Two Hydrodynamic Comparison Theorems

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1. Recently it has become evident that in the study of the classical problem of flows with free boundaries, certain hydrodynamic comparison theorems can be employed to great advantage. The first statement and use of these results occurred in the work of Lavrentieff [1], and more recently they have been presented in simpler form by Serrin [2]. The extension of these theorems to axially symmetric flows was inaugurated by Gilbarg [3], when he proved the analogues of Lavrentieff's Theorems 1 and 1' for such flows.

In the present paper we obtain for axially symmetric flows the analogues of Theorem 2 of [2]. It is important to note that while functiontheoretic methods give the simplest approach to the plane problem, in axial symmetry an entirely different, and seemingly more general, method is necessary. Finally we prove two corollaries which give quantitative estimates of flow speeds at inflection points on streamlines. These corollaries were given by Lavrentieff (Theorems 3 and 3'), but his proofs are hard to understand, apparently for lack of theorems corresponding to those of the present paper. It is assumed that the reader is familiar with [3], for many of our arguments have been shortened by references to proofs there.

In a second paper the author intends to apply the present results to obtain a number of results concerning the geometric behavior of free streamlines, including a determination of the obstacle of given dimensions with least cavity or wake drag.

2. We consider first axially symmetric irrotational flows of an ideal incompressible fluid. These flows are described by a stream function $\Psi(x, y)$, which is

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defined over the flow region of a meridian half plane and which satisfies the equation

$$(1) \quad \Psi_{xx} + \Psi_{yy} - \Psi_y/y = 0.$$

We recall that any non-constant solution of (1) can take on neither a maximum nor a minimum in the interior of its region of definition.

In Theorem 1 we consider flows which occupy an infinite region R , have a well-defined speed U at infinity, and are bounded by streamlines (sources and sinks are allowed on the boundary of R). We will also assume that $\Psi \geq 0$ in R . Such flows will be called *admissible* flows. A boundary point will be called *regular* if it is not on the axis of symmetry, and if it is touched by some circle lying entirely in R .

Theorem 1 (Under-Over Theorem). *Let R_1 and R_2 be two regions occupied by admissible flows, and let S_1 and S_2 denote the respective streamlines $\Psi = 0$. Let S_1 and S_2 have an arc MN in common such that the direction of each flow on MN is from M to N . Furthermore suppose that arcs QM of S_1 and NQ of S_2 , having only the point Q in common (Q may be the point at infinity, in which case the arc QM extends to $-\infty$, and the arc NQ extends to $+\infty$), together with MN bound a region $MNQ \equiv R_3$ interior to both R_1 and R_2 . Then if M and N are regular boundary points,*

$$(2) \quad \frac{V(M, R_1)}{V(N, R_1)} \leq \frac{V(M, R_2)}{V(N, R_2)},$$

the equality holding if and only if the flows are geometrically similar.

Proof. If the flows are similar the theorem is obvious. Otherwise let Ψ_1 and Ψ_2 be the respective stream functions and P be a variable point on MN .² We shall assume that the arc MN does not touch the axis of symmetry, and also that MN is smooth so that $V(P, R)$ exists and is continuous in P . These restrictions will later be removed. Define

$$\Omega_P \equiv V(P, R_1)\Psi_2 - V(P, R_2)\Psi_1$$

in R_3 . Then Ω_P is a solution of (1), and

$$(3) \quad \begin{aligned} \Omega_P &\geq 0 \text{ on } QM, \\ \Omega_P &= 0 \text{ on } MN, \\ \Omega_P &\leq 0 \text{ on } NQ. \end{aligned}$$

¹ $V(P, R)$ denotes the speed at a point P of a flow in R . In axial symmetry,

$$V(P, R) = \frac{1}{y_P} \frac{\partial \Psi}{\partial n} \bigg|_P,$$

where n is measured normal to the streamline passing through P .

² A figure illustrating the situation when $Q = \infty$ will be found in [2], p. 496.

Moreover

$$\left. \frac{\partial \Omega_P}{\partial n} \right|_P = V(P, R_1) \left. \frac{\partial \Psi_2}{\partial n} \right|_P - V(P, R_2) \left. \frac{\partial \Psi_1}{\partial n} \right|_P = 0,$$

so that there is a level line $\Omega_P = 0$ issuing from P into R_3 , which we denote by C_P .

We consider first the case $Q = \infty$. It will become apparent during the proof that C_P may not branch; moreover, after entering R_3 , C_P must either terminate on QM or NQ or extend to ∞ . With this in mind we introduce the set $\mathfrak{N}$ of points P such that either i) C_P terminates on MQ , or ii) C_P extends to $-\infty$, has no point in common with NQ , and

$$(4) \quad V(P, R_1)U_2 - V(P, R_2)U_1 \neq 0,$$

where U_1 and U_2 are the respective speeds at infinity. We also introduce the set $\mathfrak{N}$ of points P such that either i) C_P terminates on NQ , or ii) C_P extends to $+\infty$, has no point in common with QM , and (4) holds. Since C_P , $V(P, R_1)$, and $V(P, R_2)$ all depend continuously on P , it is clear that $\mathfrak{M}$ and $\mathfrak{N}$ are open subsets of MN . There are now three possibilities.

1) $\mathfrak{N} = MN$. Let D be the open set bounded by QM , MN , and C_N . We shall show that $\Omega_N > 0$ in D . Then it will follow that

$$\left. \frac{\partial \Omega_N}{\partial n} \right|_M > 0,^3$$

where n is the inner normal to MN . We easily compute

$$\left. \frac{\partial \Omega_N}{\partial n} \right|_M = y_M [V(N, R_1)V(M, R_2) - V(N, R_2)V(M, R_1)],$$

and (2) is obtained from the preceding inequality.

Since $N \in \mathfrak{N}$, C_N either terminates on QM or extends to $-\infty$. In the first case D is finite, and by (3), $\Omega_N \geq 0$ on the entire boundary of D . Hence by the maximum principle, $\Omega_N > 0$ in D . On the other hand, if C_N extends to $-\infty$ we can choose the origin of coordinates so that D is entirely in the second quadrant. Now it can be shown that, in R_3 ,

$$(5) \quad |\Omega_N(x, y)| \leq Ay^2, A < \infty.^4$$

For fixed $\alpha < 0$, consider the following solution of (1),

$$u_\alpha(x, y) = \alpha xy^2 + \Omega_N(x, y).$$

³ For M is a regular boundary point of D , and $\Omega_N|_M = 0$. See [3], p. 343.

⁴ This is proved in exactly the same manner as condition (2) of [3]. The ϵ of the cited equation is here replaced by the constant A , since the analogous speeds at infinity (i.e., $V(N, R_1)U_2$ and $V(N, R_2)U_1$), although not equal, are bounded.

Using (5), we have in D ,

$$\begin{aligned} u_\alpha(x, y) &\geq \alpha xy^2 - Ay^2, \\ &= y^2(\alpha x - A), \\ &\geq 0, \text{ for } x (<0) \text{ sufficiently large.} \end{aligned}$$

Thus $u_\alpha \geq 0$ on the boundary of D , and for all x sufficiently large, which implies $u_\alpha > 0$ in D . Consequently, α being arbitrary, $\Omega_N > 0$ in D .

2) $\mathfrak{N} = MN$. The proof is a duplication of the preceding one and will be omitted.

3) MN contains points of both $\mathfrak{M}$ and $\mathfrak{N}$. Since $\mathfrak{M}$ and $\mathfrak{N}$ are disjoint open sets, it follows that there is at least one point P^* on MN which is in neither $\mathfrak{M}$ nor $\mathfrak{N}$. Then C_{P^*} extends to ∞ ,⁵ and

$$(6) \quad V(P^*, R_1)U_2 = V(P^*, R_2)U_1.$$

We prove (6) by contradiction. For if (6) does not hold we have

$$\text{Slope of } C_{P^*} = -\left(\frac{\partial \Omega_{P^*}}{\partial x}\right) / \left(\frac{\partial \Omega_{P^*}}{\partial y}\right),$$

and

$$\begin{aligned} \frac{1}{y} \frac{\partial \Omega_{P^*}}{\partial x} \Big|_\infty &= 0, \\ \frac{1}{y} \frac{\partial \Omega_{P^*}}{\partial y} \Big|_\infty &= V(P^*, R_1)U_2 - V(P^*, R_2)U_1 \neq 0. \end{aligned}$$

Hence C_{P^*} is horizontal at ∞ , so that P^* must be in either $\mathfrak{M}$ or $\mathfrak{N}$, contradicting its definition.

Having proved (6) the growth condition on $\Omega_{P^*}(x, y)$ is (see footnote 4)

$$|\Omega_{P^*}(x, y)| \leq \epsilon y^2, (x, y) \in R_3,$$

where $\epsilon = \epsilon(r) \rightarrow 0$ as $r \rightarrow \infty$. Now let D_M and D_N be respectively the open regions bounded by QM , MP^* , C_{P^*} and by C_{P^*} , P^*N , NQ . With the help of (3) and the auxiliary function

$$u_\alpha(x, y) = \alpha y^2 + \Omega_{P^*}(x, y),$$

we easily obtain

$$\Omega_{P^*} > 0 \text{ in } D_M,$$

$$\Omega_{P^*} < 0 \text{ in } D_N.$$

⁵ Should C_{P^*} extend to $-\infty$ and have a point in common with NQ , the inequalities (7) and (8) are obtained as in part 1.

Evaluating the normal derivatives of Ω_{P^*} at M and N we get,

$$(7) \quad V(P^*, R_1)V(M, R_2) - V(P^*, R_2)V(M, R_1) > 0$$

$$(8) \quad V(P^*, R_1)V(N, R_2) - V(P^*, R_2)V(N, R_1) < 0.$$

Inequality (2) now follows immediately from (7) and (8).

The proof when Q is a finite point follows the same outline but is much simpler since there are no difficulties with growth conditions. We will therefore omit this result. It remains only to remove our restrictions on the arc MN . But if MN is not smooth it may be approximated by smooth (indeed analytic) arcs for each of which (2) will hold. Since the speeds at M and N will depend continuously on the alteration of non-adjacent boundary arcs, the inequality (2) must also hold for arcs MN which are not smooth.

We remark here that the above proof applies (with but trivial alteration) equally well to plane flows where Ψ satisfies Laplace's equation. However, the proof we have already given for plane flows [2] is both simpler and shorter than that above, and in addition makes no use of the fact that both flows have well-defined speeds at infinity.

Now consider flows which occupy a finite or infinite region R , are bounded by streamlines (sources and sinks are allowed on the boundary of R), and have the property that in R the stream function is bounded. These flows will be called *strip flows*, and for them we will always take $0 \leq \Psi \leq B$. Such a flow, for example, occurs in a strip domain which is a region bounded by two non-intersecting curves, each having its endpoints at $\pm \infty$. Our second theorem is an extension of Theorem 1 to strip flows, and will be stated and proved for both plane and axially symmetric flows. It is well to note, however, that the result for plane flows may be given an alternate proof following the function-theoretic method introduced in [2] for the proof of the first under-over theorem.

Theorem 2 (Under-Over Theorem). *Let R_1 and R_2 be two regions occupied by plane or axially symmetric strip flows, and let S_1 and S_2 denote the respective streamlines $\Psi = 0$. Let S_1 and S_2 have an arc MN in common such that the direction of each flow on MN is from M to N . Furthermore suppose that either of the following conditions holds:*

A) *Two arcs QM of S_1 and NQ of S_2 , having only the finite point Q in common, together with MN bound a region $MNQ \equiv R_3$ interior to both R_1 and R_2 .*

B) *The streamline $\Psi = B_1$ of the flow in R_1 and the streamline $\Psi = B_2$ of the flow in R_2 have a common arc Q_1Q_2 such that Q_1 is on S_1 , Q_2 is on S_2 , and the arcs Q_1M of S_1 , NQ_2 of S_2 bound, together with MN and Q_1Q_2 , a region $Q_1MNQ_2 \equiv R_3$ interior to both R_1 and R_2 .*

Then if M and N are regular boundary points,

$$(9) \quad \frac{V(M, R_1)}{V(N, R_1)} \leq \frac{V(M, R_2)}{V(N, R_2)},$$

the equality holding if and only if the flows are geometrically similar.

Remark. Figure 1 illustrates a geometric configuration when B) holds. It is clear that $Q_1 = \infty$ is a source for the flow in R_1 , and Q_2 is a sink for the flow in R_2 .

FIG. 1

Proof. If the flows are similar the theorem is obvious, while if A) holds the proof is the same as that for finite points Q of Theorem 1.

Thus suppose B) holds and that the flows are not similar. Let Ψ_1, Ψ_2 be the respective stream functions and P be a variable point on MN . Assuming that MN is smooth and, in the case of axial symmetry, that MN does not touch the axis of symmetry, define

$$\Omega_P \equiv V(P, R_1)\Psi_2 - V(P, R_2)\Psi_1,$$

in R_3 . Then Ω_P is a solution of (1) and

$$\begin{aligned} \Omega_P &\geq 0 && \text{on } Q_1M, \\ (10) \quad \Omega_P &= 0 && \text{on } MN, \\ \Omega_P &\leq 0 && \text{on } NQ_2, \\ \Omega_P &= V(P, R_1)B_2 - V(P, R_2)B_1 = \text{const. on } Q_1Q_2. \end{aligned}$$

Moreover, a level line $\Omega_P = 0$ issues from P into R_3 , which we denote by C_P .

As in the previous proof there are several possibilities to consider. If C_N terminates on Q_1M or if C_M terminates on NQ_2 , the argument is completed as in 1) of the previous proof. If C_N extends to $-\infty$ or terminates on Q_1Q_2 , we let D be the open region bounded by Q_1M , MN , C_N , and Q_1Q_2 . By (10), $\Omega_N \geq 0$ on the boundary of D . Upon assuming that $\Omega_N < 0$ at any point within D we obtain a contradiction (see [3], p. 314). Hence $\Omega_N > 0$, and therefore $\partial\Omega_N/\partial n|_M > 0$, from which we get (9). If C_M extends to $+\infty$ or terminates on Q_1Q_2 , a similar proof is valid.

If none of the preceding cases holds, it is evident from the continuous dependence of C_P , $V(P, R_1)$, and $V(P, R_2)$ on P that there is some point P^* on MN such that C_{P^*} terminates on Q_1Q_2 . Let D_M and D_N be the open regions

bounded by Q_1M , MP^* , C_{P^*} , Q_1Q_2 and by Q_1Q_2 , C_{P^*} , P^*N , NQ_2 . Clearly $\Omega_{P^*} = 0$ on Q_1Q_2 , so that $\Omega_{P^*} \geq 0$ on the boundary of D_M , $\Omega_{P^*} \leq 0$ on the boundary of D_N . Then, ([3], p. 314), $\Omega_{P^*} > 0$ in D_M and $\Omega_{P^*} < 0$ in D_N . The remainder of the argument is that of 3) in the previous proof, and the restrictions on MN are removed as before.

3. Several beautiful corollaries of the preceding theorems are available in case we have explicitly known flows which may be used for comparison purposes. We begin with an observation which is valid for all flows under consideration.

Suppose in the preceding theorems that MN has a continuous curvature, and let s , $0 \leq s \leq s_0$, be a parameter denoting arc length on MN increasing from M to N . If the point P has arc length s , put

$$V_1(s) \equiv V(P, R_1),$$

$$V_2(s) \equiv V(P, R_2).$$

Then we have, for small positive h , $0 < s < s_0$,

$$\frac{V_1(s)}{V_1(s+h)} \leq \frac{V_2(s)}{V_2(s+h)};$$

hence

$$\frac{V_1(s) - V_1(s+h)}{hV_1(s+h)} \leq \frac{V_2(s) - V_2(s+h)}{hV_2(s+h)},$$

and in the limit $h \rightarrow 0$,

$$\frac{d \log V_1(s)}{ds} \geq \frac{d \log V_2(s)}{ds}.$$

It does not seem to be evident that the equality holds only if the flows are similar. On the other hand, we can exclude the equality if we can find two regions R_3 and R_4 , intermediate between R_1 and R_2 , for which

$$\frac{d \log V_3(s)}{ds} > \frac{d \log V_4(s)}{ds}.$$

Corollary 1. Let R be an infinite region of the complex z -plane ($z = x + iy$), bounded by a curve S which extends from $-\infty$ to $+\infty$. Let P be an inflection point on S , and suppose that the arcs $(-\infty, P)$ and $(P, +\infty)$ of S lie on opposite sides of the tangent line L to S at P . Near P let S be represented parametrically by $z = z(s)$ and suppose that $z''(s)$ exists and satisfies a Hölder condition. We may take s to be arc length on S .

Suppose there is a plane flow in R with S as bounding streamline. If $V(s)$ repre-

sents the speed of this flow at the point $z(s)$, and if a direction is assigned to L corresponding to increasing s at P , then

$$\left. \frac{dV(s)}{ds} \right|_P > 0$$

when L is directed into R , and

$$\left. \frac{dV(s)}{ds} \right|_P < 0$$

when L is directed out of R .⁶

Proof. Let S_m , $m = 1, 2, \dots$, be a sequence of curves, with $S_1 = L$, $S_m \rightarrow S$. Since L is tangent to S , we can require that each S_m have a segment in common with L , and that each segment contain P in its interior. We can assume moreover that the arcs $(-\infty, P)$ and $(P, +\infty)$ of each S_m lie on opposite sides of S_{m-1} . Finally, if the curves S_m are given by $z = z_m(s)$, then by hypothesis on S , we can require that $z_m''(s)$ satisfy a uniform Hölder condition and that $z_m''(s) \rightarrow z''(s)$ in some m -independent neighborhood of P .

If R_m denotes the region bounded by S_m , it is evident that we can apply Theorem 1 to each pair of regions R_{m-1} , R_m , $m = 2, 3, \dots$. Thus, if L is directed into R ,

$$(11) \quad 0 = \frac{d \log V_1(s)}{ds} < \frac{d \log V_2(s)}{ds} \leq \dots \leq \frac{d \log V_m(s)}{ds} \leq \dots,$$

provided we choose S_2 to be different from L and such that $dV_2(s)/ds$ may be explicitly computed (for example, S_2 can be taken to be composed of straight segments).

Now let $z(w)$, $z_m(w)$ be the inverse complex velocity potentials of the flows in R , R_m . In virtue of a theorem of Warschawski⁷ we have at P ,

$$z_m'(w) \rightarrow z'(w), \quad z_m''(w) \rightarrow z''(w);$$

hence

$$(12) \quad \frac{d \log V_m(s)}{ds} \rightarrow \frac{d \log V(s)}{ds}.$$

From (11) and (12) we get, after cancellation of a positive factor,

$$\left. \frac{dV(s)}{ds} \right|_P > 0.$$

When L is directed out of R it is sufficient to replace s with $-s$.

⁶ This corollary and the following one are due to Lavrentieff [1]; however, insufficient hypotheses concerning the nature of the curve S at P were stated by him.

⁷ Trans. Amer. Math. Soc. **38** (1935), p. 336.

Corollary 2. Let R be a strip domain, bounded by a straight line and a curve S which satisfies the hypotheses of Corollary 1 (Figure 2).

Suppose there is a plane flow in R . Let $V(s)$ represent the speed of this flow at the point $z(s)$, and assign a direction to L corresponding to increasing s at P . If d denotes the length of the segment of L intercepted by R , then

$$\left. \frac{d \log V(s)}{ds} \right|_P > \frac{1}{d}$$

when L is directed into R , and

$$\left. \frac{d \log V(s)}{ds} \right|_P < -\frac{1}{d}$$

when L is directed out of R .

FIG. 2

Proof. Take the straight bounding streamline as the x -axis, and the intersection of L and the x -axis as the origin. Let R_1 denote that sector of the plane bounded by the x -axis and L which contains the inner normal to S at P . As in Corollary 1, we construct a sequence of regions R_m , $m = 1, 2, \dots$, $R_m \rightarrow R$. The strip flow in R_1 with source or sink at the origin is given by

$$w_1(z) = a \log z + b, \quad a, b = \text{const.}$$

Therefore, when L is directed into R ,

$$\left. \frac{d \log V_1(s)}{ds} \right|_P = \frac{1}{|z_P|} = \frac{1}{d}.$$

In view of the result stated at the beginning of this section,

$$\frac{1}{d} = \frac{d \log V_1(s)}{ds} < \dots \leq \frac{d \log V_m(s)}{ds} \leq \dots \leq \frac{d \log V(s)}{ds},$$

as required. If L is directed out of R it is only necessary to replace s with $-s$.

Assuming that an analogue of Warschawski's theorem holds in axial symmetry, we can prove

Corollary 3. *Let R be a strip domain bounded by the axis of symmetry and a curve S satisfying the hypotheses of Corollary 1. Suppose there is an axially symmetric flow in R . Then, with the terminology of the preceding corollaries,*

$$\frac{d \log V(s)}{ds} > \frac{2}{d}$$

if L is directed into R , and

$$\frac{d \log V(s)}{ds} < -\frac{2}{d}$$

if L is directed out of R .

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