*The Eigenvalues of Some Stability Problems Involving Viscosity**

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1. Introduction. Small disturbances of the form $\phi(y)e^{i\alpha(x-ct)}$ are assumed to occur in a viscous flow (1) between moving walls, (2) with a symmetrical velocity profile between two fixed walls and (3) in a boundary layer along a flat plate. Corresponding to each problem for the viscous fluid, there is a simpler problem with viscosity zero. It is natural to conjecture that, for large Reynolds numbers, the eigenvalues in the complete viscous problem are related to those in the corresponding inviscid problem. The present investigation justifies the relationship between inviscid and viscous eigenvalues in most cases and in fact expresses these viscous eigenvalues asymptotically with respect to the Reynolds number in terms of the inviscid eigenvalues.

However, in the case of Couette flow, it is known that the inviscid problem has no solution, and yet physical considerations indicate that the viscous problem should at least possess damped solutions. The existence of viscous solutions unrelated to inviscid solutions is confirmed and some of their properties are determined.

We consider cases where the following conditions hold:

(a) Each inviscid flow determines a relationship $c = c_0(\alpha)$ where to each α there are a finite number of bounded values of c . This assumption will be justified for cases (1) and (2) in the Appendix.

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(b) Each such eigenvalue has at most degeneracy two.

(c) If y_1 and y_2 denote the coordinates of the wall in (1), the wall and midstream in (2) and $w(y)$ is the velocity profile, then all of the eigenvalues $c = c_0(\alpha)$ satisfy $\text{Re} \int_{y_1}^{y_2} \sqrt{i(w - c)} dy \neq 0$. No corresponding condition is required in (3).

(d) $c = w(y)$ is assumed analytic and to have a unique inverse.

Under these assumptions for each viscous problem *an asymptotic relationship between c , α , and αR where R is the Reynolds number is determined for large values of αR .*

(I) *In all three cases there exist as many eigenvalues $c = c(\alpha, \alpha R)$ satisfying*

$$c \sim c_0(\alpha) + \frac{c_1(\alpha)}{(\alpha R)^{\frac{1}{3}}} \cdots + \frac{c_n(\alpha)}{(\alpha R)^{\frac{n}{3}}} + \cdots$$

as there are eigenvalues $c_0(\alpha)$ of the inviscid problem. Certain values of $c_0(\alpha)$ are excluded (see III below) but in general this asymptotic relation holds. The functions $c_1(\alpha)$ etc., can all be expressed in terms of the inviscid solution. The expressions for $c_1(\alpha)$ are found in Section 3. If there are no inviscid solutions, e.g., in the case of plane Couette flow, there are no eigenvalues of this type.

(II) *In cases (1) and (2) there exist other eigenvalues unrelated to the inviscid problem.* These satisfy

$$|c - c_n| < \Delta(\alpha R)^{-\frac{1}{3}},$$

where Δ is a constant and c_n is a root of

$$\text{Im} \int_{y_1}^{y_2} \sqrt{i(w - c_n)} dy = \frac{n\pi}{\sqrt{\alpha R}}, \quad \text{case (1) and case (2), antisymmetrical disturbance,}$$

$$\text{Im} \int_{y_1}^{y_2} \sqrt{i(w - c_n)} dy = \frac{(2n + 1)\pi}{2\sqrt{\alpha R}}, \quad \text{case (2), symmetrical disturbance,}$$

for any integer n . As the Reynolds number or the domain of c increases, the number of such eigenvalues increase indefinitely. Since $\text{Im } c_n < 0$ these correspond to damped disturbances.

In case (3) no such eigenvalues occur.

(III) There may or may not exist eigenvalues of the viscous problems (1) and (2) which approach as $\alpha R \rightarrow \infty$ a finite V-shaped strip in the c -plane defined by one branch of $\text{Re} \int_{y_c}^{y_1} \sqrt{i(w - c)} dy = 0$ and one branch of $\text{Re} \int_{y_c}^{y_2} \sqrt{i(w - c)} dy = 0$ where $w(y_c) = c$. Inviscid eigenvalues $c_0(\alpha)$ satisfying this condition are the exceptional cases in (I), cases (1) and (2). $c = w(y_1)$ or $c = w(y_2)$, that is, $y_c = y_1$ or $y_c = y_2$, are the only values of c corresponding to neutral disturbances that satisfy (III). The rest are all stable ($\text{Im } c < 0$).

In case (3) there may or may not be eigenvalues which approach one branch of $\text{Re} \int_{y_c}^{y_1} \sqrt{i(w - c)} dy = 0$. Any inviscid eigenvalue satisfying this condition is

an exceptional case in (I), case (3). Again all such eigenvalues would satisfy $\text{Im } c < 0$ except $c = w(y_1)$.

Finally it is proved that no other eigenvalues exist for αR sufficiently large except those given in (I)–(III).

As a result, except in a vanishingly small neighborhood of $c = w(y_1)$ or $c = w(y_2)$, there exist no unstable or neutral viscous disturbances except those corresponding to unstable or neutral inviscid solutions. The present methods cannot be applied to the neighborhoods of $c = w(y_1)$, $c = w(y_2)$.

In Section 6 the behavior of the flows satisfying (I) and (II) is described.

2. Differential Equations and Boundary Conditions. The amplitudes $\phi(y)$ of small disturbances of the form $\phi(y)e^{i\alpha(x-ct)}$ in incompressible flow satisfy the Orr-Sommerfeld equation:

$$(1) \quad \phi^{iv} - 2\alpha^2 \phi'' + \alpha^4 \phi - i\alpha R [(w - c)(\phi'' - \alpha^2 \phi) - w''\phi] = 0$$

where $w(y)$ is the velocity distribution in the undisturbed flow and R is the local Reynolds number (see [1] for the derivation of equation (1)). We assume that $w(y)$ is an analytic function of the complex variable y and that its inverse is unique in some bounded domain including y_1 and y_2 . α is real, but c and $\phi(y)$ are complex.

This equation has four formal asymptotic solutions [1, p. 126]:

$$(2) \quad A_1(y, c, \alpha, \lambda) \sim e^{\lambda Q(y, c)} \left[\eta(y, c) + \frac{\eta_{11}(y, c, \alpha)}{\lambda} + \dots \right],$$

$$(3) \quad A_2(y, c, \alpha, \lambda) \sim e^{-\lambda Q(y, c)} \left[\eta(y, c) + \frac{\eta_{21}(y, c, \alpha)}{\lambda} + \dots \right],$$

$$(4) \quad U(y, c, \alpha, \lambda) \sim u(y, c, \alpha) + \frac{u_2(y, c, \alpha)}{\lambda^2} + \dots,$$

$$(5) \quad V(y, c, \alpha, \lambda) \sim v(y, c, \alpha) + \frac{v_2(y, c, \alpha)}{\lambda^2} + \dots,$$

where $\lambda = \sqrt{\alpha R}$ and

$$(6) \quad Q(y, c) = \int_{y_c}^y \sqrt{i(w - c)} dy, \quad w(y_c) = c, \quad \eta(y, c) = (-i(w - c))^{-5/4}.$$

The functions $u(y, c, \alpha)$ and $v(y, c, \alpha)$ are any two independent solutions of the inviscid equation obtained from (1) by letting $(\alpha R)^{-1} \rightarrow 0$:

$$(7) \quad (w - c)(\psi'' - \alpha^2 \psi) - w''\psi = 0.$$

As is shown in [2] all the functions of y , c , and α occurring in the asymptotic expressions (2)–(5) are analytic in y , c , and α except at y_c . In this and the next section we shall be concerned only with the formal expansions and we shall use the expressions (2)–(5) as solutions as is done in [1].

We shall consider four different types of boundary conditions arising from three types of main flow. All details will be found in [1, p. 134 et seq.].

Case (1). Flow between solid walls in relative motion. Then $\phi(y_1) = \phi'(y_1) = \phi(y_2) = \phi'(y_2) = 0$, which leads to

$$(8) \quad F_1(\alpha, c, \lambda) = \begin{vmatrix} A_1(y_1) & A_2(y_1) & U(y_1) & V(y_1) \\ A_1'(y_1) & A_2'(y_1) & U'(y_1) & V'(y_1) \\ A_1(y_2) & A_2(y_2) & U(y_2) & V(y_2) \\ A_1'(y_2) & A_2'(y_2) & U'(y_2) & V'(y_2) \end{vmatrix} = 0.$$

Case (2a). Antisymmetrical disturbance in a symmetrical flow between solid walls. Then $\phi(y_1) = \phi'(y_1) = \phi'(y_2) = \phi'''(y_2) = 0$, or

$$(9) \quad F_2(\alpha, c, \lambda) = \begin{vmatrix} A_1(y_1) & A_2(y_1) & U(y_1) & V(y_1) \\ A_1'(y_1) & A_2'(y_1) & U'(y_1) & V'(y_1) \\ A_1'(y_2) & A_2'(y_2) & U'(y_2) & V'(y_2) \\ A_1'''(y_2) & A_2'''(y_2) & U'''(y_2) & V'''(y_2) \end{vmatrix} = 0.$$

Case (2b). Symmetrical disturbance in a symmetrical flow between solid walls. Then $\phi(y_1) = \phi'(y_1) = \phi(y_2) = \phi''(y_2) = 0$, or

$$(10) \quad F_3(\alpha, c, \lambda) = \begin{vmatrix} A_1(y_1) & A_2(y_1) & U(y_1) & V(y_1) \\ A_1'(y_1) & A_2'(y_1) & U'(y_1) & V'(y_1) \\ A_1(y_2) & A_2(y_2) & U(y_2) & V(y_2) \\ A_1''(y_2) & A_2''(y_2) & U''(y_2) & V''(y_2) \end{vmatrix} = 0.$$

Case (3). Flow in the boundary layer along a flat plate. In this case the conditions are that $\phi \rightarrow 0$ as $y \rightarrow \infty$ and $\phi(y_1) = \phi'(y_1) = 0$. There are two solutions of (1) which in general become infinite as $y \rightarrow \infty$ and two which tend to zero. Since $\phi(y) \rightarrow 0$, $\phi(y)$ must be a linear combination of any two solutions that tend to zero for large positive values of $\text{Re } y$. To satisfy the conditions at y_1 , we then obtain the determinantal equation

$$(11) \quad F_4(\alpha, c, \lambda) = \begin{vmatrix} A_1(y_1) & U(y_1) \\ A_1'(y_1) & U'(y_1) \end{vmatrix} = 0$$

for case (3), where $A_1(y)$, $U(y)$ are two solutions of (1) that tend to zero for large positive values of $\text{Re } y$. Condition (11) is somewhat different from that given in [1] but yields the same results.

3. Formal Expansion for c . The next step is to substitute the asymptotic expressions (2)–(5) in (8)–(11) and retain only the dominant terms. For this purpose we shall need to know the signs of the exponents $\operatorname{Re} Q(y_1, c)$, $\operatorname{Re} Q(y_2, c)$. We shall consider values of c lying in a domain $\mathfrak{C}$ such that the domain $\mathfrak{Y}$ of y_c where $w(y_c) = c$ is closed and singly-connected. Furthermore, in $\mathfrak{C}$ we require that we may be able to choose Q so that $\operatorname{Re} Q(y_1, c) > \operatorname{Re} Q(y_2, c)$ throughout $\mathfrak{C}$. In case (3) we require $\operatorname{Re} Q(y_2, c) < 0$ as $y_2 \rightarrow \infty$. The domain $\mathfrak{C}$ will be further described in Section 4.

Expanding (8) formally using (2)–(5) we find since $\lambda = \sqrt{\alpha R} > 0$

$$(12) \quad F_1(\alpha, c, \lambda) = \lambda^2 e^{\lambda Q(y_1, c) - \lambda Q(y_2, c)} \left\{ F_{10}(c, \alpha) + \frac{F_{11}(c, \alpha)}{\lambda} + \dots \right\}$$

with

$$\begin{aligned} F_{10}(c, \alpha) &= -\eta(y_1, c)\eta(y_2, c)Q'(y_1, c)Q'(y_2, c)[u(y_1, c)v(y_2, c) \\ &\quad - v(y_1, c)u(y_2, c)], \\ F_{11}(c, \alpha) &= \eta(y_1, c)\eta(y_2, c)Q'(y_2, c)[u'(y_1, c)v(y_2, c) - v'(y_1, c)u(y_2, c)] \\ &\quad + \eta(y_1, c)Q'(y_1, c)\eta(y_2, c)[u(y_1, c)v'(y_2, c) - v(y_1, c)u'(y_2, c)] \\ &\quad + P(c, \alpha)[u(y_1, c)v(y_2, c) - v(y_1, c)u(y_2, c)], \end{aligned}$$

where $P(c, \alpha)$ here and below represents some bounded function of c and α .

From equation (9) we find

$$(13) \quad F_2(\alpha, c, \lambda) = \lambda^4 e^{\lambda Q(y_1, c) - \lambda Q(y_2, c)} \left[F_{20}(c, \alpha) + \frac{F_{21}(c, \alpha)}{\lambda} + \dots \right]$$

with

$$\begin{aligned} F_{20}(c, \alpha) &= -\eta(y_1, c)\eta(y_2, c)Q'(y_1, c)Q'^3(y_2, c)[u(y_1, c)v'(y_2, c) \\ &\quad - v(y_1, c)u'(y_2, c)], \\ F_{21}(c, \alpha) &= +\eta(y_1, c)\eta(y_2, c)Q'^3(y_2, c)[u'(y_1, c)v'(y_2, c) \\ &\quad - u'(y_2, c)v'(y_1, c)] + P(c, \alpha)[u(y_1, c)v'(y_2, c) \\ &\quad - v(y_1, c)u'(y_2, c)]; \end{aligned}$$

and from equation (10),

$$(14) \quad F_3(\alpha, c, \lambda) = \lambda^3 e^{\lambda Q(y_1, c) - \lambda Q(y_2, c)} \left[F_{30}(c, \alpha) + \frac{F_{31}(c, \alpha)}{\lambda} + \dots \right]$$

with

$$F_{30}(c, \alpha) = -\eta(y_1, c)Q'(y_1, c)\eta(y_2, c)Q'^2(y_2, c) \\ \cdot [u(y_1, c)v(y_2, c) - u(y_2, c)v(y_1, c)],$$

$$F_{31}(c, \alpha) = \eta(y_1, c)\eta(y_2, c)Q'^2(y_2, c)[u'(y_1, c)v(y_2, c) - u(y_2, c)v'(y_1, c)].$$

Finally from equation (11)

$$(15) \quad F_4(\alpha, c, \lambda) = \lambda e^{\lambda Q(y_1, c)} \left[F_{40}(c, \alpha) + \frac{F_{41}(c, \alpha)}{\lambda} + \dots \right]$$

with

$$F_{40}(c, \alpha) = \eta(y_1, c)Q'(y_1, c)u(y_1, c),$$

$$F_{41}(c, \alpha) = \eta(y_1, c)u'(y_1, c) + P(c, \alpha)u(y_1, c).$$

Note that in F_{40} and F_{41} , $u(y, c)$ must be a solution of (7) which tends to zero as $y \rightarrow \infty$.

We now find an appropriate formal expansion for c which will make F_1 , F_2 , F_3 , or F_4 vanish formally. Details will be given only for case (1), $F_1 = 0$. We set

$$(16) \quad c = c_0 + c_1/\lambda + c_2/\lambda^2 + \dots$$

The corresponding expansions for η and Q' follow immediately from (6). The expansion for $u(y_1, c)v(y_2, c) - v(y_1, c)u(y_2, c)$ is given through a Taylor's series:

$$u(y_1, c_0)v(y_2, c_0) - v(y_1, c_0)u(y_2, c_0) + \left(\frac{c_1}{\lambda} + \frac{c_2}{\lambda^2} + \dots \right) \\ (17) \quad \cdot \frac{\partial}{\partial c} [u(y_1, c)v(y_2, c) - v(y_1, c)u(y_2, c)]_{c=c_0} + \left(\frac{c_1^2}{\lambda^2} + \dots \right) \\ \cdot \frac{\partial^2}{\partial c^2} [u(y_1, c)v(y_2, c) - v(y_1, c)u(y_2, c)]_{c=c_0} + \dots$$

The leading term F_{10} in the expansion of $F_1(\alpha, c, \lambda)$ vanishes if and only if we choose $c = c_0(\alpha)$ so that

$$(18) \quad u(y_1, c_0)v(y_2, c_0) - v(y_1, c_0)u(y_2, c_0) = 0.$$

In other words, since u and v are solutions of (7), c_0 is, as shown in [1], the eigenvalue of the inviscid problem, that is, the value of c such that (7) has a solution

which vanishes at $y = y_1$ and $y = y_2$. We may choose u to be a solution of (7) which satisfies*

$$(19) \quad u(y_1, c_0) = u(y_2, c_0) = 0.$$

The coefficient of $1/\lambda$ in the expression $F_{10} + F_{11}/\lambda + \dots$ with c satisfying (16) is found to be

$$(20) \quad \begin{aligned} & - \eta(y_1, c_0)\eta(y_2, c_0)Q'(y_1, c_0)Q'(y_2, c_0)c_1 \\ & \cdot \frac{\partial}{\partial c} [u(y_1, c)v(y_2, c) - v(y_1, c)u(y_2, c)]_{c=c_0} \\ & + \eta(y_1, c_0)\eta(y_2, c_0)Q'(y_2, c_0)[u'(y_1, c_0)v(y_2, c_0) - v'(y_1, c_0)u(y_2, c_0)] \\ & + \eta(y_1, c_0)Q'(y_1, c_0)\eta(y_2, c_0) [u(y_1, c_0)v'(y_2, c_0) - v(y_1, c_0)u'(y_2, c_0)]. \end{aligned}$$

This vanishes if we choose, using (19),

$$(21) \quad c_1 = \frac{Q'^{-1}(y_1, c_0)u'(y_1, c_0)v(y_2, c_0) - Q'^{-1}(y_2, c_0)v(y_1, c_0)u'(y_2, c_0)}{\frac{\partial}{\partial c} [u(y_1, c)v(y_2, c) - v(y_1, c)u(y_2, c)]_{c=c_0}}.$$

In continuing this process the coefficient of $1/\lambda^n$ will determine a linear relation for c_n . The coefficient of c_n in this relation will always be the same as that of c_1 in (20). It vanishes only if the inviscid solution has a multiple eigenvalue c_0 . In the case that c_0 is a double root we may choose u and v as the two eigenfunctions for $c = c_0$. Then F_{11} vanishes identically and c_1 is determined from F_{12} . We assume there are no multiple roots of higher order than two.

We shall determine c_1 from (21) in the case that c_0 is a simple root. From (19)

$$\begin{aligned} & \frac{\partial}{\partial c} [u(y_1, c)v(y_2, c) - v(y_1, c)u(y_2, c)]_{c=c_0} \\ & = \left[\frac{\partial}{\partial c} u(y_1, c_0) \right] v(y_2, c_0) - v(y_1, c_0) \frac{\partial}{\partial c} u(y_2, c_0). \end{aligned}$$

Now by differentiating (7) with respect to c we see that $\partial u/\partial c$ satisfies the non-homogeneous differential equation

$$(w - c) \left[\left(\frac{\partial u}{\partial c} \right)'' - \alpha^2 \frac{\partial u}{\partial c} \right] - w'' \frac{\partial u}{\partial c} = u'' - \alpha^2 u = \frac{w''u}{w - c}.$$

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* Note that u must be continued properly about the singular point (see [1] and [2]) in order to apply (29). Its continuation is determined by the choice of sectors in Section 4.

Thus†

$$\frac{\partial u}{\partial c} = u \int_a^y \frac{w'' w}{W(w-c)^2} dx - v \int_b^y \frac{w'' u^2}{W(w-c)^2} dx,$$

where $W = u'v - w'$ and a and b are functions of c . Using (19) we find

$$\frac{\partial u(y_1, c_0)}{\partial c} v(y_2, c_0) - v(y_1, c_0) \frac{\partial u(y_2, c_0)}{\partial c} = v(y_1, c_0) v(y_2, c_0) \int_{y_1}^{y_2} \frac{w'' u^2}{W(w-c_0)^2} dy.$$

Now W is independent of y and we may evaluate it at y_1 or y_2 :

$$W = u'(y_2, c_0) v(y_2, c_0) = u'(y_1, c_0) v(y_1, c_0).$$

Hence

$$\begin{aligned} \frac{\partial}{\partial c} [u(y_1, c) v(y_2, c) - u(y_2, c) v(y_1, c)]_{c=c_0} &= \frac{v(y_2, c_0)}{u'(y_1, c_0)} \int_{y_1}^{y_2} \frac{w'' u^2(y, c_0)}{(w-c_0)^2} dy \\ (22) \qquad \qquad \qquad &= \frac{v(y_1, c_0)}{u'(y_2, c_0)} \int_{y_1}^{y_2} \frac{w'' u^2(y, c_0)}{(w-c_0)^2} dy. \end{aligned}$$

Note that $v(y_1, c_0) \neq 0$ since u and v are independent. Substituting (22) in (21) we find a simple expression for c_1 involving only the eigenfunction $u(y, c_0)$:

$$(23) \qquad c_1 = \frac{Q'^{-1}(y_1, c_0) u'^2(y_1, c_0) - Q'^{-1}(y_2, c_0) u'^2(y_2, c_0)}{\int_{y_1}^{y_2} \frac{w'' u^2(y, c_0)}{(w-c_0)^2} dy}$$

Asymptotic expansions can be found for c for large values of λ for all the other cases in exactly the same way. The corresponding expressions for c_1 can all be formulated in terms of eigenfunctions of (7) with appropriate inviscid boundary conditions. The results are shown in Table I.

We note that case (3) is slightly different from the others since we have to determine $\left(\frac{\partial u}{\partial c}\right)_{c=c_0}$ explicitly. In

$$\frac{\partial u}{\partial c} = u \int_a^y \frac{w'' w}{W(w-c)^2} dx - v \int_b^y \frac{w'' w}{W(w-c)^2} dx$$

we note that $u(y_1, c_0) = 0$, $u(y, c) \rightarrow 0$ as $y \rightarrow \infty$ and $v(y, c) \rightarrow \infty$ as $y \rightarrow \infty$.

Since $\frac{\partial u}{\partial c} \rightarrow 0$ also we find $b = \infty$. We find then

$$\left(\frac{\partial u}{\partial c}\right)_{\substack{c=c_0 \\ y=y_1}} = -v(y_1, c) \int_{\infty}^{y_1} \frac{w'' u^2}{W(w-c_0)^2} dx = -\frac{1}{u'(y_1, c_0)} \int_{y_1}^{\infty} \frac{w'' u^2}{W(w-c_0)^2} dx.$$

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† The path of integration likewise lies in the sectors determined by u .

TABLE I

	Boundary Conditions for Inviscid Solution	c_1 in Terms of Inviscid Solution
Case (1)	$u(y_1, c_0) = u(y_2, c_0) = 0$	$- [Q'^{-1} u'^2]_{y_1}^{y_2} / \int_{y_1}^{y_2} \frac{w'' u^2}{(w - c_0)^2} dy$
Case (2a)	$u(y_1, c_0) = u'(y_2, c_0) = 0$	$(Q'^{-1} u'^2)_{y=y_1} / \int_{y_1}^{y_2} \frac{w'' u^2}{(w - c_0)^2} dy$
Case (2b)	$u(y_1, c_0) = u(y_2, c_0) = 0$	$(Q'^{-1} u'^2)_{y=y_1} / \int_{y_1}^{y_2} \frac{w'' u^2}{(w - c_0)^2} dy$
Case (3)	$u(y_1, c_0) = 0$ $u(y, c_0) \rightarrow 0$ as $y \rightarrow \infty$	$(Q'^{-1} u'^2)_{y=y_1} / \int_{y_1}^{\infty} \frac{w'' u^2}{(w - c_0)^2} dy.$

Theoretically the coefficients c_2 can also be determined in any of the cases considered, but the expressions are much more complicated since they involve higher terms in the asymptotic expansions (4) and (5) of U and V .

Table I might be used in particular to determine the conditions under which $\text{Im } c$ is positive if c_0 is real. In other words, *using known inviscid solutions in Table I we can determine whether a flow which is neutral for infinite Reynolds number is stable or unstable for large Reynolds numbers.*

4. Justification of Formal Expansion. The first step in justifying the formal asymptotic expansions of Sections 2 and 3 begins with the asymptotic expansion (2)–(5). Wasow [2] has shown that corresponding to each c and α there exists a set of solutions:

$$(24) \quad A_1(y, c, \alpha, \lambda) = e^{\lambda Q(y, c)} \left[\eta(y, c) + \frac{\eta_{11}(y, c, \alpha)}{\lambda} + \cdots + \frac{E(D_c)}{\lambda^{n+1}} \right],$$

$$(25) \quad A_2(y, c, \alpha, \lambda) = e^{-\lambda Q(y, c)} \left[\eta(y, c) + \frac{\eta_{21}(y, c, \alpha)}{\lambda} + \cdots + \frac{E(D_c)}{\lambda^{n+1}} \right],$$

$$(26) \quad U(y, c, \alpha, \lambda) = u(y, c, \alpha) + \frac{u_1(y, c, \alpha)}{\lambda^2} + \cdots + \frac{E(D_c)}{\lambda^{n+1}},$$

$$(27) \quad V(y, c, \alpha, \lambda) = v(y, c, \alpha) + \frac{v_1(y, c, \alpha)}{\lambda^2} + \cdots + \frac{E(D_c)}{\lambda^{n+1}},$$

where $E(D_c)$ represents a function of y, c, α, λ which together with all its derivatives with respect to y is bounded uniformly in λ in every closed subdomain of the open domain D_c of y which depends on c in the following way:

At the point y_c in the complex y -plane where $w(y) = c$ there are three branches of the curves $\text{Re } Q(y, c) = 0$ which issue from y_c with arguments $-\pi/2, \pi/6, 5\pi/6$, respectively.

These branches split the y -plane into three sectors S_1, S_2 and S_3 (see Figure 1). S_1 is always the lower left sector, S_2 the lower right sector and S_3 the upper

sector. D_c is the open domain lying in whichever two sectors contain y_1 and y_2 and bounded on the outside in Wasow's proof by $|Q(y, c)| = \text{constant}$.

In addition to the condition $\text{Re } Q(y_1, c) > \text{Re } Q(y_2, c)$ (see Section 3) throughout the domain $\mathcal{C}$ of c we require that D_c consists of the same two sectors $S_1 +$

FIG. 1

FIG. 2

S_2 , $S_2 + S_3$, or $S_1 + S_2$ for all $c \in \mathcal{C}$. There are then three possible choices of $\mathcal{C}$ determined in the following way.

Consider the y -plane. y_1 and y_2 are two fixed points in it. The point y_c given by $w(y) = c$ moves around the y -plane as c varies. y_c must avoid the points y_1 and y_2 since neither y_1 nor y_2 can lie on the boundary of D_c . Suppose y_c lies in the upper half plane. There are three possibilities for the positions of y_1 and y_2 relative to y_c (see Figure 2). In (a) y_1 and y_2 lie in S_2 . If y_c crosses the upper

branch of $\operatorname{Re} Q(y_1, w(y)) = 0$ into (b), y_1 lies in S_1 and y_2 lies in S_2 . If y_c crosses the upper branch of $\operatorname{Re} Q(y_2, w(y)) = 0$, then y_1 and y_2 lie in S_1 . Thus for all y_c lying in the upper half plane we can satisfy the condition that y_1 and y_2 lie in the same pair of sectors, namely $S_1 + S_2$.

In fact, this condition is also satisfied if we move from (a), (b) and (c) into the positions (d), (e) and (f); that is,

$$\begin{aligned} &\text{in (d), } y_1 \text{ and } y_2 \text{ lie in } S_2, \\ &\text{in (e), } y_1 \text{ and } y_2 \text{ lie in } S_1 + S_2, \\ &\text{in (f), } y_1 \text{ and } y_2 \text{ lie in } S_1. \end{aligned}$$

However, if y_c crosses the curve $\operatorname{Re} Q(y_1, w(y)) = 0$ into (g), y_1 and y_2 lie in $S_2 + S_3$, or if y_c continues from (g) across $\operatorname{Re} Q(y_2, w(y)) = 0$ into (h), then y_1 and y_2 lie in S_3 . Proceeding from (f) into (i) across $\operatorname{Re} Q(y_2, w(y)) = 0$ we find that y_1 and y_2 lie in $S_3 + S_1$. Combining these results we have

$$\begin{aligned} y_1, y_2 &\subset S_1 + S_2 \text{ if } y_c \subset (a) + (b) + (c) + (d) + (e) + (f) \\ y_1, y_2 &\subset S_2 + S_3 \text{ if } y_c \subset (a) + (d) + (g) + (h) \\ y_1, y_2 &\subset S_3 + S_1 \text{ if } y_c \subset (c) + (f) + (i) + (h). \end{aligned}$$

Now let us consider the positions of y_c which also satisfy the requirement $\operatorname{Re} Q(y_1, c) > \operatorname{Re} Q(y_2, c)$. We are at liberty to choose $\operatorname{Re} Q(y, c) < 0$ in either of the two sectors under consideration. For $y_1, y_2 \subset S_1 + S_2$ we choose $\operatorname{Re} Q < 0$ in S_2 . Then, obviously, for y_c in (b) or (e) we will have $\operatorname{Re} Q(y_1, c) > \operatorname{Re} Q(y_2, c)$. This condition will continue to hold for (a), (d), (c) and (f) provided we do not cross a branch of $\operatorname{Re} Q(y_1, c) = \operatorname{Re} Q(y_2, c)$. We will show in Section 5 that such branches occur only in the lower half plane in (d), (h), or (f) and may be expected to occur only for y_c in (h). We shall assume that the latter holds.

One possible choice for the domain of y_c is a closed domain lying in (d), (a), (b), (e), (c), (f). This is indicated in Figure 3 as $\mathcal{Y}_3$. Note that for $\mathcal{Y}_3$, $D_c \subset S_1 + S_2$, $\operatorname{Re} Q(y, c) < 0$ in S_2 . Another possible domain of y_c , for which $D_c \subset S_2 + S_3$ is indicated in Figure 3 as $\mathcal{Y}_1$. It lies in (a), (d), (g), (h) and is cut off by $\operatorname{Re} Q(y_1, c) = \operatorname{Re} Q(y_2, c)$. For this domain we choose $\operatorname{Re} Q(y, c) < 0$ in S_2 . A third choice for the domain of y_c for which $D_c \subset S_3 + S_1$ is indicated as $\mathcal{Y}_2$. We shall refer to any one of these overlapping domains as $\mathcal{Y}$. The corresponding domains in the c -plane are $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ and any of them is a possible choice of $\mathcal{C}$. Note that a V-shaped strip $\mathcal{V}$ defined by two sections of $\operatorname{Re} Q(y_-, c) = 0$ and $\operatorname{Re} Q(y_2, c) = 0$ is excluded as well as a strip $\mathcal{S}$ about $\operatorname{Re} Q(y_1, c) = \operatorname{Re} Q(y_2, c)$.

In case (3), we must have $\operatorname{Re} Q(y, c) < 0$ as $\operatorname{Re} y \rightarrow +\infty$, since A_1 is to tend to zero. There are thus two domains for $\mathcal{C}$ about the point $w(y_1)$. The corresponding overlapping domains $\mathcal{Y}_1$ and $\mathcal{Y}_2$ in the y_c -plane are shown in Figure 4.

We may choose the domains $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ to be sufficiently large as to contain all eigenvalues $c_0(\alpha)$ with $|c_0(\alpha)| < M$ which do not lie on the sections of

$\operatorname{Re} Q(y_1, c) = 0$ and $\operatorname{Re} Q(y_2, c) = 0$ which lie in $\mathcal{U}$. We assume no such eigenvalues lie on $\operatorname{Re} Q(y_i, c) = \operatorname{Re} Q(y_2, c)$.

We shall now show that the four solutions (24)–(27) may be chosen as analytic functions of c for $y = y_1, y = y_2$. From Wasow's proof [2, section 4] using Trjitzinsky's lemma [3, p. 24] it can be shown easily that $A_1(y, c, \alpha, \lambda)$ is that

FIG. 3

FIG. 4

solution of (1) which satisfies four initial conditions at the point p on the outer boundary of D_c where $\operatorname{Re} Q(y, c)$ has its minimum. The initial value of A_1 is given by

$$(28) \quad A_1(p, c, \alpha, \lambda) = e^{\lambda Q(p, c)} \left(\eta(p, c) + \frac{\eta_{11}(p, c, \alpha)}{\lambda} + \dots + \frac{\eta_{1, n-1}(p, c, \alpha)}{\lambda^{n-1}} \right)$$

The first three derivatives at p are given by the first n terms of the asymptotic expansions found by differentiating (24) term by term.

If $p \neq y_c$ were independent of c , $A_1(p, c, \alpha, \lambda)$ and the derivatives at p would be analytic functions of c . Since the differential equation (1) for A_1 has analytic coefficients, it would follow that A_1 would be an analytic function of c . However, p depends on D_c and hence on c . This dependence is, however, rather artificial. By choosing a different domain instead of D_c and a corresponding point instead of p it is possible to carry through Wasow's proof in such a way that the initial conditions corresponding to (28) are analytic in c .

For each c we seek a domain $\bar{D}_c$ in the y -plane with a fixed point $\bar{p}$ on its boundary at which $\operatorname{Re} Q$ is a minimum for $\bar{D}_c$. The point $\bar{p}$ will be independent of c but dependent on the choice of $\mathfrak{C}$. $\bar{p}$ is fixed as any point in the y -plane sufficiently far from y_1 and y_2 so that it lies in whichever of the two sectors of D_c for which $\operatorname{Re} Q < 0$ for all $c \in \mathfrak{C}$. Furthermore, we choose $|y_c - \bar{p}|$ for

FIG. 5

$y_c \in \mathfrak{Y}$ large enough so that $\operatorname{Re} Q(\bar{p}, c) < \operatorname{Re} Q(y_2, c) < \operatorname{Re} Q(y_1, c)$ for all $c \in \mathfrak{C}$ and at the same time choose the outer boundary of D_c , $|Q(y, c)| = \text{constant}$ large enough so that $|Q(\bar{p}, c)| < |Q(z, c)|$ where z is any point on the boundary of D_c . D_c is then always cut by the curve $\operatorname{Re} Q(y, c) = \operatorname{Re} Q(\bar{p}, c)$. We consider the domain $\bar{D}_c \subset D_c$ for which $\operatorname{Re} Q(y, c) \geq \operatorname{Re} Q(\bar{p}, c)$. Then $\bar{p}$ satisfies the required minimum conditions and $\bar{D}_c$ will fulfill the requirements of Trjitzinsky's lemma.

For example, suppose $\mathfrak{Y}$ is $\mathfrak{Y}_3$. Then $D_c \subset S_1 + S_2$ and $\operatorname{Re} Q(y, c) < 0$ in S_2 . $\bar{p}$ is chosen somewhere in the extreme lower right of the y -plane outside $\mathfrak{Y}_3$.

Wasow's proof can now be carried through using $\bar{D}_c$ and $\bar{p}$ instead of D_c and p so that the equation corresponding to (24) holds.

Since $\bar{p}$ is a fixed point lying outside of $\mathfrak{Y}$ it follows that $A_1(y, c, \alpha, \lambda)$ and its derivatives are analytic functions of c at $y = \bar{p}$. From (1) we see that $A_1(y, c, \alpha, \lambda)$ is an analytic function of c for all y , and hence, in particular, $A_1(y, c, \alpha, \lambda)$ and all its derivatives with respect to y at $y = y_1$, $y = y_2$ are analytic in c .

We can show in a similar way by adjusting another section of the boundary

of $\bar{D}_c$ —a section lying in the opposite sector from $\bar{p}$ —that $A_2(y, c, \alpha, \lambda)$, and all its derivatives are analytic functions of c at $y = y_1, y = y_2$.

We turn next to the functions U and V . From Wasow [2, p. 861], we see immediately that since A_1 and A_2 are analytic functions of c , U and V are analytic functions of c provided certain solutions of a second order equation with analytic coefficients are themselves analytic. The latter requirement may be easily proved by using again the analyticity of the initial values of these solutions through Trjitzinsky's lemma.

We therefore find that (24)–(27) hold with $\bar{D}_c$ instead of D_c and we may also conclude that all the functions $E(\bar{D}_c)$ are analytic functions of c for $c \in \mathfrak{C}$, $y = y_1, y = y_2$, since all of the terms in the asymptotic expressions are analytic for $c \in \mathfrak{C}, y = y_1, y = y_2$. This is evident in the case of the leading terms (6). That the higher terms are also analytic can be easily shown (see [1, p. 128]).

Summing up, we have

$$\begin{aligned}
 A_1(y_1, c, \alpha, \lambda) &= e^{\lambda Q(y_1, c)} \left[\eta(y_1, c) + \frac{\eta_{11}(y_1, c, \alpha)}{\lambda} + \cdots + \frac{\varepsilon}{\lambda^{n+1}} \right], \\
 A_2(y_1, c, \alpha, \lambda) &= e^{-\lambda Q(y_1, c)} \left[\eta(y_1, c) + \frac{\eta_{21}(y_1, c, \alpha)}{\lambda} + \cdots + \frac{\varepsilon}{\lambda^{n+1}} \right], \\
 (29) \quad U(y_1, c, \alpha, \lambda) &= u(y_1, c, \alpha) + \frac{u_1(y, c, \alpha)}{\lambda^2} + \cdots + \frac{\varepsilon}{\lambda^{n+1}}, \\
 V(y_1, c, \alpha, \lambda) &= v(y_1, c, \alpha) + \frac{v_1(y, c, \alpha)}{\lambda^2} + \cdots + \frac{\varepsilon}{\lambda^{n+1}},
 \end{aligned}$$

where ε is a generic symbol for a function of c, α, λ , analytic in c for $c \in \mathfrak{C}$ and bounded in λ . Since $\mathfrak{C}$ is closed, ε is uniformly bounded in λ .

Exactly similar relations hold for all derivatives with respect to y and also at the point y_2 instead of y_1 .

Substituting (29) in the determinant $F_1(c, \alpha, \lambda)$, equation (8), we find

$$(30) \quad F_1(c, \alpha, \lambda) = e^{\lambda Q(y_1, c) - \lambda Q(y_2, c)} \left[F_{10}(c, \alpha) + \frac{F_{11}(c, \alpha)}{\lambda} + \cdots + \frac{\varepsilon}{\lambda^{n+1}} \right],$$

where F_{10}, F_{11} , etc. are analytic functions of c given by (12). Note that F_1 is also analytic for $c \in \mathfrak{C}$ since A_1, A_2, U, V are analytic at $y = y_1, y = y_2$.

We now proceed to apply Rouché's theorem in the complex c -plane to find the zeros of $F_1(c, \alpha, \lambda)$ for which $c \in \mathfrak{C}$. Note that α is kept fixed and λ is chosen sufficiently large to satisfy the required conditions.

Let us first consider a fixed contour $\mathcal{O}$ lying inside $\mathfrak{C}$ that surrounds all the zeros of $F_{10}(c, \alpha)$ that lie in $\mathfrak{C}$. We know (see (18)) that the zeros of $F_{10}(c, \alpha)$ correspond to the eigenvalues of an inviscid problem and that all such eigenvalues lie either in one of the possible choices of $\mathfrak{C}$ or on those parts of the curves $\operatorname{Re} Q(y_1, c) = 0, \operatorname{Re} Q(y_2, c) = 0$ which do not lie in $\mathfrak{C}$.

Thus $F_{10}(c, \alpha)$ is bounded away from zero on $\mathcal{O}$ by m_1 where m_1 depends on α and $\mathcal{O}$. Furthermore,

$$\left| \frac{F_{11}(c, \alpha)}{\lambda} + \cdots + \frac{\varepsilon}{\lambda^{n+1}} \right| < \frac{m_2}{\lambda}$$

for c on $\mathcal{O}$ for some constant m_2 .

Hence, by Rouché's theorem for $\lambda > m_2/m_1$, $F_1(c, \alpha, \lambda)$ has the same number of zeros inside $\mathcal{O}$ as $F_{10}(c, \alpha)$. Thus, the viscous problem has the same number of eigenvalues as the inviscid problem in any of the domains $\mathcal{C}$ and hence in all of them together. The only domains excluded are the domains $\mathcal{U}$ and $\mathcal{S}$ (see Figure 3).

Now let us consider the formal asymptotic expansion (16) for the eigenvalue c of the viscous problem. We consider c inside and on the contour $\mathcal{O}_n$ in the c -plane where

$$(31) \quad \left| c - \left(c_0 + \frac{c_1}{\lambda} + \cdots + \frac{c_n}{\lambda^n} \right) \right| = \frac{\Delta}{\lambda^{n+\epsilon}} \quad \text{on } \mathcal{O}_n.$$

Here $0 \leq \epsilon < 1$, $\Delta > 0$ are constants. That is, we set

$$(32) \quad c = c_0 + \frac{c_1}{\lambda} + \cdots + \frac{c_n}{\lambda^n} + \frac{\xi}{\lambda^{n+\epsilon}},$$

where $|\xi| \leq \Delta$.

Note that $\mathcal{O}_n \subset \mathcal{C}$ for sufficiently large λ . Also we assume we may take Δ sufficiently small so that c_0 is the only eigenvalue of the inviscid problem lying inside $\mathcal{O}_n$. Then by (12)

$$(33) \quad F_1(c, \alpha, \lambda) \lambda^{-2} e^{-\lambda Q(y_1, c) + \lambda Q(y_2, c)} \\ = \frac{\xi}{\lambda^{n+\epsilon}} \left(\frac{\partial}{\partial c} [u(y_1, c), v(y_2, c) - v(y_1, c) u(y_2, c)] \right)_{c=c_0} + \frac{\varepsilon}{\lambda^{n+1}}.$$

Now ε is a uniformly bounded analytic function of c . Furthermore, we assume c_0 is a simple root so that $|(\partial/\partial c) [u(y_1, c)v(y_2, c) - u(y_2, c)v(y_1, c)]| \neq 0$ at $c = c_0$ and hence is bounded away from zero on $\mathcal{O}_n$ if Δ is sufficiently small. Hence, by Rouché's theorem, $F_1(c, \alpha, \lambda)$ and ξ have the same number of zeros inside $\mathcal{O}_n$; that is, $F_1(c, \alpha, \lambda)$ has one zero inside $\mathcal{O}_n$. We have found then an eigenvalue c of the viscous problem which lies inside the contour $\mathcal{O}_n$ and hence satisfies the asymptotic condition with arbitrary n

$$\left| c - \left(c_0 + \frac{c_1}{\lambda} + \cdots + \frac{c_n}{\lambda^n} \right) \right| \leq \frac{\Delta}{\lambda^{n+\epsilon}},$$

for a fixed Δ , arbitrary ϵ , with $0 \leq \epsilon < 1$. In other words, the eigenvalues of the viscous problem which do not lie arbitrarily close to (a) $\text{Re } Q(y_1, c) - \text{Re } Q(y_2, c) = \text{Re} \int_{y_1}^{y_2} \sqrt{i(w-c)} dy = 0$ or (b) the V-shaped curve defined by

one branch of $\operatorname{Re} Q(y_1, c) = \operatorname{Re} \int_{y_c}^{y_1} \sqrt{i(w-c)} dy = 0$ and one branch of $\operatorname{Re} Q(y_2, c) = \operatorname{Re} \int_{y_c}^{y_2} \sqrt{i(w-c)} dy = 0$ satisfy the asymptotic relationship

$$c = c_0 + \frac{c_1}{\lambda} + \cdots + \frac{c_n}{\lambda^n} + \frac{R_n}{\lambda^n},$$

where $\lim_{\lambda \rightarrow \infty} R_n = 0$. There will be as many such eigenvalue expansions as there are eigenvalues c_0 of the inviscid problem. Double eigenvalues of the inviscid problem can be treated in the same way. If a formal expansion can be found for eigenvalues corresponding to roots of higher order of the inviscid problem we may expect that the proof of their asymptotic behavior follows.

The methods used here apply equally well to cases (2a), (2b), and (3). The eigenvalues c in any of the domains $\mathfrak{C}$ are related asymptotically to the inviscid eigenvalues $c_0(\alpha)$. These are the results (I) and (III) in Section 1. We shall now prove the result (II) about $\operatorname{Re} Q(y_1, c) - \operatorname{Re} Q(y_2, c) = \operatorname{Re} \int_{y_1}^{y_2} \sqrt{i(w-c)} dy = 0$.

5. Eigenvalues Near the Curve $\operatorname{Re} Q(y_1, c) = \operatorname{Re} Q(y_2, c)$. Within $\mathfrak{U}$, the strip around part of $\operatorname{Re} Q(y_1, c) = 0$ and $\operatorname{Re} Q(y_2, c) = 0$, the asymptotic expansions for the solutions of (1) cannot be used to solve the determinants $F_1(c, \alpha, \lambda)$ etc. On either side of these curves the asymptotic expressions (2)–(5) represent different solutions of (1) since they are valid for different choices of the sectors of D_c . Therefore, the asymptotic expressions for A_1, A_2 , etc., are not analytic functions of c throughout $\mathfrak{U}$.

This is not the case near the curve $\operatorname{Re} Q(y_1, c) = \operatorname{Re} Q(y_2, c)$. On both sides of this curve y_1 and y_2 lie in the same sector. Here we are still able to find an asymptotic expression for $F_1(c, \alpha, \lambda)$ etc., which is valid on and around $\operatorname{Re} \lambda Q(y_1, c) = \operatorname{Re} \lambda Q(y_2, c)$ and is analytic in c .

Let us consider a strip $\mathfrak{S}$ cut off by $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ (see Figure 2) and excluding a small finite neighborhood of $\operatorname{Re} Q(y_1, c) = \operatorname{Re} Q(y_2, c) = 0$. If there are other branches of this curve, similar results will also hold for neighborhoods about them. For this strip we find on substituting (24)–(27) in (8):

$$(34) \quad F_1(\alpha, c, \lambda) = \lambda^2 \bar{F}_{10}(\alpha, c, \lambda) + \lambda \bar{F}_{11}(\alpha, c, \lambda),$$

where

$$\begin{aligned} \bar{F}_{10}(\alpha, c, \lambda) &= 2 \sinh \lambda [Q(y_1, c) - Q(y_2, c)] \\ &\quad \cdot [u(y_1, c)v(y_2, c) - u(y_2, c)v(y_1, c)] Q'(y_1, c)Q'(y_2, c), \\ \bar{F}_{11}(\alpha, c, \lambda) &= \varepsilon \cosh \lambda [Q(y_1, c) - Q(y_2, c)] + \varepsilon \sinh \lambda [Q(y_1, c) - Q(y_2, c)] + \varepsilon, \end{aligned}$$

while from (9)

$$(35) \quad F_{20}(\alpha, c, \lambda) = \lambda^4 \bar{F}_{20}(\alpha, c, \lambda) + \lambda^3 F_{21}(\alpha, c, \lambda),$$

where

$$\begin{aligned}\bar{F}_{20}(\alpha, c, \lambda) &= 2 \sinh \lambda [Q(y_1, c) - Q(y_2, c)] \\ &\quad [u(y_1, c)v'(y_2, c) - u'(y_2, c)v(y_1, c)] Q'(y_1, c) Q'^3(y_2, c), \\ \bar{F}_{21}(\alpha, c, \lambda) &= \varepsilon \cosh \lambda [Q(y_1, c) - Q(y_2, c)] + \varepsilon \sinh \lambda [Q(y_1, c) - Q(y_2, c)] + \varepsilon; \\ \text{and from (10)}\end{aligned}$$

$$(36) \quad F_{30}(\alpha, c, \lambda) = \lambda^3 \bar{F}_{30}(\alpha, c, \lambda) + \lambda^2 \bar{F}_{31}(\alpha, c, \lambda),$$

where

$$\begin{aligned}\bar{F}_{30}(\alpha, c, \lambda) &= 2 \cosh \lambda [Q(y_1, c) - Q(y_2, c)] \\ &\quad \cdot [u(y_1, c)v'(y_2, c) - u'(y_2, c)v(y_1, c)] Q'(y_1, c) Q'^2(y_2, c), \\ \bar{F}_{31}(\alpha, c, \lambda) &= \varepsilon \cosh \lambda [Q(y_1, c) - Q(y_2, c)] + \varepsilon \sinh \lambda [Q(y_1, c) - Q(y_2, c)] + \varepsilon.\end{aligned}$$

In each case, since there are no inviscid eigenvalues inside $\mathcal{S}$, the leading terms vanish only on the curves $\operatorname{Re} [Q(y_1, c) - Q(y_2, c)] = 0$ at the points satisfying by (6)

$$\begin{aligned}(37) \quad \sin \operatorname{Im} \lambda \int_{y_1}^{y_2} \sqrt{i(w-c)} dy &= 0 \quad \text{for } F_1 \text{ and } F_2, \\ \cos \operatorname{Im} \lambda \int_{y_1}^{y_2} \sqrt{i(w-c)} dy &= 0 \quad \text{for } F_3,\end{aligned}$$

This corresponds to

$$\operatorname{Im} \int_{y_1}^{y_2} \sqrt{i(w-c)} dy = \begin{cases} \frac{n\pi}{\lambda}, & \text{for } F_1 \text{ and } F_2 \\ \frac{(2n+1)\pi}{\lambda}, & \text{for } F_3 \end{cases} \quad \text{for } n = 0, 1, 2, \dots$$

In any given length of the curve

$$\operatorname{Re} [Q(y_1, c) - Q(y_2, c)] = 0;$$

that is,

$$(38) \quad \operatorname{Re} \int_{y_1}^{y_2} \sqrt{i(w-c)} dy = 0,$$

there will correspond to each sufficiently large λ a finite number of values of $c = c_n$ which will satisfy (37) for certain values of n . The greater this length is, or the larger λ is, the more roots c_n of (37) there will be. We shall show that

each root of (37) that lies in $\mathcal{S}$ corresponds to an eigenvalue of the viscous problem.

First of all, however, we shall investigate the locus of the curve,

$$\operatorname{Re} \int_{y_1}^{y_2} \sqrt{i(w-c)} \, dy = 0.$$

By the mean value theorem, since the integration may be taken along the real axis,

$$\operatorname{Re} \int_{y_1}^{y_2} \sqrt{i(w-c)} \, dy = (y_2 - y_1) \operatorname{Re} \sqrt{i(w(\bar{y})-c)},$$

where $y_1 \leq y \leq y_2$. The right hand side vanishes only if $\arg \sqrt{i(w(\bar{y})-c)} = \pi/2$ or $3\pi/2$, that is, $\arg (w(\bar{y})-c) = \pi/2$ or $5\pi/2$. Hence, $\operatorname{Im} c < 0$, and $w(\bar{y}) = \operatorname{Re} c$. Since y_c is single-valued, we have $w(y_1) \geq w(\bar{y}) \geq w(y_2)$. Thus, the curve $\operatorname{Re} \int_{y_1}^{y_2} \sqrt{i(w-c)} \, dy = 0$ will lie between $\operatorname{Re} c = w(y_1)$, and $\operatorname{Re} c = w(y_2)$ in the lower half c -plane. Since $\operatorname{Im} c < 0$, any eigenvalues near this curve correspond to *damped solutions*. Note further that they are independent of α .

Now we will show that there is an eigenvalue of the viscous flow problem in case (1) near each root c_n . The proof for cases (2a) and (2b) are identical. Consider a contour $\mathcal{O}_n$ surrounding a point c_n satisfying (37) and given by $c - c_n = \xi/\lambda$, $|\xi| = \Delta > 0$. Δ will be chosen small but independent of λ . On this contour we have $|u(y_1)v(y_2) - u(y_2)v(y_1)| > a_1$, where a_1 is a constant, and by (37)

$$\begin{aligned} \left| \sinh \lambda \int_{y_1}^{y_2} \sqrt{i(w-c)} \, dy \right| &= \left| \sinh \lambda \int_{y_1}^{y_2} \sqrt{i(w-c_n) + i(c_n-c)} \, dy \right|, \\ &= \left| \cosh \int_{y_1}^{y_2} \left(\frac{\xi}{\sqrt{i(w-c_n)}} + O\left(\frac{1}{\lambda}\right) \right) dy \right|, \\ &> a_2 |\xi| = a_2 \Delta \end{aligned}$$

if Δ is a sufficiently small constant, $\lambda > \lambda_0$ say. On the other hand we find

$$\left| \cosh \lambda \int_{y_1}^{y_2} \sqrt{i(w-c)} \, dy \right| < \left| \sinh \int_{y_1}^{y_2} \left(\frac{\xi}{\sqrt{i(w-c_n)}} + O\left(\frac{1}{\lambda}\right) \right) dy \right| < a_3 \Delta,$$

and

$$\left| \sinh \lambda \int_{y_1}^{y_2} \sqrt{i(w-c)} \, dy \right| < \left| \cosh \int_{y_1}^{y_2} \left(\frac{\xi}{\sqrt{i(w-c_n)}} + O\left(\frac{1}{\lambda}\right) \right) dy \right| < a_4$$

so that by (34) $|\bar{F}_{11}| < a_6$ where a_6 is a constant independent of λ . Thus we have

$$|\lambda \bar{F}_{11}| < |\lambda^2 \bar{F}_{10}|$$

provided $\lambda > a_6/(a_2 a_1)$. Thus we may apply Rouché's theorem to (34) since $\bar{F}_1$ and $\bar{F}_{10}$ are analytic in c . F_1 has the same number of zeros within $\mathcal{O}_n$ as $\bar{F}_{10}$, i.e., it has one zero given by (37).

By taking contours in $\mathcal{S}$ of the same size about points that are not roots of (37) and (38) we can show that there are no eigenvalues anywhere else in $\mathcal{S}$. Thus we have shown that the only eigenvalues of the viscous problem lying in the domain $\mathcal{S}$ satisfy

$$|c - c_n| < \frac{\Delta}{\lambda},$$

where c_n satisfies (35) and (36).

This result shows that in cases (1) and (2) for sufficiently large values of the Reynolds number $R = \lambda^2/\alpha$ there will always be eigenvalues $c = c(\alpha, \alpha R)$ lying near the curve $\text{Re} \int_{y_1}^{y_2} \sqrt{i(w - c)} dy = 0$. However, this method cannot tell us which of these eigenvalues has the smallest value of $|\text{Im } c|$, since one of them may lie arbitrarily close to the point given by $\text{Re } Q(y_1, c) = 0$, $\text{Re } Q(y_2, c) = 0$, which is excluded in the above proof.

Eigenvalues of this type have been determined for plane Couette flow between moving walls by Hopf [4] and by Southwell & Chitty [5]. However, their results hold only for sufficiently low Reynolds numbers. Southwell predicts that for larger Reynolds numbers the eigenvalues will be complex. This is confirmed by our theory since there are no real eigenvalues of the corresponding inviscid problem.

We have now proved the second result of Section 1.

6. The Behavior of the Flow. Let us consider only case (1), flow between parallel moving walls, since the other cases can be handled in the same way. As in [1] and [2] we see that any solution of (1) satisfying the appropriate boundary conditions can be represented as a constant multiple of

$$(39) \quad \phi = \begin{vmatrix} A_1(y_1) & A_2(y_1) & U(y_1) & V(y_1) \\ A_1(y) & A_2(y) & U(y) & V(y) \\ A_1(y_2) & A_2(y_2) & U(y_2) & V(y_2) \\ A_1'(y_2) & A_2'(y_2) & U'(y_2) & V'(y_2) \end{vmatrix}$$

Substituting (24)–(27) in (39) we find, using $\operatorname{Re} \lambda Q(y_1, c) \geq \operatorname{Re} \lambda Q(y_2, c) > 0$, for real y ,

$$\begin{aligned}
 \phi = & [e^{\lambda Q(y_1, c) - \lambda Q(y_2, c)} + e^{-\lambda Q(y_1, c) + \lambda Q(y_2, c)}] \\
 (40) \quad & \cdot \left\{ [\lambda \eta(y_1, c) \eta(y_2, c) Q'(y_2, c)] \left[u(y, c) v(y_2, c) - v(y, c) u(y_2, c) + \frac{E}{\lambda} \right] \right. \\
 & \quad \left. - [e^{\lambda Q(y, c) - \lambda Q(y_2, c)} + e^{-\lambda Q(y, c) + \lambda Q(y_2, c)}] \right. \\
 & \quad \cdot \left\{ \lambda \eta(y, c) \eta(y_2, c) Q'(y_2, c) \left[u(y_1, c) v(y_2, c) - u(y_2, c) v(y_1, c) + \frac{E}{\lambda} \right] \right\} \\
 & + [e^{\lambda Q(y_1, c) - \lambda Q(y, c)} - e^{-\lambda Q(y_1, c) + \lambda Q(y, c)}] \\
 & \quad \cdot \left\{ \eta(y, c) \eta(y_1, c) \left[u(y_2, c) v'(y_2, c) - u'(y_2, c) v(y_2, c) + \frac{E}{\lambda} \right] \right\}
 \end{aligned}$$

Usually, as we shall prove, we have

$$\begin{aligned}
 0 < | \operatorname{Re} Q(y_k, c) - \operatorname{Re} Q(y, c) | \\
 (41) \quad & = \left| \operatorname{Re} \int_y^{y_k} \sqrt{i(w - c)} \, dy \right| < \operatorname{Re} \int_{y_2}^{y_1} \sqrt{i(w - c)} \, dy \\
 & = \operatorname{Re} Q(y_1, c) - \operatorname{Re} Q(y_2, c),
 \end{aligned}$$

$k = 1, 2$, in which case, (40) becomes with $\operatorname{Re} \lambda Q(y_1, c) \geq \operatorname{Re} \lambda Q(y_2, c)$

$$\begin{aligned}
 (42) \quad \phi = & e^{\lambda Q(y_1, c) - \lambda Q(y_2, c)} \lambda \eta(y_1, c) \eta(y_2, c) Q'(y_2, c) \\
 & [u(y, c) v(y_2, c) - v(y, c) u(y_2, c) + E/\lambda].
 \end{aligned}$$

That is, ϕ behaves like an inviscid solution (see [1] and [2]).

However, there may be a part of the y -axis where (41) does not hold. We find then that either $e^{-\lambda Q(y_2, c) - \lambda Q(y, c)}$ or $e^{\lambda Q(y_1, c) - \lambda Q(y, c)}$ dominates; that is,

$$(43) \quad -\operatorname{Re} Q(y_2, c) + \operatorname{Re} Q(y, c) > \operatorname{Re} Q(y_1, c) - \operatorname{Re} Q(y_2, c) > 0$$

or

$$(44) \quad \operatorname{Re} Q(y_1, c) - \operatorname{Re} Q(y, c) > \operatorname{Re} Q(y_1, c) - \operatorname{Re} Q(y_2, c) > 0.$$

Let us consider the first of these two alternatives $\operatorname{Re} Q(y, c) - \operatorname{Re} Q(y_1, c) = \operatorname{Re} \int_{y_1}^y \sqrt{i(w - c)} \, dy > 0$ for some values of y between y_1 and y_2 . Since the

function is zero for $y = y_1$ and negative for $y = y_2$, not only the function itself, but its derivative vanishes somewhere between y_1 and y_2 ; i.e.,

$$\frac{d}{dy} \operatorname{Re} \int_{y_1}^y \sqrt{i(w-c)} dy = \operatorname{Re} \sqrt{i(w-c)} = 0 \text{ for } y = y^* \text{ where } y_1 < y^* < y_2.$$

This condition can occur only for $\operatorname{Im} c < 0$ and $\operatorname{Re} c = w(y^*)$ where $y_1 < y^* < y_2$. Thus, the first alternative can occur only for $\operatorname{Im} c < 0$, $w(y_1) < \operatorname{Re} c < w(y_2)$.

These requirements are also sufficient since $\operatorname{Re} c > w(y_1)$, $\operatorname{Im} c < 0$ implies, whenever the integrations can be made along the real axis,

$$\begin{aligned} \frac{d}{dy} \operatorname{Re} \int_{y_1}^y \sqrt{i(w-c)} dy &= \operatorname{Re} \sqrt{i(w-c)} \\ &= |w-c|^{\frac{1}{2}} \cos \left\{ \frac{1}{2} \arg(w-c) + \frac{\pi}{4} \right\} \end{aligned}$$

is positive for $y = y_1$. Hence, $\operatorname{Re} \int_{y_1}^y \sqrt{i(w-c)} dy$ is positive near $y = y_1$ while it is negative near $y = y_2$. For that part of the real axis where $\operatorname{Re} \int_{y_1}^{y_2} \sqrt{i(w-c)} dy > 0$ we find from (40)

$$(45) \quad \phi = e^{-\lambda Q(y,c) + \lambda Q(y_2,c)} \eta(y, c) \eta(y_2, c) Q'(y_2, c) E,$$

since $c = c_0 + O(\lambda^{-1})$.

Comparing (45) and (42) we see that a solution which converges in the range of y satisfying (41) will diverge like $\lambda^{-1} e^{-\lambda Q(y,c) + \lambda Q(y_1,c)}$ where (43) holds.

As we have seen, this situation occurs whenever we may continue solutions along the real axis and when $w(y_1) < \operatorname{Re} c < w(y_2)$, $\operatorname{Im} c < 0$. Thus, for certain parts of either of the domains $y_c \subset \mathcal{Y}_1$ or $y_c \subset \mathcal{Y}_2$ (Figure 3) there exist inner viscous regions even though the asymptotic representations (24)–(27) hold throughout the real axis (cf. [1] and [2]).

A similar situation occurs when (44) holds.

In the special case (Section 5) of eigenvalues lying very close to $\operatorname{Re} Q(y_1, c) = \operatorname{Re} Q(y_2, c)$ equation (43) holds throughout the flow since $\operatorname{Re} \lambda Q(y, c) > \operatorname{Re} \lambda Q(y_1, c) = \operatorname{Re} \lambda Q(y_2, c) > 0$. The only difference is that instead of E we would have λE since $u(y_1, c)v(y_2, c) - u(y_2, c)v(y_1, c) \neq 0$. Note that y_2 and y_1 must be interchanged as we cross $\operatorname{Re} Q(y_1, c) = \operatorname{Re} Q(y_2, c)$ in order to keep the condition $\operatorname{Re} Q(y_1, c) \geq \operatorname{Re} Q(y_2, c)$.

The stability case (2) illustrates the same phenomena. In the case of the boundary layer flow we find only solutions which behave like inviscid solutions.

Appendix

1. The Eigenvalues of the Inviscid Stability Equation are Discrete in Cases (1) and (2). The eigenvalues $c = c_0$ are such that the equation

$$(A1) \quad (w-c)(\phi'' - \alpha^2 \phi) - w''\phi = 0$$

has a solution which satisfies

$$(A2) \quad \phi(y_1) = 0,$$

$$(A3) \quad l_1\phi(y_2) + l_2\phi'(y_2) = 0,$$

where l_1 and l_2 are 0 or 1 depending on the problem in cases (1) or (2).

Let ϕ_1 be a solution of (A1) which satisfies the initial conditions

$$(A4) \quad \phi_1(y_1) = 0, \quad \phi_1'(y_1) = 1.$$

Then ϕ_1 is an analytic function of y and c for $y \subset \tilde{\mathcal{Y}}$, $c \subset \mathcal{C}$ provided $w(y) \neq c$ for $y \subset \tilde{\mathcal{Y}}$, $c \subset \mathcal{C}$ and provided $\tilde{\mathcal{Y}}$ and $\mathcal{C}$ are simply connected domains.

From (A2) and (A3) we see that if c is an eigenvalue

$$(A5) \quad l_1\phi_1(y_2, c) + l_2\phi_1'(y_2, c) = 0.$$

It is important to note one qualification on (A3) and hence on (A5). $\phi(y, c)$ is a multivalued function of y depending on how we pass around the singularity $w(y) = c$. We assume there is only one such point. This path is determined by the viscous problem. In cases (1) and (2) we may allow y_c to lie in either $\mathcal{Y}_1$, $\mathcal{Y}_2$ or $\mathcal{Y}_3$ and continue ϕ_1 from y_1 to y_2 using any path in $\mathcal{U}$. In fact, we do not need to stop at $\text{Re } Q(y_1, c) = \text{Re } Q(y_2, c)$, and $\mathcal{Y}_2$ and $\mathcal{Y}_3$ may overlap.

Then for c lying in the corresponding appropriate domains we see that $\phi_1(y_2, c)$ is analytic in c and similarly $\phi_1'(y_2, c)$ is analytic in c . Thus the left-hand side of (A5) is analytic in c and cannot vanish on any curve in these domains.

All eigenvalues of the inviscid stability problem are therefore discrete except possibly values of c lying on the branches of

$$\text{Re} \int_{y_c}^y \sqrt{i(w-c)} dy = 0 \quad \text{and} \quad \text{Re} \int_{y_c}^y \sqrt{i(w-c)} dy = 0$$

which lie in $\mathcal{U}$.

2. An Upper Bound for the Magnitudes of the Inviscid Eigenvalues. Consider any domain of c for which the inviscid solutions can be found by continuation along the real axis. This is certainly always true for $|c|$ sufficiently large. In fact, it holds except for all values of c in the triangle defined by $\text{Re } Q(y_1, c) = 0$, $\text{Re } Q(y_2, c) = 0$ and $\text{Im } c = 0$. Note that in this section we do not assume that y_c is necessarily a single valued function of c .

We use the function ϕ_1 defined by (A1) and (A4) and set

$$(A6) \quad \phi_1 = \frac{1}{\alpha} \sinh \alpha(y - y_1) + \phi_1^*.$$

Then by (A1) and (A4)

$$(A7) \quad (w - c) (\phi_1^{*''} - \alpha^2 \phi_1^*) = w'' \phi_1^* + \frac{w''}{\alpha} \sinh \alpha(y - y_1)$$

with

$$\phi_1^*(0) = 0, \quad \phi_1^{*'}(0) = 0.$$

From (A7), treated as a non-homogeneous equation corresponding to the homogeneous equation $\phi'' - \alpha^2\phi = 0$, we find

$$\begin{aligned} \phi_1^* &= \frac{\sinh \alpha(y - y_1)}{\alpha} \int_{y_1}^y \frac{w''}{w - c} \cosh \alpha(\tau - y_1) \left[\phi_1^* + \frac{\sinh \alpha(\tau - y_1)}{\alpha} \right] d\tau \\ (A8) \quad &- \frac{\cosh \alpha(y - y_1)}{\alpha} \int_{y_1}^y \frac{w''}{w - c} \sinh \alpha(\tau - y_1) \left[\phi_1^* + \frac{\sinh \alpha(\tau - y_1)}{\alpha} \right] d\tau \\ &= \frac{1}{\alpha} \int_{y_1}^y \frac{w''}{w - c} \sinh \alpha(y - \tau) \left[\phi_1^* + \frac{1}{\alpha} \sinh \alpha(\tau - y_1) \right] d\tau. \end{aligned}$$

Now we will show that $\phi_1^*(y_2) < N(\tilde{c})/c$ for $|c| > \tilde{c}$ and hence that $\phi_1(y_1)$ has no zeros for

$$(A9) \quad |c| > \max \left[\frac{N(\tilde{c})}{\frac{1}{\alpha} \sinh \alpha(y_2 - y_1)}, \tilde{c} \right].$$

Let $M(c)$ be the upper bound of $|\phi_1^*(y, c)|$ on $y_1 \leq y \leq y_2$ and let it be attained at $y = y^*$. From (A8) we find

$$\begin{aligned} (A10) \quad M(c) &\leq \frac{1}{\alpha} \left\| \frac{w''}{w - c} \right\| M(c) (y^* - y_1) \sinh \alpha(y_2 - y_1) \\ &\quad + \frac{1}{\alpha^2} \left\| \frac{w''}{w - c} \right\| (y^* - y_1) \sinh^2 \alpha(y_2 - y_1) \end{aligned}$$

where $\left\| \frac{w''}{w - c} \right\|$ is the maximum value of $\left\| \frac{w''}{w - c} \right\|$ on $y_1 \leq y \leq y_2$. For

$$(A11) \quad |c| > \frac{\|w\|}{1 - m}$$

we have $\|w/c - 1\| > m$ with arbitrary $0 < m < 1$ and hence from (A10)

$$\begin{aligned} (A12) \quad M(c) &\leq \frac{1}{\alpha} \frac{\|w''\|}{|c|m} M(c) (y_2 - y_1) \sinh \alpha(y_2 - y_1) \\ &\quad + \frac{1}{\alpha^2} \frac{\|w''\|}{|c|m} (y_2 - y_1) \sinh^2 \alpha(y_2 - y_1). \end{aligned}$$

We now require

$$(A13) \quad |c| > \frac{\|w''\| (y_2 - y_1) \sinh \alpha(y_2 - y_1)}{\alpha m(1 - n)},$$

where n is an arbitrary positive number less than 1. Then

$$1 - \frac{1}{\alpha |c|} \frac{\|w''\|}{m} \sinh \alpha(y_2 - y_1) (y_2 - y_1) > n > 0,$$

and hence from (A12) we find

$$(A14) \quad M(c) \leq \frac{1}{\alpha^2} \frac{\|w''\|}{|c| mn} (y_2 - y_1) (\sinh^2 \alpha(y_2 - y_1)),$$

and thus

$$(A15) \quad \phi_1^*(y_2, c) \leq \frac{1}{|c| \alpha^2} \frac{\|w''\|}{mn} (y_2 - y_1) \sinh^2 \alpha(y_2 - y_1)$$

for

$$(A16) \quad |c| > \max \left[\frac{\|w''\|}{\alpha m(1-n)} (y_2 - y_1) \sinh \alpha(y_2 - y_1), \frac{\|w\|}{1-m} \right]$$

with $0 < m, n < 1$. Accordingly, by (A6) and (A15) we see that $\phi_1(y_2, c)$ cannot vanish if (A16) holds and

$$(A17) \quad |c| > \frac{1}{\alpha} \frac{\|w''\|}{mn} (y_2 - y_1) \sinh \alpha(y_2 - y_1).$$

Thus there are no eigenvalues for which

$$(A18) \quad |c| > \max \left[\frac{k_1}{m(1-n)}, \frac{k_1}{mn}, \frac{k_2}{1-m} \right],$$

where $k_1 = \|w''\| \alpha^{-1} (y_2 - y_1) \sinh \alpha(y_2 - y_1)$ and $k_2 = \|w\|$. m, n are arbitrary positive numbers less than 1.

In particular, we find the lower bound of the right hand side of (A15) by choosing m and n so all three quantities are equal: $n = \frac{1}{2}$, $m = 2k_1/(2k_1 + k_2)$. Hence, *there exist no inviscid eigenvalues for which*

$$(A19) \quad |c| > \|w\| + 2 \|w''\| \frac{\sinh \alpha(y_2 - y_1)}{\alpha} (y_2 - y_1)$$

where $\|w''\|$, $\|w\|$ represent the maximum values of w'' , w on $y_1 \leq y \leq y_2$.

The inequality (A19) is first of all useful in restricting the domain of c in which we should look, by numerical or other methods, for eigenvalues of the inviscid problem.

Secondly, it shows that there are no inviscid eigenvalues on the curve $\text{Re } Q(y_1, c) = \text{Re } Q(y_2, c)$ for large values of $|c|$. Thus the theory of section 5 certainly holds for $|c|$ sufficiently large.

Finally, inequality (A19) shows that there are only a finite number of eigen-

values lying in any domain of the c -plane. Combined with the results of sections 1-5 we have a pattern for the eigenvalues $c = c(\alpha, \alpha R)$ for fixed α and large values of R . There are isolated eigenvalues c (or y_c) corresponding to the inviscid problem scattered in some bounded domain about the strip $w(y_1) \leq c \leq w(y_2)$. Then there are many eigenvalues spread along the curve $\operatorname{Re} Q(y_1, c) = \operatorname{Re} Q(y_2, c)$. Finally, there is the V-shaped domain about $\operatorname{Re} Q(y_1, c) = 0$, $\operatorname{Re} Q(y_2, c) = 0$ where nothing theoretical can be said about the eigenvalues by the present methods.

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