Uniqueness Theorems for Certain Parabolic Problems

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Two uniqueness theorems are presented for parabolic problems on a semi-infinite strip $S: 0 \leq x \leq 1, 0 \leq t$. The differential equation is $u_{xx} - u_t = N(x, t, u, u_x)$, where N may be nonlinear. The boundary conditions are linear in u_x , may contain u_t linearly, and may be nonlinear in u .

It will be proved first that certain problems of the form

$$u_{xx} - u_t = N(x, t, u, u_x), \quad 0 \leq x \leq 1, 0 < t,$$

$$u_{x0} - a_0(t)u_{x0} = n_0(t, u_0), \quad x = 0, 0 < t,$$

$$u_{x1} + a_1(t)u_{x1} = n_1(t, u_1), \quad x = 1, 0 < t,$$

$$u(x, 0+) = g(x),$$

(where subscript letters denote partial derivatives) cannot have two solutions in a certain natural class of functions u if the functions $a_0(t)$ and $a_1(t)$ are non-negative. Here $u_j \equiv u(j, t)$ for $j = 0, 1$, and similarly for u_{xj} and u_{tj} ; the subscript numbers on a and n are mere indices.

Let P denote any problem of the above form in which the following requirements are met:

1. $a_j(t)$ are continuous for $t > 0$;
 2. $n_j(t, w)$ are continuous and have partial derivatives n_{jw} that are continuous for $0 < t$ and $-\infty < w < \infty$, and are bounded uniformly in t ;
 3. $N(x, t, y, z)$, N_y , and N_z are continuous for $0 \leq x \leq 1, 0 < t$, and $-\infty < y, z < \infty$, and are bounded uniformly in x and t ;
 4. $g(x)$ is continuous.
- “Uniform boundedness” here means specifically that $|w| \leq \rho$ implies $|n_{jw}| \leq$

$\rho_j^*(\rho)$, and $|y| \leq \mu$ and $|z| \leq \nu$ imply $|N_y| \leq \mu^*(\mu, \nu)$ and $|N_z| \leq \nu^*(\mu, \nu)$; ρ_j^* , μ^* , and ν^* are independent of x and t .

Let Σ' denote the class of functions u defined on S so as to have the following properties:

1. u , u_t , u_x , and u_{xx} are continuous for $t > 0$, and $u(x, 0+) = u(x, 0)$;
2. Constants μ and ν exist such that $|u| \leq \mu$ and $|u_x| \leq \nu$ for $t > 0$.

The familiar nonvanishing "singular" solutions of certain linear homogeneous parabolic problems do not have all these properties. It is easily seen that $u(x, 0)$ is continuous if u is in Σ' .

Theorem I. *If $a_0(t) \geq 0$ and $a_1(t) \geq 0$ in a problem P , then P cannot have two solutions in the class Σ' .*

If u and v are two specific functions in Σ' that solve P , then their difference $w = v - u$ is in Σ' and solves a linear homogeneous problem Q that can be obtained by subtracting the respective equations satisfied by u from those satisfied by v . By setting

$$A(x, t; u, v) = \int_0^1 N_y(x, t, y, z) d\lambda,$$

$$B(x, t; u, v) = \int_0^1 N_z(x, t, y, z) d\lambda,$$

$$m_j(t; u, v) = \int_0^1 n_{j\zeta}(t, \zeta) d\lambda,$$

with $y = u + \lambda w$, $z = u_x + \lambda w_x$, $\zeta = u_j + \lambda w_j$ as arguments in the respective integrands, we may write Q in the form

$$w_{xx} - w_t = Aw + Bw_x, \quad 0 \leq x \leq 1, 0 < t,$$

$$w_{x0} - a_0 w_{x0} = m_0 w_0, \quad x = 0, 0 < t,$$

$$w_{x1} + a_1 w_{x1} = m_1 w_1, \quad x = 1, 0 < t,$$

$$w(x, 0+) = 0 = w(x, 0).$$

Now u and v are fixed in Σ' . Hence numbers μ and ν exist such that $|u| \leq \mu$, $|v| \leq \mu$, $|w| \leq 2\mu$, $|u_x| \leq \nu$, $|v_x| \leq \nu$, and $|w_x| \leq 2\nu$ for $0 \leq x \leq 1$ and $0 < t$. These inequalities make it possible, under the restrictions on N and the n_j in a problem P , to choose a function $F(x, t)$ with the property that the substitution $w = ze^F$ in Q yields for z a problem R for which a "maximum principle" can be developed.

First note that

$$-\infty < k_0 = -1 + \operatorname{glb}_{0 < t < \infty} m_0(t) < \infty, \quad -\infty < k_1 = 1 + \operatorname{lub}_{0 < t < \infty} m_1(t) < \infty,$$

and set

$$f(x) = \frac{1}{2}k_1x^2 - \frac{1}{2}k_0(1-x)^2.$$

Then $f'(0) = k_0$ and $f'(1) = k_1$. Observe next that

$$1 \leq c = 1 + \operatorname{lub}_{0 \leq x \leq 1, 0 < t \leq \infty} |A + Bf' - f'^2 - f''| < \infty,$$

and put

$$F(x, t) = ct + f(x).$$

Then $z = we^{-F}$ is in Σ' and solves the following problem R :

$$\begin{aligned} z_{xx} - z_t &= qz + (B - 2f')z_x, & 0 \leq x \leq 1, 0 < t, \\ z_{x0} - a_0 z_{t0} &= p_0 z_0, & x = 0, 0 < t, \\ z_{x1} + a_1 z_{t1} &= -p_1 z_1, & x = 1, 0 < t, \\ z(x, 0+) &= 0 = z(x, 0), \end{aligned}$$

where

$$\begin{aligned} q &= A + Bf' - f'^2 - f'' + c \geq 1, \\ p_0 &= m_0 - k_0 + a_0 c \geq 1, \\ p_1 &= -(m_1 - k_1 - a_1 c) \geq 1, \end{aligned}$$

Now let z solve R and choose θ so that $0 < \theta < \infty$. It will be seen that $z \equiv 0$ on the closed rectangle $D_\theta: 0 \leq x \leq 1, 0 \leq t \leq \theta$. Since θ is arbitrary it will follow that $z \equiv 0$ for $0 \leq x \leq 1, 0 \leq t$. Then $w \equiv 0$ and the theorem will be proved.

To see that $z \leq 0$ on D_θ , assume that $z > 0$ at some point on D_θ . Then $\lambda = \operatorname{lub}_{D_\theta} z(x, t) > 0$. By considering the closed (bounded) subset of D_θ for which $z(x, t) \geq \lambda/2$ one sees that $\lambda = z(\xi, \tau)$ for some point $P(\xi, \tau)$ on D_θ . Since $z(x, 0) = 0$ it is clear that $\tau > 0$. Suppose first that P is interior to D_θ . Then $z_x = 0 = z_t$ at P and, since $q \geq 1$, it then follows from the differential equation that $z_{xx} = q\lambda \geq \lambda > 0$ at P , contradicting the definition of λ . Suppose next that either $\xi = 0$ or $\xi = 1$ ($0 < \tau \leq \theta$). Interpreting the left members of the boundary conditions as directional derivatives and using the conditions $a_0 \geq 0$ and $a_1 \geq 0$ of the theorem, we see that $\xi = 0$ and $\xi = 1$ are impossible; in fact, if A is any point on the vertical boundary of D_θ at which $z > 0$ then, since $p_0 \geq 1$ and $p_1 \geq 1$, there are in any neighborhood of A points interior to D_θ at which the value of z actually exceeds the value at A . Finally, suppose that $\tau = \theta$ and $0 < \xi < 1$. Then $z_x = 0$ and $z_{xx} \leq 0$ at P and the differential equation yields.

$$z_t = z_{xx} - qz \leq -qz = -q\lambda \leq -\lambda < 0;$$

it follows that there are points interior to D_θ at which the value of z exceeds the value λ of z at P , contradicting the definition of λ . We conclude that $z \leq 0$ on D_θ .

An entirely similar argument proves that $z \geq 0$ on D_θ . Hence $z \equiv 0$ on D_θ and the theorem is established.

Application. A special case of the following nonlinear problem P , with a_0 ,

b , c , and d positive constants, is of some interest in the theory of diffusion (cf. [1]):

$$\begin{aligned} u_{xx} - u_t &= (bu - cu^2)_x, & 0 \leq x \leq 1, 0 < t, \\ u_{x0} - a_0 u_{t0} &= du_0 - cu_0^2 + f_0(t), & x = 0, 0 < t, \\ u_{x1} &= cu_1(1 - u_1), & x = 1, 0 < t \\ u(x, 0+) &= g(x). \end{aligned}$$

It follows from the theorem, if $f_0(t)$ is continuous, that such a problem cannot have more than one solution in Σ' . A boundary condition involving a similar nonlinearity, but no time derivative, occurs in recent papers of Mann & Wolf [2] and Roberts & Mann [3].

Another method. The foregoing proof, which uses a classical device in a way suggested by Professor Peter Lax, replaces an argument based on an examination of the function

$$\zeta(t) = e^{-\sigma t}(a_0 w_0^2 + a_1 w_1^2 + \int_0^1 w^2 d\xi),$$

with σ a real constant and $w = v - u$ a function of Σ' solving Q (cf. [4]). Under stronger sufficient conditions, similar to those of Theorem II, it was shown that σ can be taken so large that $\dot{\zeta}(t) \leq 0$, and it then followed that $\zeta(t) \equiv 0$ and hence $w \equiv 0$.

Now it is a disadvantage of Theorem I that membership of u in Σ' requires that u_x be uniformly bounded; this implies in turn, since $u(x, 0+) = u(x, 0)$, that $u(x, 0)$ must be continuous, and P must involve continuous initial data $g(x)$. It is possible to accept discontinuous initial data $u(x, 0+) = u(x, 0) = g(x)$, and enlarge the class Σ' to a class Σ obtained by dropping the requirement (in the definition of Σ') that u_x be bounded, in return for a strengthened condition on N requiring that N_y and N_z be bounded uniformly in x , t , and z .

In the above example, however, $N = (b - 2cu)u_x$ and N_u is not bounded uniformly for $-\infty < u_x < \infty$. Thus the theorem would not apply, even after the revision suggested in the preceding paragraph, to the above example.

It would not apply, more generally, to any problem in which

$$N(x, t, u, u_x) = \frac{dJ(x, t, u)}{dx} = J_x + u_x J_u$$

unless $J_{uu} = 0$. Theorem II below does apply to the above example and can be proved by showing that $\dot{\zeta}(t) \leq 0$.

Let P^* be a problem of the form

$$\begin{aligned} u_{xx} - u_t &= \frac{d}{dx} J(x, t, u), & 0 \leq x \leq 1, 0 < t, \\ u_{x0} - a_0 u_{t0} &= n_0(t, u_0), & x = 0, 0 < t \\ u_{x1} + a_1 u_{t1} &= n_1(t, u_1), & x = 1, 0 < t, \\ u(x, 0+) &= g(x), \end{aligned}$$

where

1. $a_j(j = 0, 1)$ have $\dot{a}_j$ continuous for $t > 0$;
2. $n_j(t, w)$ are continuous and have n_{jw} that are continuous for $0 < t$ and $-\infty < w < \infty$, and are bounded uniformly in t ;
3. $J(x, t, u)$, J_x , J_u , J_{xu} , and J_{uu} are continuous for $t > 0$ and J_u is bounded uniformly in x and t ;
4. $g(x)$ is bounded and of the first Baire category.

Let Σ denote the class of functions defined on S so as to have the following properties:

1. u , u_t , u_x , and u_{xx} are continuous for $t > 0$, and $u(x, 0+) = u(x, 0)$;
2. A constant μ exists such that $|u| \leq \mu$ for $t > 0$.

Now suppose that u and v both solve P^* and belong to Σ . Then their difference is in Σ and solves a problem Q^* which differs from Q only in that the differential equation may now be written

$$(w_x - Kw)_x = w_t,$$

where

$$K(x, t; u, v) = \int_0^1 J_y(x, t, y) d\lambda$$

with $y = u + \lambda v$ as the argument in the integrand.

In order to estimate $\dot{\zeta}$ one integrates $w\dot{w}$ by parts and uses the equations of Q^* to arrive, after a little algebraic work, at the equation

$$\begin{aligned} \dot{\zeta}e^{\sigma t} = & -[\sigma a_0 - \dot{a}_0 + 2(m_0 - K_0)]w_0^2 \\ & -[\sigma a_1 - \dot{a}_1 + 2(K_1 - m_1)]w_1^2 \\ & - \int_0^1 (\sigma - K^2)w^2 d\xi - \int_0^1 [w_x^2 + (w_x - Kw)^2] d\xi, \end{aligned}$$

where, for $j = 0, 1$, $K_j = K(j, t; u(j, t), v(j, t))$. The last term cannot be positive. Since K is bounded for $t > 0$, one can choose $\sigma \geq \text{lub } K^2$, and then the last term but one cannot be positive. Inspection of the first two terms yields the following theorem.

Theorem II. *A problem P^* cannot have two solutions in the class Σ if $a_0(t) \geq 0$ and $a_1(t) \geq 0$ and it is possible to choose a real constant σ such that both the following inequalities hold for $t > 0$:*

$$\dot{a}_0 \leq \sigma a_0 + 2(m_0 - K_0),$$

$$\dot{a}_1 \leq \sigma a_1 + 2(K_1 - m_1).$$

Engineers are often willing to neglect the time derivatives in the boundary conditions, which amounts to setting $\dot{a}_0 = 0 = \dot{a}_1$. Since $m_0 - K_0$ and $K_1 - m_1$

are bounded, it is easy to see that uniqueness follows, when a_0 and a_1 are constants, if both of the following conditions are met:

1. Either $a_0 > 0$, or $a_0 = 0$ and $n_0(t, u) - J(0, t, u)$ does not decrease with u ; and
2. Either $a_1 > 0$, or $a_1 = 0$ and $J_1(1, t, u) - n_1(t, u)$ does not decrease with u .

In the example above $a_0 > 0$ while $a_1 = 0$ and $J(1, t, u) - n_1(t, u) = (b - c)u$. Thus the sufficient condition of Theorem II will be met if $c \leq b$. In practice, in fact, $c \leq b$ holds and the problem P^* of the example therefore cannot have two solutions in Σ (or, a fortiori, in Σ').

We may notice, without elaborating the details, that an expression like $n_0(t, u) - J(0, t, u)$ may increase for u in certain intervals and decrease for u in other intervals. If it is known, as it may be in a diffusion problem for example, that u must lie in a certain interval, then uniqueness will follow if the sufficient conditions of the theorem are met for u in that interval.

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REFERENCES

1. COHEN, KARL, "The Theory of Isotope Separation" (National Nuclear Energy Series, III-1 B), New York, McGraw-Hill (1951) Chap. 3.
2. MANN, W. R., & WOLF, F., "Heat transfer between solids and gases under nonlinear boundary conditions," *Quart. of Appl. Math.*, IX (1951) 163-184.
3. ROBERTS, J. H., & MANN, W. R., "On a Certain Nonlinear Integral Equation of the Volterra Type," *Pac. Jour. of Math.*, I (1951) 431-445.
4. FICKEN, F. A., "Uniqueness Theorems for Certain Parabolic Problems," (Abstract) *Bull. Am. Math. Soc.*, 58 (1952) 192.