dx dy = \text{minimum}$$

among all functions $\psi(x, y)$ which are continuous and vanish on the boundary B^* of B , and have piece-wise continuous first derivatives.

The integrand in (4.5) can be written, by the mean value theorem,

$$\int_0^1 [\tilde{F}_{pp} \cdot (\psi_x - p_0)^2 + 2\tilde{F}_{pq} \cdot (\psi_x - p_0)(\psi_y - q_0) + \tilde{F}_{qq} \cdot (\psi_y - q_0)^2] (1 - t) dt$$

where the tildes, $\sim$, over F_{pp} , F_{pq} , F_{qq} mean that the arguments in these functions are $p_0 + t(\psi_x - p_0)$, $q_0 + t(\psi_y - q_0)$. By virtue of (4.2), the integrand lies between the $k/2$ and the $K/2$ multiple of $(\psi_x - p_0)^2 + (\psi_y - q_0)^2$. Therefore

$$(4.6) \quad \begin{aligned} \frac{K}{2} \iint_B [(\psi_x - p_0)^2 + (\psi_y - q_0)^2] dx dy &\geq I[\psi] \\ &\geq \frac{k}{2} \iint_B [(\psi_x - p_0)^2 + (\psi_y - q_0)^2] dx dy, \end{aligned}$$

so that $I[\psi]$ is finite if and only if (4.4) is finite.

There exists a function $\bar{\psi}(x, y)$ defined over B which takes the boundary values 0 and for which (4.4) is convergent. For example, enclose the boundary B^* within a large circle of radius R ; the function

$$(4.7) \quad \bar{\psi} = \lambda(r) \cdot (p_0 x + q_0 y), \quad \text{where} \quad \lambda(r) = \begin{cases} 0 & \text{for } r \leq R \\ \frac{\log(r/R)}{\log 2} & \text{for } R \leq r \leq 2R, \\ 1 & \text{for } r \geq 2R \end{cases}$$

has a convergent integral (4.4).

One can now establish, according to the known theory of double integral variational problems, that there exists a solution $\psi(x, y)$ to the minimum problem,

and this solution has continuous first and second derivatives in the domain B and satisfies the partial differential equation (4.3). For the sake of completeness, we shall give references and sketch some details of the demonstration of this in the Appendix.

5. Let $\psi(x, y)$ be a solution to the variational problem (4.6). It satisfies a variational condition in integral form which we derive now. Let $\zeta(x, y)$ be a continuous function in the closure of B , vanishing on the boundary B^* , with piecewise continuous first derivatives in B , and with finite Dirichlet integral $\iint_B (\zeta_x^2 + \zeta_y^2) dx dy$. The integral $I[\psi + \epsilon\zeta]$ can be expanded in powers of ϵ with the result

$$(5.1) \quad I[\psi + \epsilon\zeta] = I[\psi] + \epsilon \cdot \delta I + \epsilon^2 Q,$$

where

$$\delta I = \iint_B \{ [F_p(\psi_x, \psi_y) - F_p(p_0, q_0)] \zeta_x + [F_q(\psi_x, \psi_y) - F_q(p_0, q_0)] \zeta_y \} dx dy,$$

$$Q = \iint_B \left[\int_0^1 (\tilde{F}_{pp} \cdot \zeta_x^2 + 2\tilde{F}_{pq} \cdot \zeta_x \zeta_y + \tilde{F}_{qq} \cdot \zeta_y^2) (1-t) dt \right] dx dy,$$

and the arguments in $\tilde{F}_{pp}$, $\tilde{F}_{pq}$, $\tilde{F}_{qq}$ are $\psi_x + t\epsilon\zeta_x$, $\psi_y + t\epsilon\zeta_y$. By (4.2),

$$(5.2) \quad \frac{K}{2} \iint_B (\zeta_x^2 + \zeta_y^2) dx dy \geq Q \geq \frac{k}{2} \iint_B (\zeta_x^2 + \zeta_y^2) dx dy,$$

so that Q is bounded by a quantity which is independent of ϵ . Equation (5.1) shows that then

$$\lim_{\epsilon \rightarrow 0} \frac{I[\psi + \epsilon\zeta] - I[\psi]}{\epsilon} = \delta I.$$

Since $I[\psi + \epsilon\zeta] \geq I[\psi]$ for all ϵ , independent of the sign of ϵ , it follows that we must have $\delta I = 0$, or

$$(5.3) \quad \iint_B \{ [F_p(\psi_x, \psi_y) - F_p(p_0, q_0)] \zeta_x + [F_q(\psi_x, \psi_y) - F_q(p_0, q_0)] \zeta_y \} dx dy = 0.$$

This is the variational condition satisfied by $\psi(x, y)$, and is the fundamental property of $\psi(x, y)$ which will be used throughout.

The uniqueness of the solution $\psi(x, y)$ follows from (5.3). For if $\bar{\psi}(x, y)$ is any different function in B with $\iint_B [(\bar{\psi}_x - p_0)^2 + (\bar{\psi}_y - q_0)^2] dx dy$ finite, then setting $\zeta = \bar{\psi} - \psi$ and $\epsilon = 1$ in (5.1) gives

$$(5.3') \quad I[\bar{\psi}] = I[\psi] + Q > I[\psi]$$

by (5.2). This proves also that an admissible function ψ , with convergent $\iint [(\psi_x - p_0)^2 + (\psi_y - q_0)^2] dx dy$, which satisfies the variational condition (5.3) is the solution to the variational problem (4.6), for the inequality (5.3') was derived merely from the property (5.3) of ψ .

The results stated in the preceding paragraph are the uniqueness properties required in our discussion in Section 3. Note also the following evident property of the variational condition (5.3). If $\psi(x, y)$ satisfies the variational condition (5.3), then it also satisfies the variational condition for an integrand $\bar{F}(\psi_x, \psi_y)$ which coincides with $F(\psi_x, \psi_y)$ over the range of values actually taken by ψ_x, ψ_y . By the uniqueness, ψ is therefore also the solution to the variational problem with the integrand $\bar{F}(\psi_x, \psi_y)$ (provided $\bar{F}$ is itself an integrand for a regular variational problem).

If ψ has continuous second derivatives, then it follows in the usual way from (5.3) that it must satisfy the Euler-Lagrange equation

$$\frac{\partial}{\partial x} [F_p(\psi_x, \psi_y)] + \frac{\partial}{\partial y} [F_q(\psi_x, \psi_y)] = 0.$$

(Conversely, a function $\psi(x, y)$ in B which vanishes on the boundary B^* , satisfies the above partial differential equation, and makes the integral $\iint_B [(\psi_x - p_0)^2 + (\psi_y - q_0)^2] dx dy$ finite, must be the solution to the minimum problem (4.5). For it can be shown that then (4.3) is satisfied, by first proving (5.3) for smooth functions $\zeta(x, y)$ vanishing near the boundary and near infinity, and then passing to the limit.)

If we consider a subregion B_1 of B which is bounded, and select ζ to vanish on the boundary of B_1 and outside B_1 , then the extra terms $F_p(p_0, q_0)\zeta_x + F_q(p_0, q_0)\zeta_y$ in (5.3) yield 0 when integrated over the domain B_1 . The formula (5.3) becomes

$$(5.4) \quad \iint_{B_1} [F_p(\psi_x, \psi_y)\zeta_x + F_q(\psi_x, \psi_y)\zeta_y] dx dy = 0.$$

This is the form of the variational condition which we shall use in the next part II, the formula (5.3) being reserved for the discussion in the neighborhood of the point at infinity.

II. BEHAVIOR NEAR THE BOUNDARY (PART A).

6. In this part and in Part IV, we shall demonstrate the desired boundary properties of the solution $\psi(x, y)$ to the minimum problem (4.5). The culmination

in this part is Theorem 5, and in Part IV it is Theorems 9, 10, 11, 12. These parts may be considered as a continuation of the method used by the author in the paper cited in footnote 3, and of the work of Morrey also cited in this footnote.

Consider a point P_0 on the boundary B^* , and transform a neighborhood in B of P_0 into a semicircle $S_R : \eta > 0, \xi^2 + \eta^2 < R^2$, of the (ξ, η) -plane, with $\eta = 0, |\xi| < R$ corresponding to an arc of B^* containing P_0 , and $\eta = 0, \xi = 0$ to P_0 , by a single valued invertible transformation $x = x(\xi, \eta), y = y(\xi, \eta)$ with bounded, first and second derivatives and with Jacobian positive and bounded away from 0. This can clearly be done in view of the smoothness assumptions concerning the curve B^* . For example, if in the neighborhood of P_0 the curve B^* has the equation $y = Y(x)$, with the region B neighboring B^* given by $y > Y(x)$, the transformation $\xi = x - x_0, \eta = y - Y(x)$ is such a transformation. Then the Jacobian of the transformation is 1, and $\xi_x = 1, \xi_y = 0, \eta_y = 1, \eta_x = -Y'(x) = -Y'(x_0 + \xi)$. The requirement is that $Y'(x)$ be continuous and $Y''(x)$ bounded near $x = x_0$.

The function ψ becomes a function of ξ, η , and set

$$(6.1) \quad G(\xi, \eta, \psi_\xi, \psi_\eta) = F(\psi_\xi \xi_x + \psi_\eta \eta_x, \psi_\xi \xi_y + \psi_\eta \eta_y) \cdot J$$

where J is the Jacobian $\partial(x, y)/\partial(\xi, \eta)$. From the variational condition (5.4) (or the fact that ψ minimizes $I[\psi]$ in subregions, when the independent variables x, y are transformed), there follows

$$(6.2) \quad \iint_{S_R} (G_{\psi_\xi} \zeta_\xi + G_{\psi_\eta} \zeta_\eta) d\xi d\eta = 0$$

for every continuous function $\zeta(\xi, \eta)$ defined over S_R , vanishing on the boundary $\eta = 0$ and $\xi^2 + \eta^2 = R^2$ of S_R , and with finite Dirichlet integral $\iint_{S_R} (\xi_\xi^2 + \xi_\eta^2) d\xi d\eta$.

Concerning the function $G(\xi, \eta, \psi_\xi, \psi_\eta)$ in (6.1), we have

$$(6.3') \quad \begin{aligned} G_{\psi_\xi} &= J(F_{p\xi} \xi_x + F_{q\xi} \xi_y), & G_{\psi_\eta} &= J(F_{p\eta} \eta_x + F_{q\eta} \eta_y) \\ G_{\psi_\xi \psi_\xi} &= J \cdot (F_{pp} \xi_x^2 + 2F_{pq} \xi_x \xi_y + F_{qq} \xi_y^2) \\ G_{\psi_\xi \psi_\eta} &= J \cdot (F_{p p \eta} \xi_x \eta_x + F_{p q \eta} (\xi_x \eta_y + \xi_y \eta_x) + F_{q q \eta} \xi_y \eta_y) \\ G_{\psi_\eta \psi_\eta} &= J(F_{pp} \eta_x^2 + 2F_{pq} \eta_x \eta_y + F_{qq} \eta_y^2) \end{aligned}$$

so that

$$\begin{aligned} &G_{\psi_\xi \psi_\xi} \alpha^2 + 2G_{\psi_\xi \psi_\eta} \alpha \beta + G_{\psi_\eta \psi_\eta} \beta^2 \\ &= J[F_{pp}(\alpha \xi_x + \beta \eta_x)^2 + 2F_{pq}(\alpha \xi_x + \beta \eta_x)(\alpha \xi_y + \beta \eta_y) + F_{qq}(\alpha \xi_y + \beta \eta_y)^2]. \end{aligned}$$

By (4.2), this last expression lies between the k and K multiples of $J[(\alpha \xi_x + \beta \eta_x)^2 + (\alpha \xi_y + \beta \eta_y)^2]$. This last expression in turn lies between two positive multiples of $\alpha^2 + \beta^2$ throughout S_R , so that we may assert

$$(6.3) \quad K_1(\alpha^2 + \beta^2) \geq G_{\psi_\xi \psi_\xi} \alpha^2 + 2G_{\psi_\xi \psi_\eta} \alpha \beta + G_{\psi_\eta \psi_\eta} \beta^2 \geq k_1(\alpha^2 + \beta^2)$$

for all $\alpha, \beta, \psi_\xi, \psi_\eta$ and ξ, η in S_R , where k_1, K_1 are two positive constants.

We also have

$$\begin{aligned} G_{\psi_\xi \xi} &= (J\xi_x)_\xi F_p + (J\xi_y)_\xi F_q \\ &\quad + J\xi_x [F_{pp}(\psi_\xi(\xi_x)_\xi + \psi_\eta(\eta_x)_\xi) + F_{pq}(\psi_\xi(\xi_y)_\xi + \psi_\eta(\eta_y)_\xi)] \\ &\quad + J\xi_y [F_{qp}(\psi_\xi(\xi_x)_\xi + \psi_\eta(\eta_x)_\xi) + F_{qq}(\psi_\xi(\xi_y)_\xi + \psi_\eta(\eta_y)_\xi)] \end{aligned}$$

with a similar expression for $G_{\psi_\eta \xi}$. Concerning the function $F(p, q)$, we have from (4.2), that

$$(6.4) \quad |F_{pp}|, |F_{pq}|, |F_{qq}| \leq K.$$

It follows that

$$(6.4') \quad \begin{aligned} |F_p(p, q) - F_p(0, 0)|^2 &\leq 2K^2(p^2 + q^2) \quad \text{and} \\ |F_q(p, q) - F_q(0, 0)|^2 &\leq 2K^2(p^2 + q^2). \end{aligned}$$

We therefore finally see that there is a constant K_2 , depending on $K, F_p(0, 0), F_q(0, 0)$, and bounds on the second derivatives of the transformation from the x, y plane to the (ξ, η) -plane, such that

$$(6.5) \quad G_{\psi_\xi \xi}^2 + G_{\psi_\eta \xi}^2 \leq K_2^2(1 + \psi_\xi^2 + \psi_\eta^2).$$

Return to the variational condition (6.2), and denote the arguments ψ_ξ, ψ_η in $G(\xi, \eta, \psi_\xi, \psi_\eta)$ by p, q again for convenience, without confusion with the previous use of p, q . Let S' be a subregion of S_R at a positive distance from the semi-circumference $\xi^2 + \eta^2 = R^2, \eta > 0$ bounding S_R , and select $\zeta(\xi, \eta)$ arbitrary in S' and vanishing on the boundary of S' and in the complementary portion $S_R - S'$. The variational condition (6.2) is

$$(6.6) \quad \iint_{S'} (G_p \zeta_\xi + G_q \zeta_\eta) d\xi d\eta = 0.$$

Let S'' be the region obtained by translating S' h units in the ξ -direction, where $h \neq 0$ and h is so small that S'' is also contained in S_R . Let $\zeta^*(\xi, \eta)$ vanish outside S'' , while inside S'' set $\zeta^*(\xi + h, \eta) = \zeta(\xi, \eta)$. The variational condition (6.2) is

$$\iint_{S''} (G_p \zeta_\xi^* + G_q \zeta_\eta^*) d\xi d\eta = 0.$$

From the definition of ζ^* , this can be written

$$(6.6') \quad \begin{aligned} \iint_{S'} \{G_p(\xi + h, \eta, \psi_\xi(\xi + h, \eta), \psi_\eta(\xi + h, \eta)) \zeta_\xi \\ + G_q(\xi + h, \eta, \psi_\xi(\xi + h, \eta), \psi_\eta(\xi + h, \eta)) \cdot \zeta_\eta\} d\xi d\eta = 0. \end{aligned}$$

Set

$$Z(\xi, \eta) = \frac{\psi(\xi + h, \eta) - \psi(\xi, \eta)}{h}.$$

Subtraction of (6.6) from (6.6') and division by h yields

$$(6.7) \quad \iint_S \{ (\tilde{G}_{p\xi} + \tilde{G}_{pp} \cdot Z_\xi + \tilde{G}_{pq} \cdot Z_\eta) \zeta_\xi + (\tilde{G}_{q\xi} + \tilde{G}_{qp} \cdot Z_\xi + \tilde{G}_{qq} \cdot Z_\eta) \zeta_\eta \} d\xi d\eta = 0$$

where the tildes over the various quantities indicate these quantities evaluated for an argument between $\xi, \eta, \psi_\xi(\xi, \eta), \psi_\eta(\xi, \eta)$ and $\xi + h, \eta, \psi_\xi(\xi + h, \eta), \psi_\eta(\xi + h, \eta)$.

More precisely,

$$(6.8) \quad \begin{aligned} \tilde{G}_{p\xi} &= \int_0^1 G_{p\xi}(\xi + th, \eta, \mu, \nu) dt, \quad \text{where} \\ \mu &= (1 - t)\psi_\xi(\xi, \eta) + t\psi_\xi(\xi + h, \eta) \\ \nu &= (1 - t)\psi_\eta(\xi, \eta) + t\psi_\eta(\xi + h, \eta), \end{aligned}$$

with analogous expressions for the other quantities $\tilde{G}_{q\xi}, \tilde{G}_{pp}, \tilde{G}_{pq} = \tilde{G}_{qp}, \tilde{G}_{qq}$. The equations (6.7), (6.8) are derived from (6.6), (6.6') by noting that

$$\begin{aligned} \frac{1}{h} [G_p(\xi + h, \eta, \psi_\xi(\xi + h, \eta), \psi_\eta(\xi + h, \eta)) - G_p(\xi, \eta, \psi_\xi(\xi, \eta), \psi_\eta(\xi, \eta))] \\ = \frac{1}{h} \int_0^1 \frac{d}{dt} G_p(\xi + th, \eta, \mu, \nu) dt = \tilde{G}_{p\xi} + \tilde{G}_{pp} \cdot Z_\xi + \tilde{G}_{pq} \cdot Z_\eta, \end{aligned}$$

and similarly for G_q .

It is well to formulate the result we have obtained.

Theorem 1. *The difference quotient*

$$(6.9) \quad Z = \frac{\psi(\xi + h, \eta) - \psi(\xi, \eta)}{h}$$

satisfies the equation (6.7), where $\tilde{G}_{p\xi}, \tilde{G}_{q\xi}, \tilde{G}_{pp}, \tilde{G}_{pq} = \tilde{G}_{qp}, \tilde{G}_{qq}$ are given by (6.8) and analogous formulas. This can be interpreted as the variational equation for the quadratic integral

$$\iint_S \{ 2\tilde{G}_{p\xi} \cdot Z_\xi + 2\tilde{G}_{q\xi} \cdot Z_\eta + \tilde{G}_{pp} \cdot Z_\xi^2 + 2\tilde{G}_{pq} Z_\xi Z_\eta + \tilde{G}_{qq} Z_\eta^2 \} d\xi d\eta,$$

where the coefficients are given functions of ξ, η .

7. For convenience, we shall denote the quantities $\tilde{G}_{pp}, \tilde{G}_{pq} = \tilde{G}_{qp}, \tilde{G}_{qq}, \tilde{G}_{p\xi}, \tilde{G}_{q\xi}$ by a, b, c, e, f respectively. They are functions of (ξ, η) , depending on h , but they satisfy certain inequalities independent of h .

Theorem 2. *The difference quotient $Z = [\psi(\xi + h, \eta) - \psi(\xi, \eta)]/h$ has a finite Dirichlet integral and satisfies the equation*

$$(7.1) \quad \iint_{S'} \{ (aZ_\xi + bZ_\eta + e)\zeta_\xi + (bZ_\xi + cZ_\eta + f)\zeta_\eta \} d\xi d\eta = 0$$

for every function ζ in S' , vanishing on the boundary of S' and having a finite Dirichlet integral. The function Z itself satisfies

$$(7.2) \quad \iint_{S'} Z^2 d\xi d\eta \leq K_3^2,$$

while the coefficients a, b, c, e, f satisfy the inequalities

$$(7.3) \quad K_1(\alpha^2 + \beta^2) \geq a\alpha^2 + 2b\alpha\beta + c\beta^2 \geq k_1(\alpha^2 + \beta^2)$$

$$(7.4) \quad \iint_{S'} (e^2 + f^2) d\xi d\eta \leq K_4^2$$

and where k_1, K_1, K_3, K_4 are positive constants independent of h and of S' contained in S_R .

Proof. The function $\psi(x, y)$ in the original (x, y) -plane has a finite value of $\iint_B [(\psi_x - p_0)^2 + (\psi_y - q_0)^2] dx dy$. Limiting ourselves to a bounded portion

B' of the region B , we can therefore assert that $\iint_{B'} (\psi_x^2 + \psi_y^2) dx dy$ is finite.

In view of the transformation properties of the (x, y) -plane to the (ξ, η) -plane, we may assert that

$$(7.5) \quad \iint_{S_R} (\psi_\xi^2 + \psi_\eta^2) d\xi d\eta \leq K_3^2.$$

The function $Z(\xi, \eta)$ defined in (6.9) has a finite Dirichlet integral since both $\psi(\xi + h, \eta)$ and $\psi(\xi, \eta)$ have, and $h \neq 0$. It can be expressed as

$$Z = \int_0^1 \psi_\xi(\xi + th, \eta) dt,$$

so that

$$Z^2 \leq \int_0^1 \psi_\xi^2(\xi + th, \eta) dt$$

and

$$\iint_{S'} Z^2 d\xi d\eta \leq \int_0^1 \left(\iint_{S'} \psi_\xi^2(\xi + th, \eta) d\xi d\eta \right) dt \leq K_3^2,$$

which is (7.2).

That the coefficients a, b, c satisfy the inequalities (7.3) follows from (6.3) and the representations of $\tilde{G}_{pp}, \tilde{G}_{pq}, \tilde{G}_{qq}$ analogous to (6.8). For e, f , we have by applying the Schwarz inequality to (6.8),

$$\tilde{G}_{p\xi}^2 \leq \int_0^1 G_{p\xi}^2 (\xi + th, \eta, \mu, \nu) dt,$$

and an analogous inequality for $\tilde{G}_{q\xi}^2$. From (6.5), we obtain therefore

$$e^2 + f^2 \leq K_2^2 \int_0^1 [1 + \mu^2 + \nu^2] dt$$

or

$$(7.6) \quad e^2 + f^2 \leq K_2^2 [1 + \frac{1}{3} (\psi_\xi^2 + \psi_\eta^2 + \psi_\xi \bar{\psi}_\xi + \psi_\eta \bar{\psi}_\eta + \bar{\psi}_\xi^2 + \bar{\psi}_\eta^2)]$$

where $\psi_\xi, \psi_\eta, \bar{\psi}_\xi, \bar{\psi}_\eta$ represent $\psi_\xi(\xi, \eta), \psi_\eta(\xi, \eta), \psi_\xi(\xi + h, \eta), \psi_\eta(\xi + h, \eta)$. Integration of (7.6) and use of (7.5) gives (7.4) with $K_4^2 = K_2^2(\pi R^2 + K_3^2)$, where R is the radius of the semicircular region S_R .

To begin with, we shall make use of Theorem 2, but later we shall also need the inequality (7.6) in place of the weaker one, (7.4). We should like to remark that all of our later arguments which follow from Theorem 2 also apply to the case in which (7.1) is replaced by

$$(7.7) \quad \iint_{S_R} \{aZ_\xi + b_1Z_\eta + e\}\zeta_\xi + (b_2Z_\xi + cZ_\eta + f)\zeta_\eta \, d\xi \, d\eta = 0$$

with two different terms b_1, b_2 in place of the common b . Then, in (7.3) the term $2b$ should be replaced by $b_1 + b_2$ and the additional hypothesis made that b_1, b_2 are each separately uniformly bounded. This is of importance for partial differential equations that do not arise from variational problems. Likewise, there could also be a term added to the integrand of (7.7) involving $(gZ_\xi + hZ_\eta + j)\zeta$ with g, h uniformly bounded and $\iint j^2 d\xi \, d\eta$ finite.

8. We shall now select appropriate subregions S of S_R and appropriate functions ζ . For any ρ let S_ρ denote the semicircle $\xi^2 + \eta^2 < \rho^2, \eta > 0$. Limit h to be in absolute value $< R/2$. The following main theorem will be proved.

Theorem 3. *Set*

$$(8.1) \quad D_\rho[Z] = \iint_{S_\rho} (Z_\xi^2 + Z_\eta^2) \, d\xi \, d\eta.$$

There is a constant K_5 , independent of h , such that

$$(8.2) \quad D_\rho[Z] \leq K_5 \quad \text{for all} \quad \rho \leq R/8.^7$$

⁷ The constant K_5 and later constants depend on k, K_1, K_3, K_4 appearing in Theorem 2. These in turn depend on the constants k, K of (4.2), on a bound for the curvature of the boundary curve B^* , and on a bound for p_0, q_0 .

Proof. Let ρ, ρ' be two quantities such that $0 < \rho < \rho' < R/2$. In (7.1), select S' to be the semicircle $S_{\rho'}$, and ζ to be the function

$$\zeta = \mu(\sqrt{\xi^2 + \eta^2}) \cdot Z,$$

where

$$\mu(r) = \begin{cases} 1 & \text{for } r \leq \rho \\ \frac{\rho' - r}{\rho' - \rho} & \text{for } \rho \leq r \leq \rho' \\ 0 & \text{for } r \geq \rho' \end{cases}$$

The function Z is continuous and vanishes on $\eta = 0$ since ψ does, so that ζ vanishes on the boundary of $S_{\rho'}$. The function Z has a finite Dirichlet integral since both $\psi(\xi + h, \eta)$ and $\psi(\xi, \eta)$ do, while $h \neq 0$. Thus ζ has a finite Dirichlet integral and is therefore admissible to insert in (7.1). Denoting $\sqrt{\xi^2 + \eta^2}$ by r , the insertion in (7.1) gives

$$(8.3) \quad \iint_{S_{\rho'}} \mu(r) [aZ_{\xi}^2 + 2bZ_{\xi}Z_{\eta} + cZ_{\eta}^2 + eZ_{\xi} + fZ_{\eta}] d\xi d\eta \\ + \iint_{S_{\rho'}} \mu'(r) \cdot Z [(aZ_{\xi} + bZ_{\eta} + e)r_{\xi} + (bZ_{\xi} + cZ_{\eta} + f)r_{\eta}] d\xi d\eta.$$

Note that $\mu'(r) = 0$ for $r \leq \rho$, while for $\rho \leq r \leq \rho'$, $\mu'(r) = -1/(\rho' - \rho) = \text{constant}$.

Set

$$(8.4) \quad Q_r[Z] = \iint_{S_r} [aZ_{\xi}^2 + 2bZ_{\xi}Z_{\eta} + cZ_{\eta}^2 + eZ_{\xi} + fZ_{\eta}] d\xi d\eta,$$

the integrand being integrable by virtue of (7.3) and (7.4). The quantity $Q_r[Z]$ is a function of r which is continuous, vanishes for $r = 0$, and is of bounded variation (in fact, it is absolutely continuous). The first term in (8.3) represents some average of $Q_r[Z]$ for r between ρ and ρ' . Indeed, writing the area element $d\xi d\eta$ as $r d\theta dr$ and integrating first with respect to θ , the first term in (8.3) can be written $\int_0^{\rho'} \mu(r) dQ_r[Z]$,⁸ and integration by parts gives $-\int_0^{\rho'} \mu'(r) Q_r[Z] dr$, the integrated term $\mu(r) Q_r[Z] \Big|_0^{\rho'}$ vanishing because $\mu(\rho') = Q_0[Z] = 0$. Since $\mu'(r) =$

⁸ If $g(\xi, \eta)$ is absolutely integrable, then $\int \int_{S_{\rho}} \mu(r) g(\xi, \eta) d\xi d\eta = \int_0^{\rho'} \mu(r) dG(r)$, where

$G(r) = \int \int_{S_r} g(\xi, \eta) d\xi d\eta$. For the equality holds directly if $\mu(r)$ is a step function, and in general by a passage to the limit.

0 for $r < \rho$, and $\mu'(r) = \text{constant}$ for $\rho < r < \rho'$, the equation (8.3) gives the following theorem.

Theorem 4. *For the function Z satisfying (7.1), we have*

$$(8.5) \quad \int_{\rho}^{\rho'} Q_r[Z] dr = \int_{\rho}^{\rho'} \int_0^{\pi} Z[(aZ_{\xi} + bZ_{\eta} + e)r_{\xi} + (bZ_{\xi} + cZ_{\eta} + f)r] r_{\eta} d\theta dr,$$

where $Q_r[Z]$ is defined in (8.4).

The formula can be interpreted as an "averaged" Green's theorem. One deduces from (8.5) that

$$(8.6) \quad Q_r[Z] = \int_0^{\pi} Z[(aZ_{\xi} + bZ_{\eta} + e) d\eta - (bZ_{\xi} + cZ_{\eta} + f) d\xi]$$

for almost all r . This is Green's theorem for the partial differential equation $(aZ_{\xi} + bZ_{\eta} + e)_{\xi} + (bZ_{\xi} + cZ_{\eta} + f)_{\eta} = 0$ corresponding to the condition (7.1).

Formula (8.5) is the basic formula from which further estimates will be made to obtain the proof of Theorem 3. Using the notation $D_r[Z]$ of (8.1), the inequalities (7.3), (7.4) give

$$Q_r[Z] \geq k_1 D_r[Z] - \left[\iint (e^2 + f^2) d\xi d\eta \right]^{\frac{1}{2}} (D_r[Z])^{\frac{1}{2}},$$

where by the Schwarz inequality $\left[\iint (eZ_{\xi} + fZ_{\eta}) d\xi d\eta \right]^2 \leq \iint (e^2 + f^2) d\xi d\eta \cdot \iint (Z_{\xi}^2 + Z_{\eta}^2) d\xi d\eta$ so that

$$\int_{\rho}^{\rho'} Q_r[Z] dr \geq k_1 \int_{\rho}^{\rho'} D_r[Z] dr - K_4 \cdot \left[\int_{\rho}^{\rho'} D_r[Z] dr \right]^{\frac{1}{2}}.$$

For the terms on the right hand side of (8.5), we have

$$\begin{aligned} & \left(\iint Z[(aZ_{\xi} + bZ_{\eta})r_{\xi} + (bZ_{\xi} + cZ_{\eta})r_{\eta}] d\xi d\eta \right)^2 \\ & \leq \iint Z^2 d\xi d\eta \cdot \iint [(aZ_{\xi} + bZ_{\eta})^2 + (bZ_{\xi} + cZ_{\eta})^2] d\xi d\eta \\ & \leq \iint Z^2 d\xi d\eta \cdot K_1^2 \iint (Z_{\xi}^2 + Z_{\eta}^2) d\xi d\eta \leq K_1^2 K_3^2 \iint (Z_{\xi}^2 + Z_{\eta}^2) d\xi d\eta \end{aligned}$$

and

$$\left(\iint Z[er_{\xi} + fr_{\eta}] d\xi d\eta \right)^2 \leq \iint Z^2 d\xi d\eta \cdot \iint (e^2 + f^2) d\xi d\eta \leq K_3^2 K_4^2.$$

Applying these inequalities to (8.5), we obtain

$$(8.7) \quad k_1 \int_{\rho}^{\rho'} D_r[Z] dr - K_4 \left(\int_{\rho}^{\rho'} D_r[Z] dr \right)^{\frac{1}{2}} \leq K_3 K_4 + K_1 K_3 (D_{\rho'}[Z] - D_{\rho}[Z])^{\frac{1}{2}},$$

where

$$\int_{\rho}^{\rho'} \int_0^{\pi} (Z_{\xi}^2 + Z_{\eta}^2) d\xi d\eta = D_{\rho'}[Z] - D_{\rho}[Z].$$

In the inequality (8.7), now set $\rho' = \rho + s$ where s is a fixed positive number, which we will take to be $R/4$, and ρ varies in $0 \leq \rho \leq R/2 - s = R/4$. Set $\int_{\rho}^{\rho+s} D_r[Z] dr = \lambda(\rho)$, where $\lambda(\rho)$ is a continuous function of ρ with $d\lambda(\rho)/d\rho = D_{\rho+s}[Z] - D_{\rho}[Z] \geq 0$. The inequality (8.7) is

$$(8.8) \quad k_1 \lambda(\rho) - K_4 \lambda(\rho)^{\frac{1}{2}} - K_3 K_4 \leq K_1 K_3 \left(\frac{d\lambda(\rho)}{d\rho} \right)^{\frac{1}{2}}.$$

The function $\lambda(\rho)$ is a monotonically increasing function of ρ , and we shall prove as a consequence of (8.8) that $\lambda(\rho)$ must be bounded by a constant depending on the various coefficients in (8.8).

Let K_6 be a positive number depending on $k_1, K_4, K_3 K_4$ such that

$$\frac{k_1}{2} \lambda - K_4 \lambda^{\frac{1}{2}} - K_3 K_4 \geq 0 \quad \text{for all } \lambda \geq K_6.$$

Then either

$$(8.9) \quad \lambda(\rho) < K_6 \quad \text{for all } \rho \text{ in } 0 \leq \rho \leq R/8,$$

or $\lambda(R/8) \geq K_6$, in which case

$$\lambda(\rho) \geq K_6 \quad \text{for all } \rho \text{ in } R/8 \leq \rho \leq R/4.$$

In the latter case, (8.8) gives

$$(8.10) \quad \frac{k_1}{2} \lambda(\rho) \leq K_1 K_3 \left(\frac{d\lambda(\rho)}{d\rho} \right)^{\frac{1}{2}}$$

or

$$\left(\frac{k_1}{2K_1 K_3} \right)^2 \leq \frac{d}{d\rho} \left(-\frac{1}{\lambda(\rho)} \right) \quad \text{for } R/8 \leq \rho \leq R/4.$$

Integrating this inequality from $R/8$ to $R/4$ gives

$$\frac{1}{\lambda(R/8)} - \frac{1}{\lambda(R/4)} \geq \left(\frac{k_1}{2K_1 K_3} \right)^2 \cdot \frac{R}{8},$$

which implies

$$(8.10) \quad \lambda(R/8) \leq \left(\frac{2K_1 K_3}{k_1} \right)^2 \cdot \frac{8}{R}.$$

Set

$$K_7 = \max \left\{ K_6, \left(\frac{2K_1 K_3}{k_1} \right)^2 \cdot \frac{8}{R} \right\}.$$

We have established

$$(8.11) \quad \lambda(\rho) \leq K_7 \quad \text{for all } \rho \text{ in } 0 \leq \rho \leq R/8.$$

For, either (8.9) is true in which case (8.11) is satisfied, or the alternative leads to (8.10), which implies (8.11). Finally, we have

$$\lambda(\rho) = \int_{\rho}^{\rho+s} D_r[Z] dr \geq s D_{\rho}[Z] = (R/4) D_{\rho}[Z],$$

so that (8.2) is satisfied with $K_5 = 4K_7/R$. Theorem 3 is proved.

9. All of the above considerations apply in the interior as well as at the boundary, and in a somewhat easier form since no preliminary change of variable from the (x, y) -plane to the (ξ, η) -plane would be required. Instead of semi-circles about a boundary point, one considers then full circles about an interior point. As a consequence of Theorem 3 we would then obtain, by letting $h \rightarrow 0$, that $\psi(x, y)$ has continuous first derivatives in the interior of B . But we leave the discussion of these items for the Appendix. Instead, we formulate a theorem connected with our results, and apply these methods in the next part to prove the continuity of the first derivatives at infinity. We return to the discussion at the boundary in Part IV.

In Theorem 3, the bound K_5 is independent of h . Letting $h \rightarrow 0$, so that $Z = [\psi(\xi + h, \eta) - \psi(\xi, \eta)]/h \rightarrow \psi_{\xi}(\xi, \eta)$, we obtain

$$(9.1) \quad D_{\rho}[\psi_{\xi}] \leq K_5 \text{ for } \rho \leq R/8,$$

making use of the lower semi-continuity of the Dirichlet integral

$$(9.2) \quad D_{\rho}[\psi_{\xi}] \leq \liminf_{h \rightarrow 0} D_{\rho}[Z].$$

(Indeed, $Z \rightarrow \psi_{\xi}$ uniformly in any closed domain $\bar{\Gamma}$ in the interior of S_R , as follows from the continuity of ψ_{ξ} in the interior discussed in the Appendix. The proof of (9.2) is then easy. For, $D_{\Gamma}[\psi_{\xi}] = \lim_{h \rightarrow 0} D_{\Gamma}[Z]$, where the closure of Γ is contained in the interior of S_{ρ} and the subscript Γ refers to the domain Γ of integration; since $D_{\Gamma}[Z] \leq D_{\rho}[Z]$, we obtain $D_{\Gamma}[\psi_{\xi}] \leq \liminf_{h \rightarrow 0} D_{\rho}[Z]$, and (9.2) follows by letting $\Gamma \rightarrow S_{\rho}$.)

The integral in (9.1) has $\psi_{\xi\xi}^2 + \psi_{\xi\eta}^2$ as integrand. To obtain $\psi_{\eta\eta}^2$, note that $\psi(\xi, \eta)$ satisfies the partial differential equation $\partial G_{\psi\xi}/\partial\xi + \partial G_{\psi\eta}/\partial\eta = 0$ obtained as a consequence of (6.2). This partial differential equation is

$$(9.3) \quad G_{\psi\xi\psi\xi}\psi_{\xi\xi} + 2G_{\psi\xi\psi\eta}\psi_{\xi\eta} + G_{\psi\eta\psi\eta}\psi_{\eta\eta} + G_{\psi\xi\xi} + G_{\psi\eta\eta} = 0,$$

From the properties (6.3), (6.5), and (9.1),

$$\iint \psi_{\eta\eta}^2 d\xi d\eta \leq \frac{4}{k_1^2} \left\{ 4K_1^2 \iint (\psi_{\xi\xi}^2 + \psi_{\xi\eta}^2) d\xi d\eta + 2K_2^2 \iint (1 + \psi_{\xi\xi}^2 + \psi_{\eta\eta}^2) d\xi d\eta \right\}$$

Using (9.1) and (7.5), there is a positive constant K_8 which is larger than K_5 such that

$$\iint_{s_\rho} (\psi_{\xi\xi}^2 + \psi_{\eta\eta}^2) d\xi d\eta \leq K_8 \text{ for } \rho \leq R/8.$$

We therefore have the following theorem.

Theorem 5. *There is a positive constant K_8 such that*

$$(9.4) \quad D_\rho[\psi_\xi], \quad D_\rho[\psi_\eta] \leq K_8,$$

or

$$(9.5) \quad \iint_{s_\rho} (\psi_{\xi\xi}^2 + 2\psi_{\xi\eta}^2 + \psi_{\eta\eta}^2) d\xi d\eta \leq 2K_8,$$

for $\rho \leq R/8$.

III. BEHAVIOR AT INFINITY.

10. In this part, we shall prove the following main theorem.

Theorem 6. *The derivatives ψ_x, ψ_y are continuous at the point at infinity, and approach the values p_0, q_0 as (x, y) approaches infinity.*

The proof will make use of the variational condition (5.3). This will be simpler to discuss than the variational condition (6.2), where a preliminary change of variable from (x, y) to (ξ, η) was made. Let now B' be an unbounded subregion of B whose closure is contained in the interior of B , i.e., B' has a positive distance from the boundary B^* of B . By translating h units in the x -direction, where $|h|$ is sufficiently small, and setting

$$(10.1) \quad Z = \frac{\psi(x+h, y) - \psi(x, y)}{h} - p_0,$$

there results

$$(10.2) \quad \iint_{B'} [(\tilde{F}_{pp} Z_x + \tilde{F}_{pq} Z_y) \zeta_x + (\tilde{F}_{qp} Z_x + \tilde{F}_{qq} Z_y) \zeta_y] dx dy = 0,$$

exactly analogously to the derivation of (6.7) except that now no term corresponding to $\tilde{G}_{p\xi}$, $\tilde{G}_{q\xi}$ appears. The constants $F_p(p_0, q_0)$, $F_q(p_0, q_0)$ appearing in (5.3) drop out automatically in the process of subtraction in the derivation of (10.2). Thus we have (7.1) with now $e \equiv f \equiv 0$ and $k_1 = k$, $K_1 = K$. The function Z also satisfies (7.2) since Z can be written as $\int_0^1 [\psi_x(x + th, y) - p_0] dt$ and

$$\begin{aligned} \iint_{B'} Z^2 dx dy &\leq \int_0^1 \left(\iint_{B'} [\psi_x(x + th, y) - p_0]^2 dx dy \right) dt \\ &\leq \iint_{B'} [(\psi_x - p_0)^2 + (\psi_y - q_0)^2] dx dy = \bar{K}_3^2. \end{aligned}$$

Also, Z has a finite Dirichlet integral since Z can be written as

$$Z = \frac{1}{h} [\mathfrak{X}(x + h, y) - \mathfrak{X}(x, y)],$$

where

$$\mathfrak{X}(x, y) = \psi(x, y) - p_0 x - q_0 y,$$

and both $\mathfrak{X}(x + h, y)$ and $\mathfrak{X}(x, y)$ have a finite Dirichlet integral while $h \neq 0$. This completes the analogue of Theorem 2 for the unbounded domain B' .

Let R denote the radius of the circle which contains the boundary B^* of B completely in its interior. Let $S_{\rho'}$ denote the region $x^2 + y^2 > \rho'^2$ exterior to a circumference of radius ρ , where $\rho > R$. The same argument as in Section 8 leads to Theorem 4, where $R < \rho < \rho' < \infty$. Set $\bar{D}_{\rho}[Z] = \iint_{S_{\rho}} (Z_x^2 + Z_y^2) dx dy$, so that $\bar{D}_{\rho}[Z]$ is a monotonically *decreasing* function of ρ . Since $e \equiv f \equiv 0$, one obtains directly the simpler version

$$k_1 \int_{\rho}^{\rho'} \bar{D}_{\rho}[Z] d\rho \leq K_1 \bar{K}_3 (\bar{D}_{\rho}[Z] - \bar{D}_{\rho'}[Z])^{\frac{1}{2}}$$

of (8.7). As in Section 8, this leads to

$$(10.3) \quad \bar{D}_{\rho}[Z] \leq K_9 \quad \text{for } \rho \geq 2R,$$

where K_9 is independent of h , which is the analogue of Theorem 3. Indeed, one obtains the stronger result $\bar{D}_{\rho}[Z] \leq 4K_9 R^2 / \rho^2$ for $\rho \geq 2R$, but we need not use this.

From (10.2) there follows the monotonicity of Z . That is, in any subregion B'' of B' , bounded or unbounded, the function Z attains its maximum and mini-

imum⁹ on the boundary of B'' . For, if at some point P_1 in B'' the value $Z(P_1)$ is larger than the maximum of Z on the boundary of B'' , let c be a value between these two. The point set C in the closure of B'' where $Z > c$ is an open set containing P_1 and not containing any point of the boundary of B'' ; on the boundary of C we have $Z = c$. Selecting, in (10.2) $\zeta = Z - c$ in C and 0 outside C , gives by virtue of (4.2), $Z_x \equiv Z_y \equiv 0$ or $Z \equiv c$ in C . But this contradicts $Z(P_1) > c$. This contradiction, and the corresponding one for the minimum, shows that in B'' the function Z attains its maximum and minimum on the boundary of B'' .

Let S_r^* be the circumference $x^2 + y^2 = r^2$, where $r > 2R$, and let $\omega(r)$ denote the oscillation of Z on S_r^* , i.e., $\omega(r)$ is the difference between the maximum and the minimum of Z on S_r^* . Then, according to the monotonicity of Z , the oscillation of Z in the entire region exterior to S_r^* is also $\omega(r)$, and $\omega(r)$ is a monotonically decreasing function of r . We have

$$\omega(r) = \int Z_\theta d\theta$$

where the integration is taken along S_r^* between some two points. Therefore

$$\omega(r)^2 \leq 2\pi \int_{S_r^*} Z_\theta^2 d\theta$$

and

$$(10.4) \quad \int_{2R}^{\infty} \frac{\omega(r)^2}{r} dr \leq 2\pi \iint \frac{1}{r} Z_\theta^2 d\theta dr \leq 2\pi \bar{D}_{2R}[Z] \leq 2\pi K_9$$

by (10.3). But $\int_{2R}^{\rho} \omega(r)^2/r dr \geq \omega(\rho)^2 \log \rho/(2R)$, so that

$$(10.5) \quad \omega(\rho) \leq \left(\frac{2\pi K_9}{\log \rho/(2R)} \right)^{\frac{1}{2}} = \epsilon(\rho) \quad \text{for } \rho > 2R.$$

Therefore

$$(10.6) \quad |Z(P) - Z(Q)| \leq \epsilon(\rho)$$

for any two points P, Q at a distance $> \rho$ from the origin, where $\rho > 2R$ and $\epsilon(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$. Now, let $h \rightarrow 0$, so that $Z(x, y) \rightarrow \psi_x(x, y) - p_0$. Inequality (10.4) gives

$$(10.7) \quad |\psi_x(P) - \psi_x(Q)| \leq \epsilon(\rho).$$

This shows that ψ_x is continuous at the point at infinity, and approaches a limit there.

By translating h units in the y -direction instead of the x -direction, the same

⁹ For B'' unbounded, least upper bound and greatest lower bound replace maximum and minimum.

procedure can be used, with now $Z = [\psi(x, y + h) - \psi(x, y)]/h - q_0$ in place of (10.1). There results

$$(10.8) \quad |\psi_v(P) - \psi_v(Q)| \leq \epsilon(\rho),$$

which establishes the continuity of ψ_v at the point at infinity. Because of the convergence of $\iint [(\psi_x - p_0)^2 + (\psi_v - q_0)^2] dx dy$, the limits of ψ_x and ψ_v at infinity are p_0 and q_0 respectively. Theorem 6 is proved.

By letting the point Q approach the point at infinity in (10.7) and (10.8), we may now also state

$$(10.9) \quad |\psi_x(P) - p_0| \leq \epsilon(\rho), \quad |\psi_v(P) - q_0| \leq \epsilon(\rho),$$

where P is at a distance $> \rho$ from the origin, and $\epsilon(\rho)$ is given in (10.5).

11. The convergence of $\iint [(\psi_x - p_0)^2 + (\psi_v - q_0)^2] dx dy$ also leads to the fact that there is no "circulation". The conjugate function $\phi(x, y)$ to $\psi(x, y)$ is determined by

$$F_p(\psi_x, \psi_v) = -\phi_v, \quad F_q(\psi_x, \psi_v) = \phi_x.$$

If ϕ is single valued, then there is no circulation. That is, set

$$\sigma = \int_C (\phi_x dx + \phi_v dy) = \int_C [F_q(\psi_x, \psi_v) dx - F_p(\psi_x, \psi_v) dy],$$

where C is a path surrounding the boundary B^* of the domain B . The line integral is independent of the path C , and gives the "circulation". We shall prove that $\sigma = 0$. Select C to be the circumference S_r^* for $r \geq R$. We have also

$$\sigma = \int_{S_r^*} \{[F_q(\psi_x, \psi_v) - F_q(p_0, q_0)] dx - [F_p(\psi_x, \psi_v) - F_p(p_0, q_0)] dy\}$$

and

$$\sigma^2 \leq 2\pi r \cdot \int_{S_r^*} [(F_q(\psi_x, \psi_v) - F_q(p_0, q_0))^2 + (F_p(\psi_x, \psi_v) - F_p(p_0, q_0))^2] r d\theta.$$

But $|F_p(p, q) - F_p(p_0, q_0)|^2, |F_q(p, q) - F_q(p_0, q_0)|^2 \leq 2K^2[(p - p_0)^2 + (q - q_0)^2]$ which follows from (6.4) just as (6.4') does. Therefore

$$\frac{\sigma^2}{r} \leq 8\pi K^2 \int_{S_r^*} [(\psi_x - p_0)^2 + (\psi_v - q_0)^2] r d\theta.$$

Integrating with respect to r from R to ∞ gives

$$\int_R^\infty \frac{\sigma^2}{r} dr \leq 8\pi K^2 \iint [(\psi_x - p_0)^2 + (\psi_v - q_0)^2] dx dy.$$

The convergence of the integral on the right hand side of the above inequality shows that $\sigma = 0$.

We might also state that the function $\mathfrak{X}(x, y) = \psi(x, y) - p_0x - q_0y$ approaches a limit as $(x, y) \rightarrow$ the point at infinity. For, the monotonicity of $\mathfrak{X}$ follows from the variational condition (5.3) and the inequality $[F_p(p, q) - F_p(p_0, q_0)](p - p_0) + [F_q(p, q) - F_q(p_0, q_0)](q - q_0) \geq k[(p - p_0)^2 + (q - q_0)^2]$. (For this inequality, see (12.6') and the derivation immediately preceding it.) The convergence of $\mathfrak{X}(x, y)$ as $(x, y) \rightarrow \infty$ then follows as above from the finiteness of the Dirichlet integral $D[\mathfrak{X}]$.

IV. BEHAVIOR NEAR THE BOUNDARY (PART B)

12. We return to a consideration of the behavior at the boundary, in continuation of Section 8. The bound (8.2) will be made more precise by making a more careful estimate of $\iint Z^2 d\xi d\eta$ than used in obtaining (8.6). In this way, the behavior of $D_\rho[Z]$ will be discussed as a function of ρ , and in particular we shall obtain what will be called a *Morrey condition* for this behavior, i.e., there is a positive α such that

$$(12.1) \quad D_\rho[Z] \leq \text{constant} \cdot \rho^\alpha \quad \text{for all } \rho \leq R/8.$$

This will then give the desired behavior of the first derivatives of Z .

First we consider the simpler case of (8.5) in which $e \equiv f \equiv 0$, and we use the formula (8.6) which holds for almost all r . As in Section 8, we are in the (ξ, η) -plane in $\eta > 0$, S_r^* denotes the semi-circumference $\xi^2 + \eta^2 = r^2$, $\eta > 0$, S_r the semi-circle $\xi^2 + \eta^2 < r^2$, $\eta > 0$, and $D_r[Z]$ the Dirichlet integral of Z over the semi-circle S_r .

Lemma 1.
$$\int_{S_r^*} Z^2 d\theta \leq D_r[Z].$$

Proof. In the semi-circle S_r , the function Z vanishes on the diameter $\eta = 0$. Extend Z to the semicircle $\bar{S}_r: \xi^2 + \eta^2 \leq r^2$, $\eta < 0$ below $\eta = 0$ by defining $Z(\xi, \eta) = -Z(\xi, -\eta)$ for $\eta < 0$. Denote the full circle $\xi^2 + \eta^2 \leq r^2$ by C_r , the full circumference $\xi^2 + \eta^2 = r^2$ by C_r^* . The function Z is continuous over the full circle C_r and

$$\int_{C_r^*} Z d\theta = 0.$$

The lemma is then a consequence of the well-known inequality

$$(12.2) \quad \int_{C_r^*} (Z - c)^2 d\theta \leq D_{C_r}[Z] \quad \text{where} \quad c = \frac{1}{2\pi} \int_{C_r^*} Z d\theta.$$

The inequality (12.2) follows by considering the Fourier expansion of Z on C_r^* :

$$Z - c \sim \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta),$$

so that

$$\int_{C_r^*} (Z - c)^2 d\theta = \pi \sum_1^\infty (a_n^2 + b_n^2).$$

Let H be the harmonic function over C_r which coincides with Z on the circumference C_r^* bounding C_r . The harmonic function H has an expansion in C_r expressible in terms of c , a_n , b_n , $n = 1, 2, \dots$, and calculation gives

$$D_{C_r}[H] = \pi \sum_1^\infty n(a_n^2 + b_n^2).$$

Thus

$$\int_{C_r^*} (Z - c)^2 d\theta \leq D_{C_r}[H].$$

By the Dirichlet principle, the harmonic function H has the minimum value of the Dirichlet integral among all functions with the same boundary values. Thus, $D_{C_r}[H] \leq D_{C_r}[Z]$, and inequality (12.2) is obtained.

Lemma 2. *If $e \equiv f \equiv 0$ in Theorem 2, then there is a positive α , independent of h , such that $r^{-\alpha} D_r[Z]$ is an increasing function of r . In particular*

$$D_r[Z] \leq K_{10} r^\alpha \quad \text{for } r \leq R/8.$$

where K_{10} is independent of h .

Proof. In case $e \equiv f \equiv 0$, there follows from (8.6) the inequality

$$k_1^2 D_r[Z]^2 \leq \int_{s_r^*} Z^2 r d\theta \cdot K_1^2 \int_{s_r^*} (Z_\xi^2 + Z_\eta^2) r d\theta.$$

Now

$$(12.4) \quad \int_{s_r^*} (Z_\xi^2 + Z_\eta^2) r d\theta = \frac{dD_r[Z]}{dr} \quad \text{for almost all } r,^{10}$$

¹⁰ One can eliminate all measure theoretic considerations by using (8.5) with $\rho' = \rho + s$ in place of (8.6) and then allowing $s \rightarrow 0$. But this is then a subterfuge, and we prefer the procedure as used. In Section 8, however, the use of (8.5) and of a positive s was essential.

and the use of Lemma 1 gives

$$k_1^2 D_r[Z]^2 \leq K_1^2 r D_r[Z] \frac{dD_r[Z]}{dr}$$

for almost all r , or

$$\frac{k_1^2}{K_1^2} \leq \frac{d \log D_r[Z]}{d \log r}$$

Setting $\alpha = k_1^2/K_1^2$, this inequality can be written

$$\frac{d \log (r^{-\alpha} D_r[Z])}{d \log r} \geq 0$$

for almost all r , or $r^{-\alpha} D_r[Z]$, being absolutely continuous, is a monotonically increasing function of r . Therefore, for $r \leq R/8$, we obtain

$$r^{-\alpha} D_r[Z] \leq (R/8)^{-\alpha} D_{R/8}[Z] \leq K_8 (R/8)^{-\alpha}$$

by Theorem 3. This proves Lemma 2.

We now consider the actual case in which e, f do appear in Theorem 2, and Lemma 2 has been considered only as a preliminary motivation to this more general case. From (8.6), using estimations analogous to those used in proceeding from (8.6) to (8.7), we have

$$\begin{aligned} k_1 D_r[Z] - \left(\iint_{s_r} (e^2 + f^2) d\xi d\eta \right)^{\frac{1}{2}} D_r[Z]^{\frac{1}{2}} \\ (12.5) \quad \leq \left(\int_{s_r} Z^2 r d\theta \right)^{\frac{1}{2}} \cdot \left(\int_{s_r} (e^2 + f^2) r d\theta \right)^{\frac{1}{2}} \\ + \left(\int_{s_r} Z^2 r d\theta \right)^{\frac{1}{2}} \cdot K_1 \left(\int_{s_r} (Z_\xi^2 + Z_\eta^2) r d\theta \right)^{\frac{1}{2}}. \end{aligned}$$

It is necessary to estimate $\int_{s_r} (e^2 + f^2) r d\theta$ and $\iint_{s_r} (e^2 + f^2) d\xi d\eta$, and by (7.2)

this requires estimating $\int_{s_r} (\psi_\xi^2 + \psi_\eta^2) r d\theta$.

Lemma 3. *There is a positive α and a constant L such that $D_r[\psi] \leq Lr^\alpha$.*

Proof. Return to the variational condition (6.6) for the function $\psi(\xi, \eta)$, where $p = \psi_\xi$, $q = \psi_\eta$. From (6.3), we have $G_p(0, 0) = J(F_p(0, 0)\xi_x + F_q(0, 0)\xi_y)$, $G_q(0, 0) = J(F_p(0, 0)\xi_x + F_q(0, 0)\xi_y)$, so that

$$\iint_{s_r} [G_p(0, 0)\xi_\xi + G_q(0, 0)\xi_\eta] d\xi d\eta = 0.$$

Thus

$$\iint_{S'} [(G_p - G_p(0, 0))\xi_\xi + (G_q - G_q(0, 0))\xi_\eta] d\xi d\eta = 0.$$

Proceeding as in the derivation of (8.6) in Section 8 from the variational condition (7.1), by setting $\zeta = \mu(r)\psi$, one obtains

$$(12.6) \quad \iint_{S_p} [(G_p - G_p(0, 0))p + (G_q - G_q(0, 0))q] d\xi d\eta \\ \equiv \int_{S_p^*} \psi [(G_p - G_p(0, 0)) d\eta - (G_q - G_q(0, 0)) d\xi].$$

For fixed ξ, η considering $[G_p(pt, qt) - G_p(0, 0)]p + [G_q(pt, qt) - G_q(0, 0)]q$ as a function of t in $0 \leq t \leq 1$, the mean value theorem gives

$$(G_p - G_p(0, 0))p + (G_q - G_q(0, 0))q = p^2 \tilde{G}_{pp} + 2pq \tilde{G}_{pq} + q^2 \tilde{G}_{qq},$$

and the inequality (6.3) yields

$$(12.6') \quad (G_p - G_p(0, 0))p + (G_q - G_q(0, 0))q \geq k_1(p^2 + q^2).$$

Likewise,

$$(G_p - G_p(0, 0))^2 + (G_q - G_q(0, 0))^2 \leq 4K_1^2(p^2 + q^2)$$

analogously to (6.4'). Applying these inequalities to (12.6) gives

$$k_1^2 D_\rho[\psi]^2 \leq \int_{S_p^*} \psi^2 r d\theta \cdot K_1^2 \int_{S_p^*} (p^2 + q^2) r d\theta.$$

Lemma 1 gives

$$k_1^2 D_\rho[\psi]^2 \leq \rho D_\rho[\psi] \cdot K_1^2 \frac{dD_\rho[\psi]}{d\rho}$$

for almost all ρ , or

$$\frac{k_1^2}{K_1^2} \leq \frac{d(\log D_\rho[\psi])}{d(\log \rho)}.$$

This shows, exactly as in Lemma 2, that $\rho^{-\alpha} D_\rho[\psi]$ is an increasing function of ρ , where $\alpha = k_1^2/K_1^2$. Therefore

$$\rho^{-\alpha} D_\rho[\psi] \leq R^{-\alpha} D_R[\psi] \leq R^{-\alpha} K_3^2$$

by (7.5). This proves the lemma.

Lemma 4.
$$r^2 \int_{s_r^*} (\psi_\xi^2 + \psi_\eta^2) d\theta \leq Lr^\alpha.$$

Proof. Set $g(r) = \int_{s_r^*} (p^2 + q^2) d\theta$, where the integration with respect to θ is to be taken between ϵ and $\pi - \epsilon$. We have, for $\rho_1 < \rho < R/8$,

$$\begin{aligned} \rho^2 g(\rho) - \rho_1^2 g(\rho_1) &= \int_{\rho_1}^{\rho} (r^2 g)' dr = 2 \iint_{\rho_1}^{\rho} r(p^2 + q^2) dr d\theta + 2 \iint_{\rho_1}^{\rho} r^2(pp_r + qq_r) dr d\theta \\ &\leq 2 \iint_{\rho_1}^{\rho} (p^2 + q^2) r d\theta dr \cdot 2\rho + \left(\iint_{\rho_1}^{\rho} (p^2 + q^2) r dr d\theta \right)^{\frac{1}{2}} \left(\iint_{\rho_1}^{\rho} (p_r^2 + q_r^2) r dr d\theta \right)^{\frac{1}{2}} \\ &\leq 2D_\rho[\psi] + 2\rho(D_\rho[\psi]^{\frac{1}{2}}(D_\rho[p] + D_\rho[q])^{\frac{1}{2}}) \end{aligned}$$

or

$$(12.7) \quad \rho^2 g(\rho) - \rho_1^2 g(\rho_1) \leq 2D_\rho[\psi] + 2\rho(D_\rho[\psi])^{\frac{1}{2}} \cdot 2K_1^{\frac{1}{2}}$$

by (9.4) of Theorem 5. The right hand side is estimated by $L\rho^\alpha$ by Lemma 3. Since $\int_0^\rho rg(r) dr$ converges, we must have $\liminf_{r \rightarrow 0} r^2 g(r) = 0$, so that a sequence $r_i \rightarrow 0$ can be found for which $r_i^2 g(r_i) \rightarrow 0$. Selecting this sequence for ρ_1 in (12.7), one obtains

$$(12.8) \quad \rho^2 g(\rho) \leq L\rho^\alpha.$$

The integration with respect to θ in the definition of $g(r)$ was taken between ϵ and $\pi - \epsilon$. Now, let $\epsilon \rightarrow 0$ and the lemma is proved.

In the above proof, we have assumed the existence of second derivatives. Actually, if we do not wish to use this, by taking for p and q the difference quotients $[\psi(\xi + h, \eta) - \psi(\xi, \eta)]/h$ and $[\psi(\xi, \eta + h) - \psi(\xi, \eta)]/h$, and using (8.2) of Theorem 3 in place of (9.4) of Theorem 5, the inequality (12.8) is obtained with L independent of h . The limit $h \rightarrow 0$ may now be taken, and the lemma is proved.

Lemmas 1, 3, 4 provide the necessary background for the discussion of (12.5). By virtue of (7.6), inequality (12.5) becomes

$$(12.9) \quad k_1 D_r[Z] - L_1^{\frac{1}{2}} r^{\alpha/2} D_r[Z]^{\frac{1}{2}} \leq L_2^{\frac{1}{2}} r^{\alpha/2} D_r[Z]^{\frac{1}{2}} + K_1 D_r[Z]^{\frac{1}{2}} \cdot \left(r \frac{dD_r[Z]}{dr} \right)^{\frac{1}{2}}$$

or

$$(12.10) \quad \frac{k_1}{K_1} D_r[Z]^{\frac{1}{2}} - L_3 r^{\alpha/2} \leq \left(r \frac{dD_r[Z]}{dr} \right)^{\frac{1}{2}}.$$

The exponent α in Lemmas 3, 4 depends on the ratio k_1/K_1 . Select β as the smaller of the two quantities $\alpha, (k_1/2K_1)^2$, and use $L_3 r^{\alpha/2} = L_3 r^{(\alpha-\beta)/2} r^{\beta/2} \leq L_3 R^{(\alpha-\beta)/2} \cdot r^{\beta/2} =$

$L_4 r^{\beta/2}$ for $r \leq R$. Then we can replace the term $L_3 r^{\alpha/2}$ by $L_4 r^{\beta/2}$ in the inequality (12.10). Now suppose that for some r 's we would have

$$\frac{k_1}{2K_1} D_r[Z]^{\frac{1}{2}} \geq L_4 r^{\beta/2}.$$

Then for these r 's the left hand side is $\geq [k_1/(2K_1)] D_r[Z]^{\frac{1}{2}}$, and the inequality becomes

$$\left(\frac{k_1}{2K_1}\right)^2 D_r[Z] \leq r \frac{dD_r[Z]}{dr}$$

or

$$\frac{d \log (r^{-\beta} D_r[Z])}{d \log r} \geq \left(\frac{k_1}{2K_1}\right)^2 - \beta \geq 0.$$

Thus $r^{-\beta} D_r[Z]$ is a monotonically increasing function of r , and once it exceeds $((2K_1 L_4)/k_1)^2$ it remains larger than this quantity, and then

$$r^{-\beta} D_r[Z] \leq (R/8)^{-\beta} D_{R/8}[Z] \leq (R/8)^{-\beta} \cdot K_5$$

by (8.2). Thus

$$r^{-\beta} D_r[Z] \leq \max \left[\left(\frac{2K_1 L_4}{k_1}\right)^2, (R/8)^{-\beta} \cdot K_5 \right],$$

and we have proved the following main theorem.

Theorem 7. *We have*

$$D_r[Z] \leq K_{11} r^{\beta} \quad \text{for all } r \leq R/8,$$

where K_{11} , β are positive and independent of h .

13. In the above considerations, we have been within the semicircle $\xi^2 + \eta^2 < R^2$, $\eta > 0$ of the (ξ, η) -plane. We have taken semicircles about the point $(0, 0)$ of the boundary. We could equally well take semicircles about any point $(\xi_0, 0)$ of the boundary with exactly the same conclusion given in Theorem 7 and with the same constants K_{11} , β .

Likewise, we could have been considering full circles about a point (ξ_0, η_0) with radii $\leq \eta_0$. For a full circle, there is just one difference in all of the above considerations and that concerns the appearance of the term $\int Z^2 d\theta$ in (12.5) and the use of Lemma 1. The basic inequality (12.5) was obtained from (8.6), which in turn was obtained from (7.1) by setting

$$\zeta = \mu(\sqrt{(\xi - \xi_0)^2 + (\eta - \eta_0)^2}) \cdot Z.$$

In place of this, set $\zeta = \mu(\sqrt{(\xi - \xi_0)^2 + (\eta - \eta_0)^2}) \cdot (Z - c)$ where c is any constant. The result is (8.6) with Z replaced by $Z - c$. In the inequality (12.5), the terms $\int_{S_r^*} Z^2 r d\theta$ are to be replaced by $\int_{S_r^*} (Z - c)^2 r d\theta$, where S_r^* denotes now the full circumference $(\xi - \xi_0)^2 + (\eta - \eta_0)^2 = r^2$, and $r \leq \eta_0$. Select $c = (1/2\pi) \int_{S_r^*} Z d\theta$, and now (12.2) applies in place of Lemma 1. The same remark applies to (12.6), with $\psi - c'$ replacing ψ , where c' is selected as $(1/2\pi) \int_{S_r^*} \psi d\theta$. There results Lemma 3 and inequality (12.9) leading to Theorem 7 with the same constants K_{11} , β , but now with circles taken about (ξ_0, η_0) .

Limit ξ_0, η_0 to lie in $\xi_0^2 + \eta_0^2 \leq (R/16)^2$, $\eta_0 > 0$. Then

$$(13.1) \quad D_{C_r}[Z] \leq Kr^\beta \text{ for all } r \leq \eta_0.$$

For $r > \eta_0$, let $\bar{C}_r$ denote the intersection of the circle C_r with the region $\eta > 0$, i.e., $\bar{C}_r$ is the point set (ξ, η) satisfying $(\xi - \xi_0)^2 + (\eta - \eta_0)^2 \leq r^2$, $\eta > 0$. The set $\bar{C}_r$ lies in the semicircular region $S_{2r}: (\xi - \xi_0)^2 + \eta^2 \leq 4r^2$, $\eta > 0$ of radius $2r$ and center $(\xi_0, 0)$ on the boundary. Therefore by Theorem 7,

$$D_{\bar{C}_r}[Z] \leq D_{S_{2r}}[Z] \leq K(2r)^\beta$$

as long as $2r \leq R/8$. Thus, with $K' = K_{11} \cdot 2^\beta$, we may state the following theorem.

Theorem 8.

$$(13.2) \quad D_{\bar{C}_r}[Z] \leq K' \cdot r^\beta \text{ for all } r \leq R/16,$$

where $\bar{C}_r$ is the region $(\xi - \xi_0)^2 + (\eta - \eta_0)^2 \leq r^2$, $\eta > 0$, and $\xi_0^2 + \eta_0^2 \leq (R/16)^2$, while β and K' are independent of h and of ξ_0, η_0 .

The same discussion applies to ψ in place of Z , and we obtain

$$(13.3) \quad D_{\bar{C}_r}[\psi] \leq L' r^\beta \text{ for } r \leq R/16$$

in continuation of Lemma 3, with a new constant L' .

In Theorem (13.2), let $h \rightarrow 0$, so that $Z \rightarrow \psi_\xi$. Theorem 8 gives

$$D_{\bar{C}_r}[\psi_\xi] \leq K' \cdot r^\beta.$$

Finally, using the same argument as in Section 9 in proceeding from integrals of $\psi_{\xi\xi}^2 + \psi_{\xi\eta}^2$ to an integral of $\psi_{\eta\eta}^2$, we obtain

$$D_{\bar{C}_r}[\psi_\eta] \leq K'' \cdot r^\beta.$$

We therefore have the following theorem.

Theorem 9. *There are positive constants β and K such that*

$$(13.4) \quad D\bar{c}_r[\psi_\xi] \leq Kr^\beta, \quad D\bar{c}_r[\psi_\eta] \leq Kr^\beta$$

for all r in $r \leq R/16$, with $\bar{C}_r$ defined in Theorem 8.

Finally, Theorem 9 permits the application of the growth theorem of Morrey. The theorem asserts that a function which satisfies a Morrey condition, which ψ_ξ and ψ_η do as a consequence of Theorem 9, likewise satisfies a Hölder condition. For the sake of completeness, the growth theorem of Morrey (obtained in the works of Morrey cited in footnotes 3 and 15) is stated and proved in the Appendix. One obtains the following theorem.

Theorem 10. *There are positive constants β and K such that*

$$(13.5) \quad |\psi_\xi(P) - \psi_\xi(Q)| \leq K \cdot \overline{PQ}^{\beta/2}, \quad |\psi_\eta(P) - \psi_\eta(Q)| \leq K \cdot \overline{PQ}^{\beta/2},$$

where P, Q are two points in $\xi^2 + \eta^2 \leq (R/16)^2$, $\eta > 0$, and $\overline{PQ}$ is the distance between P and Q .

The constant K in (13.5) depends only on K in (13.4) and on β . Theorem 10 asserts in particular that the quantities ψ_ξ, ψ_η are uniformly continuous functions throughout $\xi^2 + \eta^2 \leq (R/16)^2$, $\xi > 0$; they therefore approach limits as $\eta \rightarrow 0$ which are continuous functions of ξ . Since $\psi(\xi'', \eta) - \psi(\xi', \eta) = \int_{\xi'}^{\xi''} \psi_\xi(\xi, \eta) d\xi$, the passage to the limit $\eta \rightarrow 0$ and the fact that $\psi = 0$ for $\eta = 0$ shows that the limit of $\psi_\xi(\xi, \eta)$ as $\eta \rightarrow 0$ is indeed zero. Thus, ψ_ξ, ψ_η exist and are continuous in the closed domain $\xi^2 + \eta^2 \leq (R/16)^2$, $\eta \geq 0$ including the boundary, and satisfy (13.5).

Transforming back to the (x, y) -plane in place of the (ξ, η) -plane establishes the fact that ψ_x and ψ_y are continuous in the closure of a neighborhood of the point P_0 of the boundary B^* , and satisfy a Hölder condition. By a Heine-Borel argument, there is a strip about the boundary in which this is true, and we have the following theorem.

Theorem 11. *There is a strip in the region B surrounding the boundary B^* , and positive constants β and K , such that*

$$(13.6) \quad |\psi_x(P) - \psi_x(Q)| \leq K \cdot \overline{PQ}^{\beta/2}, \quad |\psi_y(P) - \psi_y(Q)| \leq K \cdot \overline{PQ}^{\beta/2}$$

for any points P, Q in the strip.

We have been considering the behavior of ψ_x, ψ_y near the boundary. The same discussion, however, applies in a much simpler version in the interior as well. For then, we remain in the (x, y) -plane and $e \equiv f \equiv 0$. See the Appendix. The following theorem is obtained.

Theorem 12. *If B_1 is any bounded subregion of B , there are positive constants β and K such that*

$$(13.7) \quad |\psi_x(P) - \psi_x(Q)| \leq K \cdot \overline{PQ}^{\beta/2}, \quad |\psi_y(P) - \psi_y(Q)| \leq K \cdot \overline{PQ}^{\beta/2}$$

for any points P, Q in B_1 .

V. DEPENDENCE ON PARAMETERS. COMPLETION OF PROOF

14. The case will now be considered in which the integrand of the variational problem (4.5) depends on parameters, which we shall denote by a . The integrand is

$$(14.1) \quad F(p, q; a) - F(p_0, q_0; a) - F_p(p_0, q_0; a)(p - p_0) - F_q(p_0, q_0; a)(q - q_0).$$

The following will be supposed:

- i) the parameters a vary in a compact range of values;
- ii) $F(p, q; a)$, $F_p(p, q; a)$, $F_q(p, q; a)$ depend continuously on p, q, a ;
- iii) p_0, q_0 depend continuously on a , and vary over a bounded range of values;
- iv) there are common positive constants k, K such that (4.2) remains valid for all a .

As a consequence of ii), iii) we may say that the integrand (14.1) depends continuously on p, q, a .

The integral (4.5) with the integrand (14.1) will be denoted by $I[\psi; a]$ and the unique solution to the variational problem (4.5) by $\psi(x, y; a)$. We have

$$(14.2) \quad I[\psi(x, y; a); a] \leq I[\bar{\psi}(x, y); a]$$

for any admissible function $\bar{\psi}(x, y)$. Selecting $\bar{\psi}(x, y)$ to be the specific function constructed in (4.7), inequality (14.2) yields

$$(14.3) \quad \iint_B [(\psi_x - p_0)^2 + (\psi_y - q_0)^2] dx dy \leq \frac{K}{k} \iint_B [(\bar{\psi}_x - p_0)^2 + (\bar{\psi}_y - q_0)^2] dx dy \leq \bar{K}_3^2.$$

For all a in its range of values, the right hand integral in (14.3) can be estimated in terms of a single bound K_3^2 , by virtue of iii). Thus, the bound K_3^2 in (7.5) is obtained independently of a .

All of the constants appearing in the theorems of Parts II, III, IV depend on the constants k_1, K_1, K_3, K_4 appearing in Theorem 2, and these in turn depend on k, K of (4.2), on the boundary B^* of B , and on the bound K_3^2 for the Dirichlet integral of ψ given in (7.5). Because of (14.3) and of iii), this bound K_3^2 is independent of a . Thus, all of the bounds obtained in Parts II, III, IV and in the Appendix are independent of a .

The first derivatives $\psi_x(x, y; a)$, $\psi_y(x, y; a)$ are equicontinuous throughout B by virtue of Theorem 12 and inequalities (10.7), (10.8) near the point at infinity. They are also uniformly bounded throughout B because of this and of (10.9). The functions $\psi(x, y; a)$, which vanish on B^* , are therefore also uniformly bounded and equicontinuous in any bounded subdomain of B .

Let $a_1, i = 1, 2, \dots, n, \dots$ be a sequence of a 's converging to a given value $\bar{a}$.

There is by the theorem of Arzela a subsequence, again denoted by a_i , such that $\psi(x, y; a_i)$ and $\psi_x(x, y; a_i)$, $\psi_y(x, y; a_i)$ converge, uniformly in every bounded subdomain of B , to a function $\psi(x, y)$ and to functions which are necessarily the derivatives $\psi_x(x, y)$, $\psi_y(x, y)$ of ψ .¹¹ We shall prove that $\psi(x, y)$ is the solution $\psi(x, y; \bar{a})$ to the minimum problem for $I[\psi; \bar{a}]$.

Let $\chi_1(x, y)$, $\chi_2(x, y)$ be the functions $x \cdot \lambda(r)$, $y \cdot \lambda(r)$, respectively, where $\lambda(r)$ is the specific function defined in (4.7). Then

$$\iint_B \left[\left(\frac{\partial \chi_1}{\partial x} - 1 \right)^2 + \left(\frac{\partial \chi_1}{\partial y} \right)^2 \right] dx dy \quad \text{and} \quad \iint_B \left[\left(\frac{\partial \chi_2}{\partial x} \right)^2 + \left(\frac{\partial \chi_2}{\partial y} - 1 \right)^2 \right] dx dy$$

are finite. Every function admissible in the variational problem (4.5) which makes $I[\phi]$ finite has the form

$$\chi(x, y) + p_0 \chi_1(x, y) + q_0 \chi_2(x, y)$$

where $\chi(x, y)$ vanishes on the boundary B^* and has a finite Dirichlet integral $\iint_B (\chi_x^2 + \chi_y^2) dx dy$. For the values a_i , $i = 1, \dots$ and $\bar{a}$ of a let p_0 , q_0 have the values p_0^i , q_0^i and $\bar{p}_0$, $\bar{q}_0$. For a fixed such function $\chi(x, y)$, inequality (14.2) is

$$(14.4) \quad I[\psi(x, y; a_i); a_i] \leq I[\chi + p_0^i \chi_1 + q_0^i \chi_2; a_i].$$

Now, the integrand in $I[\chi + p_0^i \chi_1 + q_0^i \chi_2; a_0]$ is

$$\leq K \left\{ \left[\frac{\partial \chi}{\partial x} + p_0^i \left(\frac{\partial \chi_1}{\partial x} - 1 \right) + q_0^i \left(\frac{\partial \chi_2}{\partial x} \right) \right]^2 + \left[\frac{\partial \chi}{\partial y} + p_0^i \frac{\partial \chi_1}{\partial y} + q_0^i \left(\frac{\partial \chi_2}{\partial y} - 1 \right) \right]^2 \right\},$$

and inserting the fact that p_0^i , q_0^i are uniformly bounded, we see that the integrand is $\leq$ a summable function. The limit $i \rightarrow \infty$ can therefore be performed under the integral sign with the result that

$$(14.5) \quad I[\chi + p_0^i \chi_1 + q_0^i \chi_2; a_i] \rightarrow I[\chi + \bar{p}_0 \chi_1 + \bar{q}_0 \chi_2; \bar{a}].$$

In any bounded subregion B_1 of B , we have

$$I_{B_1}[\psi(x, y; a_i); a_i] \leq I[\psi(x, y; a_i); a_i] \leq I[\chi + p_0^i \chi_1 + q_0^i \chi_2; a_i],$$

where the subscript B_1 refers to the region over which the integration is to be performed. In B_1 , $\psi(x, y; a_i)$ together with its first derivatives converge uniformly to $\psi(x, y)$ and its first derivatives. Therefore

$$I_{B_1}[\psi(x, y; \bar{a})] = \lim_{i \rightarrow \infty} I_{B_1}[\psi(x, y; a_i); a_i] \leq I[\chi + \bar{p}_0 \chi_1 + \bar{q}_0 \chi_2; \bar{a}]$$

¹¹ Because $\psi(x'', y; a_i) - \psi(x', y; a_i) = \int_{x'}^{x''} \psi_x(x, y; a_i) dx$ becomes on letting $i \rightarrow \infty$ $\psi(x'', y) - \psi(x', y) = \int_{x'}^{x''} [\lim_{i \rightarrow \infty} \psi_x(x, y; a_i)] dx$. Similarly for ψ_y .

by (14.5). Letting B_1 become larger and larger, approaching the entire region B , one obtains

$$(14.6) \quad I[\psi(x, y; \bar{a})] \leq I[\chi + \bar{p}_0\chi_1 + \bar{q}_0\chi_2; \bar{a}]$$

The inequality (14.6) holds for every function χ which vanishes on B^* and has a finite Dirichlet integral over B . Inequality (14.6) asserts that $\psi(x, y)$ is the solution to the minimum problem for $I[\psi; \bar{a}]$, so that $\psi(x, y) \equiv \psi(x, y; \bar{a})$.

Since the above is true for any sequence $a_i \rightarrow \bar{a}$, it follows that $\psi(x, y; a)$, $\psi_x(x, y; a)$, $\psi_y(x, y; a)$ converge uniformly in every bounded subdomain of B to $\psi(x, y; \bar{a})$, $\psi_x(x, y; \bar{a})$, $\psi_y(x, y; \bar{a})$ for arbitrary $a \rightarrow \bar{a}$. The following main theorem has been obtained.

Theorem 13. *The solution $\psi(x, y; a)$ to the variational problem for $I[\psi; a]$ and its first derivatives $\psi_x(x, y; a)$, $\psi_y(x, y; a)$ are continuous functions of a and x, y , for all x, y in the closure of the domain B .*

In the neighborhood of the point at infinity, the discussion of Section 10 and in particular the inequalities (10.5), (10.2), together with (14.3) show that the first derivatives $\psi_x(x, y; a)$, $\psi_y(x, y; a)$ are equicontinuous in the neighborhood of the point at infinity as well. Likewise, one shows that the function $\psi(x, y; a) - p_0x - q_0y$ is equicontinuous in the neighborhood of the point at infinity. Theorem 13 may therefore be strengthened to the following theorem.

Theorem 14. *The functions $\psi(x, y; a) - p_0x - q_0y$, $\psi_x(x, y; a)$, $\psi_y(x, y; a)$ are uniformly continuous functions of a and x, y for all (x, y) in the closure of B , including the point at infinity.*

The difference between Theorem 13 and Theorem 14 occurs because B is an unbounded region. From Theorem 13, it would follow that ψ , ψ_x , ψ_y are uniformly continuous functions of a, x, y for x, y in a bounded subset of the closure of B . Actually, for the main purpose of the present paper it is only required to know that the first derivatives ψ_x , ψ_y are continuous functions of a, x, y for (x, y) on the boundary B^* . This is contained in Theorem 13.

15. We have seen in Section 10 that the difference quotient

$$Z = [\psi(x + h, y) - \psi(x, y)]/h$$

is a monotonic function, i.e., in any subdomain Z attains its maximum and minimum on the boundary of the subdomain. Letting $h \rightarrow 0$, we see that ψ_x has this monotonicity property. Thus, throughout the domain $B + B^*$, the function ψ_x attains its maximum and its minimum on the boundary B^* . Similarly for ψ_y , and, by making a linear coordinate transformation, for the derivative $\psi_{x'}$ in any fixed direction x' .

The quantity $|\text{grad } \psi| = (\psi_x^2 + \psi_y^2)^{\frac{1}{2}}$ attains its maximum on the boundary B^* . For, at any particular point P in B , choose x' in the direction of $\text{grad } \psi$ at P . Then

$$|\text{grad } \psi| \text{ at } P = |\psi_{x'}(P)| \leq \max_{B^*} |\psi_{x'}| \leq \max_{B^*} |\text{grad } \psi|.$$

This shows that the maximum of $|\operatorname{grad} \psi|$ is attained on B^* .

Theorem 15. *The quantity $V(a) = \max_B |\operatorname{grad} \psi(x, y; a)|$ is a continuous function of a .*

Proof. The theorem is a consequence of Theorem 14, or of Theorem 13 if one makes use of the fact that the maximum of $|\operatorname{grad} \psi|$ is attained on the boundary B^* .

This theorem was required in Part I, Section 3. The argument of Section 3 establishes the main theorem of the paper, stated in Section 1.

VI. FLOW WITH CIRCULATION. EXTENSIONS.

16. We have considered flows with no circulation, i.e., for which the conjugate function ϕ to ψ is single-valued. To obtain the case in which the conjugate function ϕ is not single-valued, we introduce a variational problem different from (4.5) in that a different expression is subtracted from $F(\psi_x, \psi_y)$. We shall first construct a function $S(x, y)$ which will describe the behavior of $\psi(x, y)$ at infinity and which will be called the singularity function.

The differential equation to be satisfied by ψ is (4.3). Consider instead, for the determination of $S(x, y)$, the linear differential equation

$$(16.1) \quad F_{pp}(p_0, q_0)S_{xx} + 2F_{pq}(p_0, q_0)S_{xy} + F_{qq}(p_0, q_0)S_{yy} = 0$$

with constant coefficients. A solution of this is

$$\log [F_{qq}(p_0, q_0)x^2 - 2F_{pq}(p_0, q_0)xy + F_{pp}(p_0, q_0)y^2].$$

Accordingly, select the origin of the coordinate system (x, y) so as to lie inside the finite region bounded by B^* . Define the singularity function $S(x, y)$ by

$$(16.2) \quad S(x, y) = p_0 x + q_0 y - \frac{\sigma}{4\pi\Delta^{\frac{1}{2}}} \log [F_{qq}(p_0, q_0)x^2 - 2F_{pq}(p_0, q_0)xy + F_{pp}(p_0, q_0)y^2]$$

where $\Delta = F_{pp}(p_0, q_0)F_{qq}(p_0, q_0) - F_{pq}(p_0, q_0)^2$ and σ is a constant determining the circulation. Concerning $S(x, y)$, we see that $S_x \rightarrow p_0, S_y \rightarrow q_0$ as $x^2 + y^2 \rightarrow \infty$; and S has no singularities in B , with S_x, S_y uniformly bounded in B .

In place of (4.3), consider the variational problem of minimizing

$$(16.3) \quad J[\psi] = \iint_B [F(\psi_x, \psi_y) - F(S_x, S_y) - A(x, y)(\psi_x - S_x) - B(x, y)(\psi_y - S_y)] dx dy$$

among all functions $\psi(x, y)$ which vanish on the boundary B^* , where

$$(16.4) \quad \begin{aligned} A(x, y) &= F_p(p_0, q_0) + F_{pp}(p_0, q_0)(S_x - p_0) + F_{pq}(p_0, q_0)(S_y - q_0), \\ B(x, y) &= F_q(p_0, q_0) + F_{qp}(p_0, q_0)(S_x - p_0) + F_{qq}(p_0, q_0)(S_y - q_0). \end{aligned}$$

The functions $A(x, y)$, $B(x, y)$ have the following properties:

$$(16.5) \quad A_x + B_y = 0,$$

$$(16.6) \quad \iint_B [[A - F_p(S_x, S_y)]^2 + [B - F_q(S_x, S_y)]^2] dx dy < \infty,$$

$$(16.7) \quad \iint_B [A_x^2 + A_y^2 + B_x^2 + B_y^2] dx dy < \infty.$$

The first (16.5) follows from the fact that $S(x, y)$ satisfies the differential equation (16.1). The second (16.6) follows from the fact that near the point at infinity, at which $S_x \rightarrow p_0$, $S_y \rightarrow q_0$, $F_p(S_x, S_y)$ and $F_q(S_x, S_y)$ differ from $A(x, y)$ and $B(x, y)$ respectively by terms which are quadratic in $S_x - p_0$, $S_y - q_0$; therefore $[F_p(S_x, S_y) - A(x, y)]^2$ and $[F_q(S_x, S_y) - B(x, y)]^2$ are of the fourth degree in $S_x - p_0$, $S_y - q_0$, and a simple calculation shows that integrals over the domain B of fourth degree expressions in $S_x - p_0$, $S_y - q_0$ are convergent. A calculation likewise establishes (16.7). The properties (16.5), (16.6), (16.7) are the principal properties of the functions $A(x, y)$, $B(x, y)$.

The Euler equation for the variational problem (16.3) coincides with the Euler equation for the integrand $F(\psi_x, \psi_y)$ alone, the extra terms in (16.3) having no influence because of (16.5).

By the mean value theorem, we may write

$$\begin{aligned} J[\psi] &= \iint_B [\tilde{F}_{pp}(\psi_x - S_x)^2 + 2\tilde{F}_{pq}(\psi_x - S_x)(\psi_y - S_y) + \tilde{F}_{qq}(\psi_y - S_y)^2] dx dy \\ &\quad + \iint_B [(F_p(S_x, S_y) - A)(\psi_x - S_x) + (F_q(S_x, S_y) - B)(\psi_y - S_y)] dx dy. \end{aligned}$$

By the inequality (4.2) and the Schwarz inequality, we obtain

$$(16.8) \quad KD[\psi - S] + M\dot{D}[\psi - S]^\dagger \geq J[\psi] \geq kD[\psi - S] - MD[\psi - S]^\dagger,$$

where $D[\psi - S]$ is the Dirichlet integral $\iint_B [(\psi_x - S_x)^2 + (\psi_y - S_y)^2] dx dy$, and M^2 is a bound on the integral appearing in (16.6). The inequality (16.8) shows that $J[\psi]$ is convergent if and only if $D[\psi - S]$ is convergent. The existence of a solution to the variational problem (16.3) may be established, as in the Appendix.

The variational condition for the solution $\psi(x, y)$ to the variational problem (16.3) is

$$(16.9) \quad \iint_B \{ (F_p(\psi_x, \psi_y) - A(x, y))\zeta_x + (F_q(\psi_x, \psi_y) - B(x, y))\zeta_y \} dx dy = 0$$

among all functions $\zeta(x, y)$ vanishing on B^* and having a finite Dirichlet integral $\iint (\zeta_x^2 + \zeta_y^2) dx dy$. This is obtained exactly as in Section 5, and uniqueness of a solution also follows. For a bounded subdomain B_1 of B , the additional terms $A(x, y)\zeta_x + B(x, y)\zeta_y$ integrate out to zero by integration by parts and use of (16.5), and the variational condition (5.4) is obtained. Thus, the solution $\psi(x, y)$ has continuous first and second derivatives, as in the Appendix, and all the boundary theorems in Sections 6, 7, 8, 9, 12, 13, 14, 15 apply.

In the neighborhood of the point at infinity, one proceeds from (16.9) and translates h units in the x -direction. There results

$$(16.10) \quad \begin{aligned} & \iint_{B'} (\tilde{F}_{pp}Z_x + \tilde{F}_{pq}Z_y)\zeta_x + (\tilde{F}_{qp}Z_x + \tilde{F}_{qq}Z_y)\zeta_y dx dy \\ & - \iint_{B'} \left[\frac{A(x+h, y) - A(x, y)}{h} \zeta_x + \frac{B(x+h, y) - B(x, y)}{h} \zeta_y \right] dx dy = 0 \end{aligned}$$

where $Z = [\psi(x+h, y) - \psi(x, y)]/h$. The function $\psi(x, y) - S(x, y)$ has a bounded Dirichlet integral; and so has $\psi(x+h, y) - S(x+h, y)$. The function $S(x+h, y) - S(x, y)$ has a bounded Dirichlet integral since $\iint_B (S_{xx}^2 + S_{xy}^2) dx dy$ is convergent. Therefore the function Z has a bounded Dirichlet integral and the first integral in (16.10) is a convergent integral.

Also

$$\iint_B \left[\left(\frac{A(x+h, y) - A(x, y)}{h} \right)^2 + \left(\frac{B(x+h, y) - B(x, y)}{h} \right)^2 \right] dx dy$$

is convergent since $\iint (A_x^2 + B_x^2) dx dy$ is convergent, by property (16.7). The second integral (16.10), which is a convergent integral, is in fact zero by property (16.5). For, consider first the integral taken over the portion of B' inside a circle of radius r . Integration by parts would yield, because of (16.5), and the fact that ζ vanishes on the boundary of B' :

$$\int_r \zeta \left[\frac{A(x+h, y) - A(x, y)}{h} dy - \frac{B(x+h, y) - B(x, y)}{h} dx \right].$$

This is equal to

$$\int_r (\zeta - c) \left[\frac{A(x+h, y) - A(x, y)}{h} dy - \frac{B(x+h, y) - B(x, y)}{h} dx \right]$$

for any constant c since the integral $\int_c (A dy - B dx)$ is independent of the path

by virtue of (16.5). Selecting $c = 1/(2\pi) \int_0^{2\pi} \zeta(r, \theta) d\theta$, the Schwarz inequality shows that the square of this integral is

$$\begin{aligned} &\leq \int_r (\zeta - c)^2 r d\theta \cdot \int \left[\left(\frac{A(x+h, y) - A(x, y)}{h} \right)^2 + \left(\frac{B(x+h, y) - B(x, y)}{h} \right)^2 \right] r d\theta \\ &\leq r D'_r[\zeta] \cdot f(r) \leq M r f(r) \end{aligned}$$

by inequality (12.2), where $D'_r[\zeta]$ is the Dirichlet integral of ζ over the exterior of the circle of radius r , which is bounded by a constant M , and $f(r)$ is the remaining integral involving A, B . Since $\int_r^\infty f(r) dr$ is convergent, which can be

written $\int_r^\infty r f(r) dr / r < \infty$, we must have $\liminf_{r \rightarrow \infty} r f(r) = 0$. Thus, there is a sequence of $r_i, i = 1, 2, \dots$, with $r_i \rightarrow \infty$, such that $r_i f(r_i) \rightarrow 0$. Selecting r to be this sequence of r_i , it follows that the second integral in (16.10) vanishes.

Thus (10.2) again holds, and Section 10 now shows that ψ_x, ψ_y approach a limit as $(x, y) \rightarrow \infty$, and this limit must necessarily be p_0, q_0 respectively.

The circulation of the flow is exactly σ . For, denote the circulation by Σ , so that

$$\Sigma = \int_c (\phi_x dx + \phi_y dy) = \int_c [F_q(\psi_x, \psi_y) dx - F_p(\psi_x, \psi_y) dy]$$

(see Section 11). Also, a simple calculation shows that

$$\int_c (B dx - A dy) = \sigma$$

where σ is the constant used in (16.2) for determining the singularity function $S(x, y)$. Therefore

$$\Sigma - \sigma = \int_c [(F_q(\psi_x, \psi_y) - B) dx - (F_p(\psi_x, \psi_y) - A) dy].$$

But

$$\iint [(F_p(\psi_x, \psi_y) - A)^2 + (F_q(\psi_x, \psi_y) - B)^2] dx dy$$

is convergent, by virtue of (16.6) and of the convergence of the Dirichlet integral of $\psi - S$. The argument in Section 11 then shows that $\Sigma - \sigma = 0$, so that the circulation is indeed σ .

By using the procedure of Section 3, we have therefore proved the following theorem for the case of the flow of an adiabatic fluid. *Let $c_0, u_0, v_0 = 0$ be given at the point at infinity, and let the circulation be σ . Let L be a typical length of the*

obstacle. Then there is a region, depending on the obstacle, in a u_0/c_0 , $\sigma/(c_0L)$ plane, including the origin, such that for any point in this region there exists a unique purely subsonic flow about the obstacle with these values of u_0/c_0 , $\sigma/(c_0L)$. Denote the maximum local Mach number throughout the fluid by M_1 . Then, as the point u_0/c_0 , $\sigma/(c_0L)$ approaches the boundary of this region, M_1 approaches 1.

17. The method we have used makes no use of the polytropic law for the equation of state of the fluid. For an arbitrary equation of state, with $c^2 = dp/d\rho > 0$, then a similar theorem holds. *There is a region C in the (u_0, v_0, σ) -space, containing the origin $(0, 0, 0)$, such that for given (u_0, v_0, σ) in C there exists a unique subsonic flow about the obstacle; denoting the maximum of the local Mach number of this flow by $M_1(u_0, v_0, \sigma)$, then $M_1(u_0, v_0, \sigma) \rightarrow 1$ as $(u_0, v_0, \sigma) \rightarrow$ the boundary of C .*

For an arbitrary variational problem of the form (2.1) which is always of elliptic type, the following theorems apply. If there are positive constants k, K such that (4.2) holds for all p, q , where $F(p, q) = G(p^2 + q^2)$, then a solution exists for all u_0, v_0, σ . If the positive constants k, K do not exist, as occurs in the minimal surface case when $G = \sqrt{1 + p^2 + q^2}$, then we have: *there is a region C , bounded or unbounded, in the (u_0, v_0, σ) -space containing the origin $(0, 0, 0)$, such that a unique solution exists for given (u_0, v_0, σ) in C ; denoting the maximum of $|\text{grad } \Phi|$ by $V(u_0, v_0, \sigma)$, then $V(u_0, v_0, \sigma) \rightarrow \infty$ as $(u_0, v_0, \sigma) \rightarrow$ the boundary of C .* For the case of minimal surfaces, see the work of Bers cited in footnote 1.

18. Consider the variational problem (4.1) in which a parameter a appears in integrand, in the conditions at infinity and on the boundary, and in the boundary B^* (provided that the curvature of B^* remains uniformly bounded). The procedure of the present paper can also be used to study the dependence of the solution on the parameter a , by forming differences in Section 6, etc. with respect to a instead of first displacing in a given direction and then forming differences. This gives properties of the difference quotient $Z = [\psi(x, y; a + h) - \psi(x, y; a)]/h$ and thus of $\psi_a(x, y; a)$.

The general procedure is applicable to other boundary value problems in the flow of a compressible fluid, to integrands of the form $G(x, y, \psi, \psi_x, \psi_y)$, and to other partial differential equations.

VII. APPENDIX

A1. In the Appendix, we discuss briefly some theorems that are relevant for the present work and are known in the literature. We begin with the existence of a solution to the minimum problem (4.5). The existence is established by Tonelli

within a wide class of admissible functions introduced by Tonelli.¹² This class of functions $\psi(x, y)$ is defined as follows:

- i) ψ is continuous in B
- ii) for almost all x , $\psi(x, y)$ is an absolutely continuous function of y ; and for almost all y , $\psi(x, y)$ is an absolutely continuous function of x .
- iii) in any bounded subregion of B , $\iint (\psi_x^2 + \psi_y^2) dx dy$ is convergent.

The condition ii) implies that both ψ_x and ψ_y exist almost everywhere, so that the integral in iii) has a meaning.

Likewise, the integral in (4.5) taken over a bounded region has a meaning and is convergent by virtue of (4.2). When taken over the unbounded region B , the integral $I[\psi]$ is convergent if and only if $\iint_B [(\psi_x - p_0)^2 + (\psi_y - q_0)^2] dx dy$ is convergent.

Consider now the minimum problem (4.5) among all the functions satisfying i), ii), iii) above and vanishing on the boundary B^* of B . Within the class of admissible functions, Tonelli has shown that a solution to the minimum problem exists. This is obtained from a minimizing sequence by theorems of levelling, compactness, and lower semi-continuity, and it is best to refer to footnote 12 for these. The solution is also unique within this class of admissible functions, as in the discussion in Section 5 (because of ii, an admissible function ψ for which $\psi_x = \psi_y = 0$ almost everywhere must be identically zero).

A2. The fact that the solution $\psi(x, y)$ has first derivatives everywhere in B which are continuous is proved by Morrey and also by the author in the works listed in footnote 3. Reference can be made to the cited work of the author, but since this is contained within the body of the present work, we shall give the proof here.

The solution $\psi(x, y)$ satisfies the variational condition (5.3) or (5.4). Remaining in (x, y) -plane, in the neighborhood of a given point in B , we proceed as in Sections 6, 7, 8. These paragraphs have been obtained from the variational condition (6.2), while the variational condition (5.3) or (5.4) is in fact simpler since the independent variables x, y are not involved explicitly. Thus, Theorem 1, 2 result, where F, x, y replaces G, ξ, η and in Theorem 2, $a = \tilde{F}_{pp}, b = \tilde{F}_{pq}, c = \tilde{F}_{qq}, e \equiv f \equiv 0$. Also, consider full circles about a given point instead of semicircles, and Theorem 3 results. Indeed, Theorem 3 is more easily obtained for the present case since $e \equiv f \equiv 0$.

We may proceed as in Sections 12, 13 for the difference quotient Z , but we prefer the following procedure. Consider a closed bounded subdomain B_1 of B , and let the distance from every point of B , to every point of the boundary B^* of B be $>$ a positive quantity R . From the variational condition (7.2), with $e \equiv f \equiv 0$, for the difference quotient $Z = [\psi(x + h, y) - \psi(x, y)]/h$, one

¹² L. TONELLI, *Acta Math.*, **53** (1929), 325-346, and *Ann. scuola norm. sup. Pisa*, **2** (1933), 89-130.

proceeds as in Section 10. But use squares about the point (x_0, y_0) ,

$$S_r^*: \text{Max} (|x - x_0|, |y - y_0|) = r$$

in place of circles, where r varies in the range $0 < r < R/8$. Let $w(r)$ denote the oscillation of Z on S_r^* . Because Z satisfies i), ii) of the class of admissible functions, we have for almost all r

$$w(r) = \int (Z_x dx + Z_y dy),$$

the integration being taken along S_r^* between two points. Then, for almost all r

$$w(r)^2 \leq 8r \int_{S_r^*} (Z_x^2 + Z_y^2) ds$$

and

$$\int_0^{R/8} \frac{w(r)^2}{r} dr \leq 8 \int_0^{R/8} \int_{S_r^*} (Z_x^2 + Z_y^2) ds dr = 8 \iint_{S_{R/8}} (Z_x^2 + Z_y^2) dx dy \leq 8K_5,$$

where ds is the element of arc length on S_r^* , and $S_{R/8}$ is the square region bounded by $S_{R/8}^*$. Using $w(r) \geq w(\rho)$ for $r \geq \rho$, which is a consequence of the monotonicity of Z (see Section 10), one obtains

$$w(\rho)^2 \log \frac{R}{8\rho} \leq 8K_5$$

or

$$(A2.1) \quad w(\rho) \leq \left(\frac{8K_5}{\log [R/(8\rho)]} \right)^{1/2} = \epsilon_1(\rho) \quad \text{for } \rho \leq R/8.$$

Therefore,

$$(A2.2) \quad |Z(P) - Z(Q)| \leq \epsilon_1(\rho) \quad \text{if } \overline{PP_0} \leq \rho, \quad \overline{QP_0} \leq \rho,$$

where P_0 is the point (x_0, y_0) . The bound $\epsilon_1(\rho)$ is independent of the point P_0 in B_1 , so that (A2.2) holds for any two points P, Q in B_1 with $\overline{PQ} \leq \rho$.

The quantity $\epsilon_1(\rho)$ is independent of h , and $\epsilon_1(\rho) \rightarrow 0$ if $\rho \rightarrow 0$. The family of functions $Z = [\psi(x + h, y) - \psi(x, y)]/h$ depending on the parameter h therefore form, by (A2.2), an equicontinuous family of functions. By the theorem of Arzela, a sequence $h_1, h_2, \dots \rightarrow 0$ can be selected so that $[\psi(x + h, y) - \psi(x, y)]/h$ converges uniformly in B_1 to a continuous function $g(x, y)$. It is easily shown¹³ that then $[\psi(x + h, y) - \psi(x, y)]/h$ converges uniformly to $g(x, y)$ for arbitrary $h \rightarrow 0$. Thus, ψ_x exists and is continuous in B_1 . Since B_1 is any closed bounded subdomain of B , this statement holds everywhere in B .

¹³ See page 283 in M. SHIFFMAN, *Annals of Math.*, **48** (1947).

Likewise, by displacing h units in the y -direction instead of the x -direction, and using $Z = [\psi(x, y + h) - \psi(x, y)]/h$, one obtains the same estimate (A2.2) and proves that ψ_y exists and is continuous in B .

For the subdomain B_1 , one obtains from (A2.2) by a passage to the limit $h \rightarrow 0$,

$$(A2.3) \quad |\psi_x(P) - \psi_x(Q)| \leq \epsilon_1(\rho), \quad |\psi_y(P) - \psi_y(Q)| \leq \epsilon_1(\rho) \text{ if } \overline{PQ} \leq \rho.$$

Here $\epsilon_1(\rho)$ is given in (A2.1) and depends on the subdomain B_1 .

To discuss second derivatives of $\psi(x, y)$, we now make use of Lemma 2 in Section 12. This, together with Theorem 3, gives

$$D_r[Z] \leq Kr^\alpha$$

where K, α are independent of h and of the point (x_0, y_0) in B_1 . The Morrey theorem gives the Hölder condition

$$|Z(P) - Z(Q)| \leq K \cdot \overline{PQ}^{\alpha/2}$$

for any two points P, Q in B_1 . Letting $h \rightarrow 0$, one obtains

$$(A2.4) \quad |\psi_x(P) - \psi_x(Q)| \leq K \cdot \overline{PQ}^{\alpha/2}, \quad |\psi_y(P) - \psi_y(Q)| \leq K \cdot \overline{PQ}^{\alpha/2}$$

depending on whether Z has been obtained by displacement in the x -direction or in the y -direction. Therefore, the first derivative ψ_x, ψ_y satisfy a Hölder condition in B_1 . The inequality (A2.4) is a stronger form of (A2.3).

The fact that $\psi(x, y)$ has continuous second derivatives now follows from a theorem of E. Hopf.¹⁴ The solution $\psi(x, y)$ to our variational problem, initially known by the existence theorem of Tonelli to be absolutely continuous in the sense of Tonelli, therefore does indeed have continuous first and second derivatives and satisfies the Euler equation for the variational problem.

A3. Because the theorem of Morrey is of such importance in the present work, we shall state and prove it for the sake of completeness.¹⁵ It is given here for the entire region, and not merely for an interior subregion.

Theorem (Morrey). *Let $Z(x, y)$ be a function of the class considered in A1 above, defined for (x, y) in the interior of a square Σ . Let $\bar{C}_r$ denote the intersection of Σ with a circle of radius r and center in Σ . Suppose that there are positive constants L, α such that*

$$D_{\bar{C}_r}[Z] \leq Lr^\alpha$$

for all $\bar{C}_r$, where $D_{\bar{C}_r}[Z]$ is the Dirichlet integral $\iint_{\bar{C}_r} (Z_x^2 + Z_y^2) dx dy$. Then

¹⁴ E. HOPF, *Math. Zeits.*, **30** (1929), 404–413.

¹⁵ See C. B. MORREY, *Trans. Amer. Math. Soc.*, **43** (1938), 126–166, and the work cited in footnote 3.

$$|Z(P) - Z(Q)| \leq L' \cdot \overline{PQ}^{\alpha/2}$$

for any two points P, Q in Σ , where $L' = 16 (L/\alpha)^{1/3} / \sqrt{\pi}$.

Proof. Suppose first that the line segment joining P, Q is parallel to one of the sides of Σ . Let A be the midpoint of $\overline{PQ}$, and L the perpendicular bisector of $\overline{PQ}$. We can clearly proceed along L in at least one of the two directions from A a distance equal to $\frac{1}{2} \overline{PQ}$ and remain completely in Σ . Let B be the resulting point on L , so that $\overline{AB} = \frac{1}{2} \overline{PQ}$. Select a point S on AB , and draw the lines PS, QS . We have

$$(A3.1) \quad Z(Q) - Z(P) = \int_{PS} Z_r dr - \int_{QS} Z_r dr$$

where the integrations are taken along PS and QS , respectively, and r in the two integrals represent distances from P and Q , respectively. The angles $\widehat{SPA}$, $\widehat{SQA}$ are equal to each other, and will be denoted by θ . If S moves from A to B , θ varies from 0 to $\pi/4$.

Integrate (A3.1) with respect to θ from 0 to $\pi/4$. There results

$$\frac{\pi}{4} [Z(Q) - Z(P)] = \iint_{PAB} Z_r dr d\theta - \iint_{QAB} Z_r dr d\theta,$$

so that

$$(A3.2) \quad |Z(Q) - Z(P)| \leq \frac{4}{\pi} \iint_{PAB} |\text{grad } Z| dr d\theta + \frac{4}{\pi} \iint_{QAB} |\text{grad } Z| dr d\theta.$$

By the Schwarz inequality, we have

$$(A3.3) \quad \left(\iint_{PAB} |\text{grad } Z| dr d\theta \right)^2 \leq \iint_{PAB} \frac{1}{r^{1-(\alpha/2)}} dr d\theta$$

$$\iint_{PAB} |\text{grad } Z|^2 r^{1-(\alpha/2)} dr d\theta \leq \frac{\pi}{4} \cdot \frac{2}{\alpha} \overline{PB}^{\alpha/2} \cdot \iint_{PAB} \frac{|\text{grad } Z|^2 \cdot r dr d\theta}{r^{\alpha/2}}.$$

Let $D_r'[Z]$ be the Dirichlet integral of Z over the region obtained as the intersection of the triangle PAB with the circle of radius r and center P . Then

$$(A3.4) \quad \iint_{PAB} \frac{|\text{grad } Z|^2 r dr d\theta}{r^{\alpha/2}} \cdot \int_0^{\overline{PB}} \frac{dD_r'[Z]}{r^{\alpha/2}}.$$

By the hypothesis of the theorem, we have $D_r'[Z] \leq Lr^\alpha$, so that $1/(r^{\alpha/2}) \leq (L/D_r'[Z])^{1/2}$. Equation (A3.4) gives

$$\iint_{PAB} \frac{|\text{grad } Z|^2 r dr d\theta}{r^{\alpha/2}} \leq 2L^{1/2} (D_{\overline{PB}}'[Z])^{1/2} \leq 2L \cdot \overline{PB}^{\alpha/2}.$$

Inserting in (A3.3), we obtain

$$(A3.5) \quad \iint_{PAB} |\operatorname{grad} Z| \, dr \, d\theta \leq \left(\frac{\pi L}{\alpha} \cdot \overline{PB}^\alpha \right)^{\frac{1}{\alpha}} \leq \left(\frac{\pi L}{\alpha} \cdot \overline{PQ}^\alpha \right)^{\frac{1}{\alpha}}.$$

The same inequality is obtained for the corresponding integral over the triangle QAB . Insertion in (A3.2) gives

$$(A3.6) \quad |Z(Q) - Z(P)| \leq \frac{8}{\pi^{\frac{1}{\alpha}}} \cdot \left(\frac{L}{\alpha} \right)^{\frac{1}{\alpha}} \cdot \overline{PQ}^{\alpha/2}.$$

The same inequality applies if the line segment PQ is parallel to the other pair of sides of Σ . Finally, the inequality of the theorem is obtained for any pair of points P, Q in Σ , and the theorem is proved.

The theorem and the above proof hold as well if the region over which the function Z is defined is an arbitrary bounded region satisfying the following condition: the boundary is a curve with piecewise continuous tangent line, forming angles at the corners which are not zero or 2π . (The angle 2π may be admitted, if the distance $\overline{PQ}$ is understood within the region). The constant L' then has a different numerical multiple appearing in it.