*The Elementary Basic Principles of the Unified Theory of Relativity. A**

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INTRODUCTION

The unified theory of relativity recently proposed by Einstein† is based on three principles:

A) The introduction of a metric tensor $g_{\lambda\mu}$ which is neither symmetric nor skew symmetric:

$$(0.1) \quad g_{\lambda\mu} \neq \pm g_{\mu\lambda} .$$

B) The introduction of a non-symmetric connection $\Gamma_{\lambda\mu}{}^\nu$ by means of

$$(0.2) \quad \frac{\partial}{\partial x^\omega} g_{\lambda\mu} = \Gamma_{\lambda\omega}{}^\alpha g_{\alpha\mu} + \Gamma_{\omega\mu}{}^\alpha g_{\lambda\alpha} .$$

C) Provided that the equations (0.2) may be solved for $\Gamma_{\lambda\mu}{}^\nu$ (which is not always the case), one imposes some intrinsic conditions on the Γ 's which yield the components $g_{\lambda\mu}$ as solutions.

In a series of three papers, of which this is the first, I shall deal with each of these principles from the geometrical point of view. However, some physical conclusions may be obtained from the geometrical results.

In this first paper we are concerned only with the tensor $g_{\lambda\mu}$. The chief idea may be roughly described as follows. A symmetric quadratic tensor defines a cone at each point of the space. The question arises whether a tensor $g_{\lambda\mu}$ satisfying (0.1) does not in some way privilege certain elements of the space, thus leading to results which may have a physical interpretation. The first five

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† A. EINSTEIN, *The Meaning of Relativity*, Princeton University Press (1950), Appendix II.

sections, which are purely geometrical, answer this question, while the sixth section tentatively describes the possible physical conclusions.

1. The tensor $g_{\lambda\mu}$. Let X_4 be our space referred to the coördinates x^{**} and $g_{\lambda\mu}$ a real tensor which is neither symmetric nor skew symmetric. Then we may put

$$(1.1a) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},$$

$$(1.1b) \quad h_{\lambda\mu} = g_{(\lambda\mu)},$$

$$(1.1c) \quad k_{\lambda\mu} = g_{[\lambda\mu]}.$$

We denote by g, h, k the determinants of $g_{\lambda\mu}, h_{\lambda\mu}$ and $k_{\lambda\mu}$, respectively.

Agreement (1.1). Unless otherwise stated we assume throughout this paper

$$(1.2) \quad g h k \neq 0,$$

so that in particular there is one tensor $h^{\lambda\mu} = h^{\mu\lambda}$ only which satisfies the condition

$$h_{\lambda\mu} h^{\lambda\nu} = \delta_{\mu}^{\nu}.$$

Moreover we assume in the first three sections

$$(1.3) \quad \frac{1}{4} h (h^{\omega\mu} h^{\lambda\nu} k_{\omega\lambda} k_{\mu\nu})^2 \neq 4 k.$$

This agreement enables us to prove the following

Theorem (1.1). *The roots $\rho_1, \dots, \rho_4$ of the equation*

$$(1.4a) \quad \text{Det} | g_{\lambda\mu} - \rho g_{\mu\lambda} | = 0$$

have the following properties:

- a) they are distinct;
- b) none of them is equal to 0, +1, or -1;
- c) their product is equal to 1.

Proof. Suppose $\rho = 0, +1$, or -1 . Then (1.4a) reduces to $g = 0, k = 0$, or $h = 0$ respectively, and these cases are excluded by (1.2). The last and the first member of the left hand side of (1.4a) are $\rho^4 g$ and g respectively, so that the statement c) holds. By the statement b), already proved, the equation

$$(1.5) \quad \lambda = \frac{1 - \rho}{1 + \rho}$$

is not meaningless. From (1.5) and (1.1a) we obtain

$$(1.6) \quad g_{\lambda\mu} - \rho g_{\mu\lambda} = h_{\lambda\mu} (1 - \rho) - k_{\mu\lambda} (1 + \rho) = (1 + \rho)[\lambda h_{\mu\lambda} - k_{\mu\lambda}],$$

so that (1.4a) is equivalent to

$$(1.4b) \quad \text{Det} | \lambda h_{\mu\lambda} - k_{\mu\lambda} | = 0,$$

* Greek indices run from 1 to 4.

or

$$(1.4c) \quad \lambda^4 h + \frac{\lambda^2}{2} h h^{\omega\mu} h^{\lambda\nu} k_{\omega\lambda} k_{\mu\nu} + k = 0.$$

Put

$$(1.4d) \quad D = \frac{1}{4} h^2 (h^{\omega\mu} h^{\lambda\nu} k_{\omega\lambda} k_{\mu\nu})^2 - 4 h k.$$

We have $D \neq 0$ by virtue of (1.3), and consequently (1.4c) has four different roots $\lambda_1, \dots, \lambda_4$. Going back to (1.4a), by means of (1.5) we see that this equation has four different roots $\rho_1, \dots, \rho_4$.

Since $g \neq 0$ there is only one tensor $g^{\lambda\nu}$ satisfying the condition

$$g_{\lambda\mu} g^{\lambda\nu} = g_{\mu\lambda} g^{\nu\lambda} = \delta_\mu^\nu,$$

while the tensor

$$(1.7a) \quad G_\lambda^\nu = g_{\mu\lambda} g^{\nu\mu}$$

is in general different from δ_λ^ν .

Theorem (1.2). *The projectivity*

$$(1.8a) \quad {}^{\nu}e^{\nu} = \rho G_\lambda^\nu e^\lambda$$

has the following properties:

- a) It has exactly four eigenvalue vectors ${}_1e^\nu, \dots, {}_4e^\nu$, and they are linearly independent.
- b) The eigenvalue vectors satisfy the conditions*

$$(1.9a) \quad g_{\lambda\mu} {}_ae^\lambda {}_be^\mu \neq 0 \quad \text{for } ab = 12, 21, 34, 43.$$

$$(1.9b) \quad g_{\lambda\mu} {}_ae^\lambda {}_be^\mu = 0 \quad \text{in all remaining cases.}$$

Proof. The eigenvalue vectors of (1.8a) satisfy the equations

$$(1.10a) \quad e^\nu = \rho G_\lambda^\nu e^\lambda,$$

or

$$(1.10b) \quad e^\lambda g_{\lambda\mu} = \rho e^\lambda g_{\mu\lambda}.$$

The corresponding equation (1.4a) for the eigenvalue roots has four distinct roots $\rho_1, \dots, \rho_4$ different from zero and consequently the statement a) holds. Let ${}_ae^\nu$ be the eigenvalue vector corresponding to ρ_a . Then we have from (1.10b) the conditions

$$(1.11) \quad {}_ae^\lambda {}_be^\mu g_{\lambda\mu} (1 - \rho_a \rho_b) = 0, \quad {}_be^\lambda {}_ae^\mu g_{\lambda\mu} (1 - \rho_a \rho_b) = 0.$$

* The Latin indices run from 1 to 4.

Since $g \neq 0$ and since the vectors ${}_a e^\nu$ are linearly independent the equations (1.9b) cannot be satisfied for all values of a, b . Suppose that for $ab = 12$, we have (1.9a). Then (1.11) yields

$$(1.12a) \quad 1 = \rho_1 \rho_2.$$

Since the roots are all different and since none of the roots is equal to ± 1 we must have

$$(1.12b) \quad \begin{aligned} 1 &\neq \rho_1 \rho_b, & b &= 1, 3, 4, \\ 1 &\neq \rho_a \rho_2, & a &= 2, 3, 4. \end{aligned}$$

Hence we obtain from (1.11) the equations (1.9b) for

$$\begin{aligned} a &= 1, & b &= 1, 3, 4, & a &= 2, 3, 4, & b &= 2, \\ b &= 1, & a &= 1, 3, 4, & b &= 2, 3, 4, & a &= 2. \end{aligned}$$

Finally, the same method yields (1.9b) for $a = b = 3, 4$. Introduce the notation

$${}^*g_{ab} = g_{\lambda\mu} {}_a e^\lambda {}_b e^\mu.$$

Then we have according to our results

$$\text{Det } |{}_a e^\lambda| \text{ Det } |g_{\lambda\mu}| = {}^*g_{12} {}^*g_{21} {}^*g_{34} {}^*g_{43}.$$

The left hand side being different from zero, we must have also ${}^*g_{21} {}^*g_{34} {}^*g_{43} \neq 0$; or we must have (1.9a) also for $ab = 21, 34, 43$.

Note. If we put

$$(1.7b) \quad {}^*G_\lambda{}^\nu = g_{\lambda\mu} g^{\mu\nu},$$

then we obtain from (1.10b)

$$(1.10c) \quad e^\nu = \frac{1}{\rho} {}^*G_\lambda{}^\nu e^\lambda.$$

The vectors e^ν satisfying this equation are eigenvalue vectors of the projectivity

$$(1.8b) \quad {}'e^\nu = \frac{1}{\rho} {}^*G_\lambda{}^\nu e^\lambda.$$

Since both equations (1.10c) and (1.10a) are equivalent to (1.10b), the eigenvalue vectors (1.10c) of (1.8b) are the same as the eigenvalue vectors (1.10a) of (1.8a) (cf. (1.12a)).

Theorem (1.3). *The bivectors ${}_1 e^{[\nu} {}_2 e^{\lambda]}$ and ${}_3 e^{[\nu} {}_4 e^{\lambda]}$ are totally perpendicular.**

Proof. First of all we have

$$({}_a e^\lambda {}_b e^\mu + {}_a e^\mu {}_b e^\lambda) g_{\lambda\mu} = ({}_a e^\lambda {}_b e^\mu + {}_a e^\mu {}_b e^\lambda) (h_{\lambda\mu} + k_{\lambda\mu}).$$

* The metric as well as the angular metric is taken with respect to $h_{\lambda\mu}$.

Remembering that $h_{\lambda\mu}$ ($k_{\lambda\mu}$) is symmetric (skew symmetric), we may simplify this equation to

$$g_{\lambda\mu} ({}_ae^\lambda {}_be^\mu + {}_ae^\mu {}_be^\lambda) = 2 {}_ae^\lambda {}_be^\mu h_{\lambda\mu}.$$

The statement of the theorem follows from this equation as well as from (1.9b).

Theorem (1.4). *The tensor $g_{\lambda\mu}$ may be expressed as follows:*

$$(1.13a) \quad g_{\lambda\mu} = \alpha {}_1e_\lambda {}_2e_\mu + \beta {}_2e_\lambda {}_1e_\mu + \gamma {}_3e_\lambda {}_4e_\mu + \delta {}_4e_\lambda {}_3e_\mu,$$

where

$$(1.13b) \quad {}_ae_\lambda = g_{\mu\lambda} {}_ae^\mu \quad ({}_ae^\nu = {}_ae_\lambda g^{\nu\lambda})$$

and

$$(1.13c) \quad \alpha = \rho_2 e_{21}, \quad \beta = \rho_1 e_{12}, \quad \gamma = \rho_4 e_{43}, \quad \delta = \rho_3 e_{34},$$

$$(1.13d) \quad e_{ab} = (g_{\lambda\mu} {}_ae^\lambda {}_be^\mu)^{-1}, \quad ab = 12, 21, 34, 43.$$

Proof. The tensor $g_{\lambda\mu}$ may be expressed in the following way:

$$g_{\lambda\mu} = {}^ac^b {}_ae_\lambda {}_be_\mu.$$

where ${}^ac^b$ are properly chosen scalars. From this equation as well as from (1.9) and (1.13b) one obtains (1.13a), and this latter equation yields then

$$(1.14a) \quad G_\mu{}^\nu = \alpha {}_1e^\nu {}_2e_\mu + \beta {}_2e^\nu {}_1e_\mu + \gamma {}_3e^\nu {}_4e_\mu + \delta {}_4e^\nu {}_3e_\mu.$$

Consequently we obtain from (1.10a)

$${}_1e^\nu = \rho_1 (\alpha {}_1e^\nu {}_2e_\mu + \beta {}_2e^\nu {}_1e_\mu + \gamma {}_3e^\nu {}_4e_\mu + \delta {}_4e^\nu {}_3e_\mu) {}_1e^\mu = \rho_1 \alpha {}_2e_\mu {}_1e^\mu {}_1e^\nu,$$

so that

$$(1.14b) \quad 1 = \rho_1 \alpha g_{\lambda\mu} {}_2e^\lambda {}_1e^\mu = \frac{\rho_1 \alpha}{e_{21}}.$$

Comparing this equation with (1.12a) one obtains the first equation of (1.13c) (with (1.13d) for $ab = 21$). The remaining equations may be proved in a similar way.

Agreement (1.2). *Throughout this paper we assume that $-h_{\lambda\mu}$ is of the signature $+++ -$.*

This agreement enables us to prove the following

Theorem (1.5). *The vectors of one of the two couples ${}_1e^\nu, {}_2e^\nu$ and ${}_3e^\nu, {}_4e^\nu$, say ${}_1e^\nu, {}_2e^\nu$, are complex conjugates, while the remaining vectors are real.*

Proof. To prove this statement it is obviously sufficient to prove that two of the roots ρ_a are complex conjugates and the remaining two real. The determinant k is skew symmetric, and since $n = 4$, it is equal to a square. Hence k is positive.

On the other hand h is negative by virtue of the agreement (1.2). Hence D as defined by (1.4d) is a sum of two squares and consequently positive, and

$$2\sqrt{D} > |h h^{\omega\mu} h^{\lambda\nu} k_{\omega\lambda} k_{\mu\nu}|.$$

Consequently two roots of (1.4c) are complex conjugates; the remaining two are real. The same must hold by virtue of (1.5) for the roots ρ .

Theorem (1.6). *The tensor $-g_{\lambda\mu}$ may be written*

$$(1.15) \quad \begin{aligned} -g_{\lambda\mu} = & {}^1u_\lambda {}^1u_\mu + {}^2u_\lambda {}^2u_\mu + {}^3u_\lambda {}^3u_\mu - {}^4u_\lambda {}^4u_\mu \\ & + 2i \left(\frac{\beta - \alpha}{\beta + \alpha} {}^1u_{[\lambda} {}^2u_{\mu]} - i \frac{\delta - \gamma}{\delta + \gamma} {}^3u_{[\lambda} {}^4u_{\mu]} \right), \end{aligned}$$

where the real vectors ${}^a u_\lambda$ are defined by

$$(1.16) \quad \begin{aligned} {}^1e_\lambda &= \mu({}^1u_\lambda + i {}^2u_\lambda), & {}^2e_\lambda &= \bar{\mu}({}^1u_\lambda - i {}^2u_\lambda), \\ {}^3e_\lambda &= \sigma({}^3u_\lambda + {}^4u_\lambda), & {}^4e_\lambda &= \tau({}^3u_\lambda - {}^4u_\lambda) \end{aligned}$$

and the scalars $\mu, \bar{\mu}, \sigma, \tau$ satisfy the equation

$$(1.17) \quad (\alpha + \beta) \mu \bar{\mu} = (\gamma + \delta) \sigma \tau = -1.$$

Proof. The vectors ${}^1e^\nu {}^2e^\nu ({}^3e^\nu {}^4e^\nu)$ being complex conjugate (real), the same holds for ${}^1e_\lambda {}^2e_\lambda ({}^3e_\lambda {}^4e_\lambda)$. Hence the vectors ${}^a u_\lambda$ in (1.16) are real provided that $\mu, \bar{\mu}$ are complex conjugate and σ, τ are real. One obtains (1.15) from (1.13a), (1.16), and (1.17).

In the following theorem we introduce the scalars

$$N = i \frac{\alpha - \beta}{\alpha + \beta}, \quad Z = \frac{\gamma - \delta}{\gamma + \delta}.$$

Theorem (1.7). *If the vectors ${}^a u_\lambda$ are gradient vectors,*

$${}^a u_\lambda = \frac{\partial^a x}{\partial x^\lambda},$$

then the components of $g_{\lambda\mu}$ in the coördinate system ${}^a x$ are

$$(1.18) \quad ((g_{\lambda\mu})) = \begin{pmatrix} -1, & N & 0 & 0 \\ -N & -1 & 0 & 0 \\ 0 & 0 & -1 & Z \\ 0 & 0 & -Z & 1 \end{pmatrix}.$$

Proof. The vectors ${}^a u_\lambda$ are obviously linearly independent. Hence the Jacobian $|\partial^a x / \partial x^\lambda|$ is different from zero and consequently ${}^a x$ may be regarded as coördinates of a new coördinate system. In it we have

$$(1.19) \quad {}^a u_\lambda = \delta_\lambda^a,$$

so that according to (1.15)

$$(1.20) \quad g_{\lambda\mu} = -\delta_{\lambda}^1 \delta_{\mu}^1 - \delta_{\lambda}^2 \delta_{\mu}^2 - \delta_{\lambda}^3 \delta_{\mu}^3 + \delta_{\lambda}^4 \delta_{\mu}^4 + 2N \delta_{[\lambda}^1 \delta_{\mu]}^2 + 2Z \delta_{[\lambda}^3 \delta_{\mu]}^4.$$

The equation (1.18) results at once from (1.20).

2. Geometrical consequences

In this section we shall derive some geometrical consequences of the previous results for the tensors $h_{\lambda\mu}$ and $k_{\lambda\mu}$.

Theorem (2.1). *The eigenvalue vectors e^r of the projectivity (1.8) satisfy the equation*

$$(2.1) \quad \lambda e^{\alpha} h_{\mu\alpha} = e^{\alpha} k_{\mu\alpha},$$

where λ is given by (1.5).

Proof. The equation (2.1) follows from (1.10b), (1.5), and (1.1a).

Note. Denote by $T_4(P)$ the tangential (linear) space of X_4 at its general point P . Let $I_3(P)$ be the ideal space of $T_4(P)$. The tensor $h_{\lambda\mu}$ ($k_{\lambda\mu}$) defines in I_3 a quadric H (a linear line complex K) whose homogeneous coördinates are $h_{\lambda\mu}$ ($k_{\lambda\mu}$). On the other hand the vector e^r (e_{λ}) defines a point (a plane) in $I_3(P)$ whose homogeneous coördinates are e^r (e_{λ}).

Theorem (2.2). *The tetrahedron with the vertices ${}_ae^r$ in $I_3(P)$ has the following properties:*

- a) *The edges of its skew quadrilateral ${}_1e {}_3e {}_2e {}_4e$ are rulings of H as well as of K and each of them is a self conjugate polar of H and K . They are always imaginary.*
- b) *The remaining edges ${}_1e {}_2e$ and ${}_3e {}_4e$ of the tetrahedron are (skew) conjugate polars with respect to H and K . They are both real. The edges of the tetrahedron ${}_1e {}_3e {}_2e {}_4e$ are the only conjugate polars with respect to H and K .*

Proof. The geometrical interpretation of (2.1) is as follows: the polar plane of e^{λ} with respect to H is the same as the polar plane of e^{λ} with respect to K . Since this latter is incident with e^{λ} this point must be incident with H . From the statement a) of Theorem (1.2) we see that there are only four points

$${}_1e^r, \dots, {}_4e^r$$

satisfying (2.1) and that they are vertices of a tetrahedron. From the equation preceding (1.13a) as well as from (1.9b) we obtain

$$(2.2) \quad \text{a) } h_{\lambda\mu} {}_ae^{\lambda} {}_be^{\mu} = 0, \quad \text{b) } k_{\lambda\mu} {}_ae^{\lambda} {}_be^{\mu} = 0, \quad ab \neq 12, 21, 34, 43.$$

Consider (2.2a) for $ab = 13, 14, 23, 24$. It says that two points ${}_ae^r$ and ${}_be^r$, both on H , are polar conjugate with respect to H . Hence these two points are on the same ruling of H . We obtain in this way the edges of the skew quadrilateral ${}_1e {}_3e {}_2e {}_4e$. Since they are rulings of H each of them is a self-conjugate polar with respect to H . Consider now (2.2b) for $ab = 13, 14, 23, 24$. It says that the point

${}_a e''$ (${}_b e''$) is incident with the polar plane of ${}_b e''$ (${}_a e''$) with respect to K . Consequently these two points are incident with the same ruling ${}_a e {}_b e$ of K . Hence each edge of the aforementioned skew quadrilateral is a ruling of K , and as such it is a self-conjugate polar with respect to K . Since $-h_{\lambda\mu}$ has the signature $+++ -$, the rulings of H are all imaginary.

In order to prove the statement b) we proceed as follows: The polar plane of $\left\{ \begin{smallmatrix} 1e'' \\ 2e'' \end{smallmatrix} \right\}$ with respect to H is the tangential plane of H in this point and as such it contains the rulings $\left\{ \begin{smallmatrix} 1e {}_3e \\ 2e {}_3e \end{smallmatrix} \right\}$ and $\left\{ \begin{smallmatrix} 1e {}_4e \\ 2e {}_4e \end{smallmatrix} \right\}$ of H . Hence the tangential planes of $1e$ and $2e''$ meet in ${}_3e {}_4e$, so that $1e {}_2e$ and ${}_3e {}_4e$ are conjugate polars with respect to H . The lines $\left\{ \begin{smallmatrix} 1e {}_3e \\ 2e {}_3e \end{smallmatrix} \right\}$ and $\left\{ \begin{smallmatrix} 1e {}_4e \\ 2e {}_4e \end{smallmatrix} \right\}$ are rulings of K , so that $\left\{ \begin{smallmatrix} 1e {}_3e {}_4e \\ 2e {}_3e {}_4e \end{smallmatrix} \right\}$ is the polar plane of $\left\{ \begin{smallmatrix} 1e'' \\ 2e'' \end{smallmatrix} \right\}$ with respect to K . The planes $1e {}_3e {}_4e$ and $2e {}_3e {}_4e$ meet in ${}_3e {}_4e$, so that $1e {}_2e$ and ${}_3e {}_4e$ are conjugate polars with respect to K^* . The statement about their reality follows from Theorem (1.5).

Since the equation (1.4b) has all its four roots distinct and none of them equal to zero, the equation (2.1) yields only four points $1e, \dots, 4e$. Hence we cannot have more than six lines conjugate with respect to H and K .

Note. The case (1.19) leads to the space X_4 of the special theory of relativity, where there is only one ideal space I_3 of all tangential spaces $T_4(P) = X_4$. In this case the quadric H given by constant components $h_{\lambda\mu}$ is the same, independently of the point P we select. On the other hand $k_{\lambda\mu} = k_{\lambda\mu}(P)$ are in general not constant. Hence one would expect that the tetrahedron $\tau \equiv 1e {}_3e {}_2e {}_4e$ as described in Theorem (2.2) will change from point to point of X_4 , $\tau = \tau(P)$. However, this is not true as one may convince himself by substituting from (1.19) into (1.16): the tetrahedron is the same for all points P of X_4 , while the complex $K = K(P)$ depends on the choice of P (cf. Theorem (3.4), which shows clearly that all complexes $K(P)$ belong to the same pencil). On the other hand one has to stress that (1.19) is a sufficient but in general not a necessary condition for X_4 to be the space of the special theory of relativity.

The vectors ${}_a e'$ in X_4 are known up to factors of proportionality which may be chosen arbitrarily. We choose them in such a way that

$$(2.3) \quad h_{\lambda\mu} 1e^\lambda 2e^\mu = h_{\lambda\mu} 3e^\lambda 4e^\mu = -1.$$

Theorem (2.3). *The tensors $h_{\lambda\mu}$ and $k_{\lambda\mu}$ may be expressed as follows:*

$$(2.4a) \quad -2h_{\lambda\mu} = (2 + \rho_1 + \rho_2) 1e_{[\lambda} 2e_{\mu]} + (2 + \rho_3 + \rho_4) 3e_{[\lambda} 4e_{\mu]}, \dagger$$

$$(2.4b) \quad -2k_{\lambda\mu} = (\rho_2 - \rho_1) 1e_{[\lambda} 2e_{\mu]} + (\rho_4 - \rho_3) 3e_{[\lambda} 4e_{\mu]}.$$

* As such they must be skew because they are distinct.

† The indices 2 and 4 are not affected by the round [square] brackets.

Proof. Put $ab = 12, 21, 34, 43$. Then we have from (2.1)

$$(2.5a) \quad \lambda_a h_{\mu\alpha} a e^\alpha b e^\mu = a e^\alpha b e^\mu k_{\mu\alpha} \neq 0.$$

On the other hand, by virtue of (1.13d) and (1.1a)

$$(2.5b) \quad 1/e_{ab} = h_{\lambda\mu} a e^\lambda b e^\mu + k_{\lambda\mu} a e^\lambda b e^\mu,$$

so that we obtain from (2.5a) and (2.5b)

$$1/e_{ab} = h_{\lambda\mu} a e^\lambda b e^\mu (1 - \lambda_a) = h_{\lambda\mu} a e^\lambda b e^\mu \frac{2\rho_a}{1 + \rho_a}.$$

Taking in account (2.3) we have

$$(2.6a) \quad -e_{ab} = \frac{1 + \rho_a}{2\rho_a} = (b e^\lambda a e_\lambda)^{-1},$$

and consequently by virtue of (1.13c)

$$(2.6b) \quad -2\alpha = 1 + \rho_2, \quad -2\beta = 1 + \rho_1, \quad -2\gamma = 1 + \rho_4, \quad -2\delta = 1 + \rho_3.$$

Substituting these values into (1.13a) one obtains

$$(2.7) \quad \begin{aligned} -2g_{\lambda\mu} = & (1 + \rho_2) {}_1e_\lambda {}_2e_\mu + (1 + \rho_1) {}_2e_\lambda {}_1e_\mu \\ & + (1 + \rho_4) {}_3e_\lambda {}_4e_\mu + (1 + \rho_3) {}_4e_\lambda {}_3e_\mu, \end{aligned}$$

and the theorem follows easily from (2.7).

Note. Compare (2.7) and (1.18).

Theorem (2.4). Assume $(\rho_1)^m \neq (\rho_2)^{m*}$ and put

$${}^m G_\lambda{}^\nu = {}^{m-1} G_\alpha{}^\nu G_\lambda{}^\alpha, \quad {}^1 G_\lambda{}^\nu = G_\lambda{}^\nu, \quad m = 2, 3, \dots$$

The tensors $G_\lambda{}^\nu$, ${}^m G_\lambda{}^\nu$ have the same eigenvalue vectors as $k_{\alpha\lambda} h^{\alpha\nu}$.

Proof. First we have from (1.13b) and (2.7)

$$(2.8a) \quad \begin{aligned} -2G_\lambda{}^\nu = & (1 + \rho_2) {}_1e^\nu {}_2e_\lambda + (1 + \rho_1) {}_2e^\nu {}_1e_\lambda \\ & + (1 + \rho_4) {}_3e^\nu {}_4e_\lambda + (1 + \rho_3) {}_4e^\nu {}_3e_\lambda, \end{aligned}$$

and more generally

$$(2.8b) \quad \begin{aligned} (-1)^m 2 {}^m G_\lambda{}^\nu = & \rho_2^{m-1} (1 + \rho_2) {}_1e^\nu {}_2e_\lambda + \rho_1^{m-1} (1 + \rho_1) {}_2e^\nu {}_1e_\lambda \\ & + \rho_4^{m-1} (1 + \rho_4) {}_3e^\nu {}_4e_\lambda + \rho_3^{m-1} (1 + \rho_3) {}_4e^\nu {}_3e_\lambda, \quad m = 1, 2, \dots \end{aligned}$$

The roots of the equation

$$\text{Det} | \sigma \delta_\lambda{}^\nu - {}^m G_\lambda{}^\nu | = 0$$

* The condition $(\rho_3)^m \neq (\rho_4)^m$ for the real roots is always satisfied.

are

$$\sigma_a = -(-\rho_a)^m, \quad (a = 1, \dots, 4),$$

so that there are only 4 eigenvalue vectors of ${}^mG_\lambda$. On the other hand one sees from (2.8b) that four eigenvalue vectors of ${}^mG_\lambda$ are ${}_1e, \dots, {}_4e$. They also satisfy (2.1), which is equivalent to

$$(2.9) \quad \lambda e^\nu = e^\alpha k_{\mu\alpha} h^{\mu\nu},$$

and this proves the last part of our theorem.

3. Privileged elements

Theorem (3.1). *If the agreement (1.1) holds then the only elements defined uniquely and intrinsically by means of the quadric H and the complex K are the vertices, edges, and sides of the tetrahedron referred to in Theorem (2.2).*

The proof will be accomplished by means of line geometry.* Denote by P_5 the projective five dimensional space which contains the four dimensional quadric Q_4 , the image of the line space of I_3 . (To avoid any misunderstanding we put the elements of P_5 in quotation marks and speak of the "quadric" Q_4 , and so on.) The complex K is mapped on a three dimensional "quadric" K_3 in Q_4 , which is the intersection of a four dimensional "space" S_4 and Q_4 . We denote its "pole" with respect to Q_4 by K_0 †. The quadric H consists of two reguli R and R' which are mapped on two non-degenerate‡ "conic sections" R_1 and R_1' in Q_4 . The "planes" S_2 and S_2' of these "conic sections" are polar conjugate with respect to Q_4 and consequently do not meet at all. In order to prove our theorem it suffices to find all "elements" on Q_4 common to K_3 , R_1 , R_1' and to find all "elements" on Q_4 uniquely defined by means of K_3 , R_1 , R_1' .

The "plane" S_2 (S_2') is not contained in S_4 . If it were, then the "conic section" R_1 (R_1') would belong to K_3 . In this case every ruling of the regulus R (R') would also be a ruling of K and consequently self conjugate with respect to H and K . Because of the agreement (1.1), there are only two such rulings on R (R') (cf. Theorem (2.2)). Consequently S_2 (S_2') is not contained in S_4 , and thus meets it in a "line" which intersects K_3 and Q_4 in two "points" A_0, B_0 , (A_0', B_0'). The "lines" A_0B_0 and $A_0'B_0'$ incident with S_2 and S_2' , respectively, are skew and define a three dimensional "space" S_3' . Its polar conjugate "line" L_1 is incident with K_0 and intersects Q_4 in two points E_0, F_0 which are not "points" of K_3 .

The only "points" on Q_4 common to K_3 , R_1 , and R_1' are obviously the "points"

* The reader unfamiliar with line geometry might either pass over this proof (and take Theorem (3.1) for granted) or he might acquaint himself with the elements of line geometry from my book "Differential line geometry," translated by H. LEVY (Noordhoff, to be published soon) pp. 17-44, or from the German edition "Differentialle Liniengeometrie" translated by M. PINL (Noordhoff, 1945) pp. 19-50.

† K_0 is not on Q_4 because $k \neq 0$.

‡ This follows from $h \neq 0$.

A_0, B_0, A_0', B_0' . The only "points" on Q_4 defined by means of K_3, R_1 and R_1' are are E_0, F_0 . From the manner in which these points were obtained it is obvious that they are maps of six edges of a tetrahedron whose two skew edges (mapped on E_0, F_0) are conjugate polars with respect to K while the remaining skew quadrilateral consists of two (skew) rulings of R and K (which are mapped on A_0, B_0) and two (skew) rulings of R' and K (which are mapped on A_0', B_0'). Hence this tetrahedron is the one mentioned in Theorem (2.2).

Among all "lines" defined by six linearly independent "points" $A_0, B_0, A_0', B_0', E_0, F_0$ we select three different types

$$1) \quad A_0 B_0, \quad 2) \quad A_0 E_0, \quad 3) \quad E_0 F_0.$$

The "line" $A_0 B_0$ ($E_0 F_0$) intersects Q_4 only in "points" A_0, B_0 (E_0, F_0) which are maps of two skew edges of the aforementioned tetrahedron. The "points" A_0, E_0 on Q_4 are polar conjugate with respect to Q_4 . Consequently $A_0 E_0$ is a ruling of Q_4 , and as such it is a map of a pencil of lines defined by two intersecting edges of our tetrahedron. These latter are the only ones uniquely defined in the pencil by K_3, R_1, R_1' . Similar results hold for all the remaining "lines" defined by $A_0, \dots, F_0$.

Among all "planes" defined by these "points" we select three different types

$$1) \quad A_0 B_0 A_0', \quad 2) \quad A_0 B_0 E_0, \quad 3) \quad E_0 F_0 A_0.$$

Consider first the "plane" $A_0 B_0 A_0'$, and denote by K_1 the "conic section" common to it and to Q_4 . The points A_0', B_0, A_0 are on K_1 and A_0' is polar conjugate to B_0 and A_0 with respect to K_1 . Hence K_1 consists of two "lines" $A_0' A_0$ and $A_0' B_0$. Each of these lines is a map of a pencil defined by two edges of our tetrahedron. Only these three edges are uniquely determined in the corresponding pencils. Consider now the plane $A_0 B_0 E_0$ and denote again by K_1 the "conic section" common to it and to Q_4 . The same consideration as before shows that K_1 degenerates into two "lines", $E_0 A_0$ and $E_0 B_0$, so that we obtain again three edges of our tetrahedron as uniquely defined in the corresponding pencils. Consider now the plane $E_0 F_0 A_0$. Here again we are dealing with two rulings, $A_0 E_0$ and $A_0 F_0$ of Q_4 , and the situation is the same as before. All other cases of "planes" lead to similar results.

Among all three dimensional "spaces" defined by four of the six linearly independent points $A_0, \dots, F_0$ we select four different types, namely

$$1) \quad A_0 B_0 A_0' B_0', \quad 2) \quad A_0 B_0 E_0 A_0', \quad 3) \quad A_0 B_0 E_0 F_0, \quad 4) \quad E_0 F_0 A_0 A_0'.$$

Consider first the case of the "space" S_3 defined by A_0, B_0, A_0', B_0' . It intersects Q_4 in a "quadric" Q_2 which is a map of a linear line congruence, whose axes are obviously the skew edges of our tetrahedron which are mapped on E_0, F_0 . Only these axes and four rulings of this congruence are uniquely determined by K_3, R_1 and R_1' . Consider now the S_3 "space" $A_0 B_0 E_0 A_0'$. Its intersection "quadric" with Q_4 consists of two "planes" $A_0 E_0 A_0'$ and $B_0 E_0 A_0'$ contained in Q_4 . One of them is a map of a bundle of lines through one of the vertices of our tetra-

hedron; the other is a map of the set of all lines in one of the sides of the tetrahedron. The only lines uniquely determined in the bundle (in the set) are those whose maps are the points A_0, B_0, E_0, A_0' . Case 3) is similar to case 1) (the axes of the congruence having the maps A_0', B_0') while case 4) is similar to case 2) (the "planes" in Q_4 being $E_0 A_0 A_0'$ and $F_0 A_0 A_0'$). All other "spaces" defined by four of the "points" $A_0, \dots, F_0$ give similar results.

Finally we have to consider the four dimensional spaces defined by five of the six points $A_0, \dots, F_0$. We select two different types, namely

$$1) \quad A_0 B_0 A_0' E_0 F_0, \quad 2) \quad A_0 B_0 A_0' B_0' E_0.$$

In the first case the three dimensional intersection "quadric" Q_3 with Q_4 is a three dimensional "cone" with the "vertex" A_0' (for this "point" on Q_3 is polar conjugate to four remaining "points" on Q_3), and consequently it is a map of a special linear complex whose axis is the edge of the tetrahedron whose map is A_0' . Beside this edge only four other edges are uniquely determined as rulings of the special complex. In the second case the situation is similar. We have a special linear complex whose axis is the edge of our tetrahedron whose map is E_0 . All remaining cases lead to similar results.

Definition (3.1). Any element of $I_3(P)$ which is real, intrinsically defined by means of given data and isolated*, is termed a privileged element of $I_3(P)$. The corresponding element in $T_4(P)$ † will be called a privileged element in $T_4(P)$. If $I_3(P)$ has no privileged elements for any P then X_4 will be referred to as isotropic.

Theorem (3.2). X_4 is not isotropic. In each $T_4(P)$ there is a couple of privileged totally perpendicular planes through P and in one of them there is a couple of privileged lines through P .

Proof. The privileged planes are the planes of the bivectors $1e^{lv} 2e^{lv}$ and $3e^{lv} 4e^{lv}$, the privileged lines are given by the vectors $3e^v, 4e^v$. The given data reduces here to $g_{\lambda\mu}$ and according to the previous theorem there are no other privileged elements intrinsically defined by $g_{\lambda\mu}$.

Theorem (3.3). If in the case referred to in Theorem (1.7) we take the privileged planes for coördinate planes of a coördinate system x, y, z, t then any other system $'x, 'y, 'z, 't$ with this property (and which preserves each of the lines defined by the vectors $1e^v, \dots, 4e^v$) is obtained by a subgroup of Lorentz transformations:

$$\begin{aligned} \epsilon_1' x &= x \cos \phi + \epsilon_3 y \sin \phi, & * \epsilon_1' z &= z \cosh \psi + * \epsilon_3 t \sinh \psi, \\ (3.1) \quad \epsilon_2' y &= -x \sin \phi + \epsilon_3 y \cos \phi, & * \epsilon_2' t &= z \sinh \psi + * \epsilon_3 t \cosh \psi \\ & & (\epsilon_1, \dots, * \epsilon_3 &= \pm 1, \quad \epsilon_1 = \epsilon_2 \epsilon_3, \quad * \epsilon_1 = * \epsilon_2 * \epsilon_3). \end{aligned}$$

Moreover in each of these systems we have

$$(3.2) \quad k_{\lambda\mu} = 0 \quad \text{for } \lambda\mu \neq 12, 21, 34, 43,$$

* By this statement we mean that in a suitably defined sufficiently small neighborhood there is no other element having these properties.

† One obtains it by means of the projection from P of the privileged element of $I_3(P)$.

while the remaining components k_{12} and k_{34} are (relative) scalars with respect to (3.1).

Proof. The components $h_{\lambda\lambda}$ are given in this special case by the main diagonal of the matrix in (1.18), while the remaining $h_{\lambda\mu}$ are equal to zero. This situation is preserved by a group of Lorentz transformations. Its subgroup (3.1) preserves the privileged planes as coördinate planes xy , zt and in particular it is the only one which preserves each of the lines defined by the vectors ${}_1e''$, $\dots$, ${}_4e''$. The remaining statement following from the matrix in (1.18) and from (3.1) is obvious.

In the following theorem, which we shall use later on, we denote by K_1 (K_2) the special complex in $I_3(P)$ whose axis is the line ${}_1e_{[\lambda} {}_2e_{\mu]}$ (${}_3e_{[\lambda} {}_4e_{\mu]}$).

Theorem (3.4). *The complex K is contained in the pencil*

$$(3.3) \quad \lambda K_1 + \mu K_2.$$

There is only one complex K' in this pencil which is projectively orthogonal to K , namely

$$(3.4) \quad K' \equiv (\rho_1 - \rho_2)K_1 + (\rho_4 - \rho_3)K_2.$$

Proof. The first part follows at once from the equation (2.4b). The general complex of the pencil (3.3) has the projective homogeneous coördinates λ , μ . Consequently we have

$$\text{for } \begin{cases} K_1 \\ K_2 \\ K \\ K' \end{cases} \text{ the coördinates } \begin{cases} 1 & 0 \\ 0 & 1 \\ \rho_2 - \rho_1 & \rho_4 - \rho_3 \\ \xi & \eta \end{cases}$$

A necessary and sufficient condition for K' to be projectively orthogonal to K is

$$(K_1 K_2, K K') = -1,$$

where the left hand member denotes the cross ratio of these four complexes. This condition leads at once to

$$\xi : \eta = (\rho_1 - \rho_2) : (\rho_4 - \rho_3).$$

4. Special cases

All our previous results have been obtained under the assumption that Agreement (1.1) holds. As far as the condition (1.2) is concerned, the requirement

$$(4.1) \quad gh \neq 0$$

is necessary for later purposes (principle B). We will now drop the condition (1.3) and start with an almost trivial

Theorem (4.1). *If (cf. eq. (1.4d))*

$$(4.2) \quad \text{a) } D = 0, \quad \text{b) } k \neq 0,$$

then the roots of the equation (1.4a) are not all distinct.

Proof. If (4.1) and (4.2) hold, then the equation (1.4c) has only two different roots $\lambda \neq 0$, and the proof follows easily.

Theorem (4.2). *There is no real field $k_{\lambda\mu}$ satisfying (4.1) and (4.2).*

Proof. In the proof of Theorem (1.5) we have shown that for a real field $k_{\lambda\mu}$ the discriminant is a sum of two squares, one of them being a positive multiple of k , which is a square. Hence if (4.2b) holds the conditions (4.2a) cannot be satisfied by a real field $k_{\lambda\mu}$.

Note. Since we confine ourselves to real $g_{\lambda\mu}$ only, we must disregard the case mentioned in Theorem (4.1).

In the next theorem we shall consider another case where Agreement (1.1) does not hold.

Theorem (4.3a). *Assume*

$$(4.3) \quad \text{a) } k = 0, \quad \text{b) } g h h^{\omega\mu} h^{\lambda\nu} k_{\omega\lambda} k_{\mu\nu} \neq 0.$$

Then K is a special complex and its axis $k_{\lambda\mu}$ meets H in two different points ${}_3e^\nu$, ${}_4e^\nu$. If we denote by ${}_1e^\nu$, ${}_2e^\nu$ the intersection points of the conjugate polar to $k_{\lambda\mu}$ with H , then Theorem (2.2) holds. Moreover we have*

$$(4.4) \quad -2 k_{\lambda\mu} = (\rho_2 - \rho_1) {}_1e_{[\lambda} {}_2e_{\mu]}^\dagger.$$

Proof. If $k = 0$ then K is a special complex and $k_{\omega\lambda}$ are the (plane) Plücker coördinates of its axis. The polar conjugate line of $k_{\lambda\mu}$ with respect to H has the Plücker (point) coördinates $p^{\mu\nu} = k_{\omega\lambda} h^{\omega\mu} h^{\lambda\nu}$. By (4.3b) we have

$$p^{\mu\nu} k_{\mu\nu} = k_{\omega\lambda} k_{\mu\nu} h^{\omega\mu} h^{\lambda\nu} \neq 0.$$

Hence these two lines do not meet, and consequently the line $k_{\lambda\mu}$ is not tangent to H and intersects it in two distinct points ${}_3e^\nu$ and ${}_4e^\lambda$. Suppose that they are real, and denote by ${}_1e^\nu$, ${}_2e^\lambda$ the intersection points of H and p^ν . Then we have

* The polar plane with respect to K of an arbitrary point of its axis is indeterminate. We therefore assume that the polar plane with respect to H of this point is also the polar plane with respect to K .

† In this theorem we assume that ${}_3e^\nu$ and ${}_4e^\nu$ are real points,

$$h^{\omega\mu} h^{\lambda\nu} k_{\omega\lambda} k_{\mu\nu} > 0.$$

If, however, the intersection points of $k_{\lambda\mu}$ and H are complex conjugate we call them ${}_1e^\nu$, ${}_2e^\nu$ and denote by ${}_3e^\nu$, ${}_4e^\nu$ the real intersection points of H and of the conjugate polar of $k_{\lambda\mu}$ (with respect to H). Then we obtain instead of (4.4) the equation

$$(4.4)' \quad -2 k_{\lambda\mu} = (\rho_4 - \rho_3) {}_3e_{[\lambda} {}_4e_{\mu]}.$$

again the intrinsically and uniquely defined tetrahedron and Theorem (2.2) may easily be proved. Moreover we have two roots of (1.4c) equal to zero while the remaining two are different. The roots $\lambda = 0$ lead to $e^\alpha k_{\alpha\mu} = 0$ in (2.1). It may be proved that not only ${}_3e'$ and ${}_4e'$ but every point of $k_{\lambda\mu}$ belongs to $\lambda = 0$. The corresponding roots ρ are both equal to one. Hence we have either $\rho_3 = \rho_4$ or $\rho_1 = \rho_2$. The points ${}_3e'$ and ${}_4e'$ being real, we have the situation described in Theorem (2.2). Moreover $k_{\lambda\mu}$ is polar conjugate to ${}_1e^{[\nu} {}_2e^{\lambda]}$, and consequently its plane Plücker coördinates $k_{\lambda\mu}$ are proportional to ${}_1e_{[\lambda} {}_2e_{\mu]}$. Hence we have to put $\rho_3 = \rho_4$ in (2.4) and obtain in this way (4.4)

Theorem (4.3b). *In the case referred to in the previous theorem, Theorem (3.3) holds with the single exception that only one of the components of K , namely k_{12}^* , is different from zero, while the remaining $k_{\lambda\mu} = 0$.*

We leave the proof to the reader.

Theorem (4.4a). *If*

$$(4.5) \quad a) \quad k = 0, \quad b) \quad h^{\omega\mu} h^{\lambda\nu} k_{\omega\lambda} k_{\mu\nu} = 0, \quad c) \quad g \neq 0,$$

then K is a special complex, whose axis $k_{\lambda\mu}$ is tangent to H . The tensor $k_{\lambda\mu}$ may be written in one of the forms

$$(4.6) \quad k_{\lambda\mu} = 2\gamma {}^2u_{[\lambda} ({}_3u_{\mu]} + \epsilon {}^4u_{\mu}), \quad \epsilon = \pm 1,$$

where ${}_2u_\lambda$, ${}_3u_\lambda$, ${}_4u_\lambda$ are properly chosen perpendicular unit vectors

$$(4.7) \quad {}^2u_\lambda {}^2u_\mu h^{\lambda\mu} = {}^3u_\lambda {}^3u_\mu h^{\lambda\mu} = -{}^4u_\lambda {}^4u_\mu h^{\lambda\mu} = -1.$$

Proof. From (1.5) we obtain $\lambda = 1$ for $\rho = 0$ and consequently we have from (1.6) and (4.5a, b)

$$g = \text{Det} | h_{\mu\lambda} - k_{\mu\lambda} | = h.$$

Hence we obtain from this equation by virtue of (4.5c)

$$(4.8) \quad h \neq 0^\dagger.$$

From (4.5a) we see that K is a special complex. Its axis satisfies (4.5b), which is the necessary and sufficient condition that it meets its conjugate polar with respect to H . Hence $k_{\lambda\mu}$ is tangent to H by virtue of (4.5b). Now let ${}_3u^\nu$, ${}_4u^\nu$ be two polar conjugate points such that the line ${}_3u^{[\nu} {}_4u^{\lambda]}$ contains the tangential point e' of $k_{\lambda\mu}$ and H . Moreover, let ${}_1u^\nu$ be a point on $k_{\lambda\mu}$ polar conjugate to ${}_3u^\nu$ and ${}_4u^\nu$; and finally let ${}_2u^\nu$ be the unique polar conjugate point to ${}_1u^\nu$, ${}_3u^\nu$, ${}_4u^\nu$. Without loss of generality we may put

$$-{}_a u^\lambda {}_b u^\mu h_{\lambda\mu} = \begin{cases} 0 & \text{for } a \neq b, \\ 1 & \text{for } a = b \neq 4, \\ -1 & \text{for } a = b = 4. \end{cases}$$

* Or k_{34} if (4.4)' holds (cf. the footnote on page 552).

† (4.8) follows immediately from the Agreement (1.2). We could not have used it before having proved (4.8).

If we put ${}^a u_\lambda = {}^a u^\mu h_{\lambda\mu}$ then we see that one of the two planes ${}^3 u_\lambda + \epsilon {}^4 u_\lambda$ ($\epsilon = \pm 1$) is incident with $k_{\lambda\mu}$, because it is the tangential plane to H in e^ν . Moreover ${}^2 u_\lambda$ is also incident with $k_{\lambda\mu}$ for ${}^2 u_\lambda e^\lambda = {}^2 u_\lambda ({}^3 u^\lambda + \epsilon {}^4 u^\lambda) = 0$ and ${}^2 u_\lambda {}^1 u^\lambda = 0$. Hence $k_{\lambda\mu}$ must be proportional to ${}^2 u_{[\lambda} ({}^3 u_{\mu]} + \epsilon {}^4 u_{\mu})$. If we now interpret ${}^2 u_\lambda$, ${}^3 u_\lambda$, ${}^4 u_\lambda$ as perpendicular unit vectors in X_4 which satisfy (4.7), we obtain (4.6).

Theorem (4.4b). *Let ${}^1 u_\lambda$ be a unit vector field perpendicular to ${}^2 u_\lambda$, ${}^3 u_\lambda$, ${}^4 u_\lambda$. If*

$$(4.9) \quad {}^a u_\lambda = \frac{\partial}{\partial x^\lambda} {}^a x, \quad a = 1, 2, 3, 4,$$

then we have in the coördinate system ${}^a x$

$$(4.10) \quad \begin{aligned} \text{a)} \quad & k_{23} = \epsilon k_{24} = \gamma, \quad \text{the remaining } k_{\lambda\mu} = 0; \\ \text{b)} \quad & h_{11} = h_{22} = h_{33} = -h_{44} = -1, \quad \text{the remaining } h_{\lambda\mu} = 0. \end{aligned}$$

It is not possible to reduce more components (4.10a) to zero and to have (4.10b). The situation as described in (4.10) is preserved under the Lorentz transformations

$$(4.11) \quad \begin{aligned} {}^1 x &= {}^1 x, \quad {}^2 x = {}^2 x, \\ \epsilon_1 {}^3 x &= {}^3 x \cosh \psi + \epsilon_3 {}^4 x \sinh \psi \\ \epsilon_2 {}^4 x &= {}^3 x \sinh \psi + \epsilon_3 {}^4 x \cosh \psi \quad (\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1, \epsilon_1 = \epsilon_2 \epsilon_3). \end{aligned}$$

Proof. Using the perpendicular unit vector fields ${}^a u_\lambda$, we may always write

$$(4.12) \quad -h_{\lambda\mu} = {}^1 u_\lambda {}^1 u_\mu + {}^2 u_\lambda {}^2 u_\mu + {}^3 u_\lambda {}^3 u_\mu - {}^4 u_\lambda {}^4 u_\mu.$$

If (4.9) holds, then we have

$${}^a u_\lambda = \delta_\lambda^a \quad \text{in the coördinate system } {}^a x,$$

and this equation together with (4.6) and (4.12) leads to (4.10a, b) and to the immediately following statement. The statement about (4.11) follows also at once from (4.12) and (4.6).

Theorem (4.5). *The only possible cases with $k = 0$ are the cases described in Theorems (4.3) and (4.4).*

Proof. The equation $k = 0$ is a necessary and sufficient condition that K be a special complex. Then $k_{\lambda\mu}$ is its axis. Since $g_{\lambda\mu}$ is supposed to be real, $k_{\lambda\mu}$ must be real, too. A real line $k_{\lambda\mu}$ is either tangent to H or it intersects it in two distinct (real or complex conjugate) points.* These cases were dealt with in Theorems (4.3) and (4.4).

* Since $-h_{\lambda\mu}$ is supposed to be of signature $+++-$, it has no real rulings and consequently $k_{\lambda\mu}$ can not be the ruling of H .

5. Canonical form of $k_{\lambda\mu}$.

Theorem (5.1). *If Agreement (1.1) and equations (2.3) hold and if we put*

$$(5.1) \quad \begin{aligned} {}^1e_\lambda &= \frac{\sqrt{2}}{1+\rho_1} ({}^1u_\lambda + i {}^2u_\lambda), & {}^3e_\lambda &= \frac{\sqrt{2}}{1+\rho_3} ({}^3u_\lambda + {}^4u_\lambda), \\ {}^2e_\lambda &= \frac{\sqrt{2}}{1+\rho_2} ({}^1u_\lambda - i {}^2u_\lambda), & {}^4e_\lambda &= \frac{\sqrt{2}}{1+\rho_4} ({}^3u_\lambda - {}^4u_\lambda), \end{aligned}$$

then the vectors ${}^a u_\lambda$ are perpendicular unit vectors and

$$(5.2) \quad -h_{\lambda\mu} = {}^1u_\lambda {}^1u_\mu + {}^2u_\lambda {}^2u_\mu + {}^3u_\lambda {}^3u_\mu - {}^4u_\lambda {}^4u_\mu.$$

Proof. First we have from (1.13b), (2.1), and (1.5)

$$(5.3a) \quad {}^a e_\lambda = h_{\lambda\mu} {}^a e^\mu (1 - \lambda_a) = h_{\lambda\mu} {}^a e^\mu \frac{2\rho_a}{1 + \rho_a},$$

and consequently in virtue of (2.3), (1.12a) and Theorem (1.3)

$$(5.3b) \quad \begin{aligned} -h^{\lambda\mu} {}^a e_{\lambda} {}^b e_{\mu} &= \frac{4}{(1 + \rho_a)(1 + \rho_b)} = \frac{4}{2 + \rho_a + \rho_b} & \text{for } a \neq b = 1, 2, 3, 4, \\ h^{\lambda\mu} {}^a e_{\lambda} {}^b e_{\mu} &= 0 & \text{for } a = b = 1, 2, 3, 4. \end{aligned}$$

On the other hand

$$\begin{aligned} {}^1u_\lambda &= \frac{2 + \rho_1 + \rho_2}{4} \left(\frac{\sqrt{2}}{1 + \rho_2} {}^1e_\lambda + \frac{\sqrt{2}}{1 + \rho_1} {}^2e_\lambda \right), \\ {}^2u_\lambda &= \frac{2 + \rho_1 + \rho_2}{4i} \left(\frac{\sqrt{2}}{1 + \rho_2} {}^1e_\lambda - \frac{\sqrt{2}}{1 + \rho_1} {}^2e_\lambda \right). \end{aligned}$$

These equations together with (5.3) yield

$$(5.4a) \quad -h^{\lambda\mu} {}^a u_{\lambda} {}^b u_{\mu} = \delta_{ab} \quad a, b = 1, 2.$$

In a similar manner we obtain

$$(5.4b) \quad h^{\lambda\mu} {}^a u_{\lambda} {}^b u_{\mu} = \begin{cases} -1 & \text{for } a = b = 3, \\ +1 & \text{for } a = b = 4, \\ 0 & \text{for } a \neq b, \quad a, b = 1, \dots, 4. \end{cases}$$

Hence ${}^a u_\lambda$ are perpendicular unit vectors. The equations (5.2) may be obtained either from (5.4) or from (2.4a) and (5.1).

Note. This theorem holds whenever we have a tetrahedron ${}^1e {}^2e {}^3e {}^4e$ whose elements satisfy (5.3), where the ρ 's (the λ 's) are roots of (1.4a) ((1.4c)) and ${}^a e^\nu$ is the eigenvalue point corresponding to ρ_a . In the case described in Theorem (4.4a) we have $\rho_a = 1$, $\lambda_a = 0$, $a = 1, \dots, 4$. Even in this case we are able to

find a tetrahedron ${}^1e\,{}^2e\,{}^3e\,{}^4e$ by means of (5.1), where now ${}^1u_\lambda, \dots, {}^4u_\lambda$ are supposed to be four given mutually perpendicularly unit vector field and $\rho_a = 1$. Hence Theorem (5.1) holds in every case we have discussed in previous sections.

Theorem (5.2)†. *The field $k_{\lambda\mu}$ may always be put into one of the following canonical forms:*

$$(5.5a) \quad k_{\lambda\mu} = 2{}^1u_{[\lambda}{}^2u_{\mu]} \operatorname{tg} \xi - 2{}^3u_{[\lambda}{}^4u_{\mu]} \operatorname{tgh} \eta \quad (\text{eq. (2.4b)}),$$

$$(5.5b_1) \quad k_{\lambda\mu} = 2{}^1u_{[\nu}{}^2u_{\mu]} \operatorname{tg} \xi \quad (\text{eq. (4.4)}),$$

$$(5.5b_2) \quad k_{\lambda\mu} = -2{}^3u_{[\lambda}{}^4u_{\mu]} \operatorname{tgh} \eta \quad (\text{eq. (4.4)'}),$$

$$(5.5c) \quad k_{\lambda\mu} = 2\gamma{}^2u_{[\lambda}({}^3u_{\mu]} + \epsilon{}^4u_{\mu]}) \quad (\text{eq. (4.6)}),$$

where ${}^a u_\lambda$ are perpendicular unit vector fields satisfying (5.2).

Proof. Assume first that all roots $\rho_1, \dots, \rho_4$ are different. Since ρ_1 and ρ_2 (ρ_3 and ρ_4) are supposed to be complex conjugate (real) roots and since

$$\rho_1 \rho_2 = \rho_3 \rho_4 = 1,$$

we may put

$$(5.6) \quad \begin{aligned} \rho_1 &= \cos 2\xi + i \sin 2\xi, & \rho_3 &= \cosh 2\eta + \sinh 2\eta, \\ \rho_2 &= \cos 2\xi - i \sin 2\xi, & \rho_4 &= \cosh 2\eta - \sinh 2\eta. \end{aligned}$$

Then we have from (5.1) and (5.6)

$$\begin{aligned} (\rho_2 - \rho_1) {}^1e_{[\lambda}{}^2e_{\mu]} &= \frac{-4 \sin 2\xi}{1 + \cos 2\xi} {}^1u_{[\lambda}{}^2u_{\mu]} = -4 {}^1u_{[\lambda}{}^2u_{\mu]} \operatorname{tg} \xi, \\ (\rho_4 - \rho_3) {}^3e_{[\lambda}{}^4e_{\mu]} &= + \frac{4 \sinh 2\eta}{1 + \cosh 2\eta} {}^3u_{[\lambda}{}^4u_{\mu]} = + 4 {}^3u_{[\lambda}{}^4u_{\mu]} \operatorname{tgh} \eta. \end{aligned}$$

Substituting these expressions into (2.4b), one obtains (5.5a). The equations (5.5b₁, b₂) may be obtained from (4.4) and (4.4)' in a similar manner. The equation (5.5c) is identical with (4.6). The first equation refers to the case $ghk \neq 0$, $D \neq 0^*$; the next two equations refer to the case $gh \neq 0$, $k = 0$, $D \neq 0$; while the last equation refers to the case $gh \neq 0$, $k = 0$, $D = 0$. The first case exhausts all possibilities connected with $ghk \neq 0$; the latter three cases exhaust all possibilities with $gh \neq 0$ and $k = 0$ (Theorem (4.5)).

Note. According to Theorem (5.2) the right hand sides of the equations (5.5) may be looked upon as canonical forms of $k_{\lambda\mu}$ in the most general case.

6. Physical conclusions

1) Let us first consider the cases we have obtained by virtue of Agreement (1.1) and the inequality (4.1). Starting with $g_{\lambda\mu}$ we defined two projectively

* The case $ghk \neq 0$, $D = 0$ was excluded by virtue of Theorem (4.2).

† Note added for the reprinting, 1964. From now on we consider only the case $g/h > 0$.

orthogonal complexes K and K' in $I_3(P)$ which according to Theorem (3.4) intersect in a non-parabolic congruence (whose axes are K_1 and K_2). The line manifolds K_1 , K_2 , K , and K' are defined intrinsically, so that they do not depend on the choice of coördinate system.

In another paper* I have shown that the projective three dimensional spinor space S_3 , associated with the (centered) four dimensional vector space of the special theory of relativity is fully characterized—as far as spinor theory is concerned—by two projectively perpendicular linear complexes intersecting in a non-parabolic linear congruence. These manifolds play the rôle of the “absolute” in the spinor space S_3 and are defined intrinsically. Identifying these manifolds with K , K' and with the congruence of the axes K_1 and K_2 , we may identify $I_3(P)$ and S_3 . This identification is not affected by the coördinate transformations (which are different in both spaces) because in both cases the manifolds are defined intrinsically. Hence *the introduction of $g_{\lambda\mu}$ in X_4 leads in the most natural way to the theory of spinor space of the relativity theory*. With each point P we associated the space $I_3(P)$ which we identify with the spinor space associated with $T_4(P)$. Hence we may denote this space by $S_3(P)$. The totality of all $S_3(P)$ is the spinor space associated with the space X_4 of the general theory of relativity. This identification of line manifolds in $I_3(P)$ with the corresponding line manifolds in $S_3(P)$ could be taken as a basis for an *intrinsic definition of a connection in the spinor space attached to the space X_4 † of the general (unified) theory of relativity*.

2) Consider now the case where the ${}^a u_\lambda$ involved in Theorem (5.2) are gradient vectors, so that in a properly chosen coördinate system we have ${}^a u_\lambda = \delta_\lambda^a$. This can occur only in the case of the special theory of relativity. Identify $k_{\lambda\mu}$ with the electromagnetic field. We see from Theorem (5.2) that there are only three different classes of all possible electromagnetic fields (one of them being split in two subclasses) and in our special case the components of the field are given in the following table:

k_{12}	k_{23}	k_{31}	k_{14}	k_{24}	k_{34}	k	D
$+ \operatorname{tg} \xi$	0	0	$\operatorname{tgh} \eta$	0	0	$\neq 0$	$\neq 0$
$+ \operatorname{tg} \xi$	0	0	0	0	0	$= 0$	$\neq 0$
0	0	0	$\operatorname{tgh} \eta$	0	0	$= 0$	$= 0$
0	γ	0	0	0	$\epsilon\gamma$	$= 0$	$= 0$

The first three columns give us the electric fields, the next three columns give us the magnetic fields, and the last two columns point out the behavior of k

* V. HLAVATÝ, “Spinor space and line geometry” (*Canadian Journal of Math.* **3**, 1950, pp. 442–459) and “Spinor space and line geometry II” (*Journal of Rational Mechanics and Analysis* **1**, 1951, pp. 49–72) where the DIRAC equations are interpreted by means of the line manifolds mentioned above.

† For this definition one has to use the method described in a paper by my friend the late E. BORTOLOTTI and myself: *Contributi alla teoria delle connessioni* (Annali di Mat. (IV) **15**, pp. 1–17). I will deal with this problem in another paper on Spinor Analysis.

and D^* . According to the Theorems (3.3) and (4.3b), any one of the first three electromagnetic fields has the property that *at each point of the space X_4 it gives rise to totally perpendicular planes, in each of which it defines a coördinate system known up to Lorentz transformations. If the field moves along either of these planes according to these transformations the electric (magnetic) field remains electric (magnetic) and its canonical components are invariants*†. This does not contradict the results of the special theory of relativity: if in an arbitrary system of coördinates x, y, z, t we apply Lorentz transformations which do not affect x and y , then the six transformation formulae for $k_{\lambda\mu}$ split into two groups. One of them consists of two equations, namely $*k_{12} = k_{12}$ and $*k_{34} = k_{34}$, while in the remaining four equations the remaining $*k_{\lambda\mu}$ are expressed as a linear combination of the remaining $k_{\lambda\mu}$. Hence if by chance these latter are equal to zero each of the components $*k_{23}, *k_{14}, *k_{31}, *k_{24}$ is also equal to zero. The electric (magnetic) field remains purely electric (magnetic), and its components are invariant $*k_{12} = k_{12}$ ($*k_{34} = k_{34}$). The italicized statement above asserts only that in our special case ${}^a u_\lambda = \delta_\lambda^a$ a similar situation may be enforced for every electromagnetic field $k_{\lambda\mu}$ if one uses the totally perpendicular planes and the corresponding Lorentz transformations to which $k_{\lambda\mu}$ gives rise at each point of the space.

Before we discuss electromagnetic fields whose canonical form is given by the last row of the table, let us remember how we have obtained the canonical forms given by the first three rows. Starting with the given data only, namely with $g_{\lambda\mu}$, we found at each point P of X_4 four directions defined uniquely and intrinsically‡, and these four directions defined two sets of real bivectors $B_1(P)$, $B_2(P)$ whose planes are totally perpendicular. In either one of these two sets we have chosen *arbitrarily* two fields of perpendicular unit vectors which we denoted by ${}^a u_\lambda$ and obtained in this way (5.5a, b_1, b_2). Finally, in the special theory of relativity we assumed these vectors to be gradient vectors and obtained the three first rows of the table.

The situation changes with any electromagnetic field whose canonical form is given by the last row as is, for example, the electromagnetic field of a plane wave in the electromagnetic theory of light. Here we have only one direction uniquely and intrinsically defined, namely the tangential direction of the plane of the bivector $k_{\lambda\mu}$. Let us call its corresponding vector e' . Moreover there is another plane beside the one of $k_{\lambda\mu}$ which is uniquely and intrinsically defined,

* k and D may be expressed in terms of the three dimensional electric and magnetic vectors respectively. If E, M, ϕ denote their length and their angle respectively then

$$k = E^2 M^2 \cos^2 \phi \quad D = (E^2 - M^2)^2 + 4 E^2 M^2 \cos^2 \phi$$

(the expressions $E^2 - M^2, E^2 M^2 \cos^2 \phi$ are obviously LORENTZ-invariants.)

† In other words: If the field moves according to these rules an observer at rest with respect to one of these coördinate systems does not perceive any change of the field.

‡ Namely the directions of the vectors ${}_1 e'(P), \dots, {}_4 e'(P)$. In the special theory of relativity each point $Q \neq P$ on either one of them is known to be physically characterized by the fact that the points P and Q can be connected (in vacuo) by a light signal. However, only two of these four directions are real.

namely the plane of the bivector $p_{\lambda\mu}$ containing e' and totally perpendicular to $k_{\lambda\mu}^*$. Let us choose arbitrarily a unit vector ${}^1u'$ in the plane of $k_{\lambda\mu}$ and a unit vector ${}^2u'$ in the plane of $p_{\lambda\mu}$. Then the plane totally perpendicular to ${}^1u' \wedge {}^2u'$ is uniquely determined. In it we choose arbitrarily two perpendicular unit vectors ${}^3u'$, ${}^4u'$. They obviously satisfy the condition $e' = {}^3u' + \epsilon {}^4u'$, where ϵ is either $+1$ or -1 , provided that the factor of proportionality of e' is conveniently chosen. Putting ${}^au_\lambda = h_{\lambda\mu} {}^au^\mu$, we obtained (4.6) in this way; assuming ${}^au_\lambda = \delta_\lambda^a$ in the special theory of relativity, we obtained the corresponding canonical form in the table. The vectors ${}^1u'$ and ${}^2u'$ being chosen arbitrarily in the planes of $k_{\lambda\mu}$ and $p_{\lambda\mu}$, respectively, we see that the plane of ${}^3u' \wedge {}^4u'$ is not defined by means of given data $g_{\lambda\mu}$ only. Hence the Lorentz transformation referred to in Theorem (4.4b) is not defined intrinsically by our electromagnetic field.

3) Let us go back to the general case where the vectors ${}^au_\lambda$ are not gradient vectors. Then the table gives us the nonholonomic components (with respect to the ${}^au_\lambda$) of the field. This fact, however, obviously has neither a special geometrical nor physical meaning.

Nevertheless there is a consequence not devoid of interest which might have physical importance: every electromagnetic field is of one of the forms mentioned in Theorem (5.2). Hence in each of the first three cases a given electromagnetic field $k_{\lambda\mu}$ gives rise at each point P of X_4 to two real and totally perpendicular bidirections $B_1(P)$ and $B_2(P)$. The set S_1 (S_2) of all bidirections $B_1(P)$ ($B_2(P)$) as P moves along X_4 is in general a nonholonomic surface. Now the possibility is not *a priori* excluded† that S_1 (S_2) reduces to a set of holonomic surfaces σ_1 (σ_2). In this case the analytic expression for S_1 and S_2 is given by

$$(6.1) \quad x' = x'(\xi^I, \xi^{II}, \xi^{III}, \xi^{IV}),$$

where each individual surface σ_1 (σ_2) of S_1 (S_2) is characterized by some constant values of ξ^{III} , ξ^{IV} (ξ^I , ξ^{II}). Since the sets S_1 and S_2 consist of surfaces whose tangential planes are totally perpendicular we have also $\text{Det} |(\partial x')/(\partial \xi^i)| \neq 0$ ($i = I, \dots, IV$). Now it is quite natural to admit only such transformations of parameters ξ which preserve the sets S_1 and S_2 in (6.1):

$$(6.2) \quad {}^*\xi^a = {}^*\xi^a(\xi^I, \xi^{II}), \quad {}^*\xi^e = {}^*\xi^e(\xi^{III}, \xi^{IV})^\ddagger \quad (a, b = I, II; e, f = III, IV).$$

Choose now the vectors ${}^au'$ (${}^eu'$) in S_1 (S_2). Then the equations

$$(6.3) \quad {}^au_e = 0, \quad {}^eu_a = 0 \quad (a, b = I, II; e, f = III, IV),$$

* Totally perpendicular in the sense that the ideal lines of these bivectors are conjugate polars with respect to H .

† We will come to this question in another paper. Here we assert only that this possibility is not *a priori* excluded, without claiming that it really could happen.

‡ We disregard the possibility of interchanging S_1 and S_2 , similarly to the requirement of the special theory of relativity that the transformation of coördinates must not interchange the reguli of H .

are preserved under (6.2). We see that the m -th row of the table ($m = 1, 2, 3$) gives us substantially the ξ -components of the field in the m -th equation of (5.5)*. Moreover, because of the transformation formulae

$${}^*k_{12} = \frac{\partial(\xi^I, \xi^{II})}{\partial({}^*\xi_I, {}^*\xi_{II})} k_{12}, \quad {}^*k_{34} = \frac{\partial(\xi^3, \xi^4)}{\partial({}^*\xi_3, {}^*\xi_4)} k_{34}, \quad {}^*k_{a'} = \frac{\partial\xi^b}{\partial{}^*\xi^a} \frac{\partial\xi^f}{\partial{}^*\xi^b} k_{bf},$$

the canonical form already obtained for the first three cases given by (5.5) is independent of (6.2). Hence it follows that in the case when S_1 and S_2 constitute a set of holonomic surfaces, the electromagnetic field which is responsible for this situation may be split into two fields. It remains to be seen first whether this situation is geometrically possible and if so, what is the physical meaning of this splitting.

4) It is not difficult to find out to which class (5.5) a given electromagnetic field belongs. We will give here some examples in the special theory of relativity. For the sake of convenience we introduce the contravariant (dual) components $k^{\lambda\nu}$ of $k_{\omega\mu}$ by

$$k^{12} = k_{34}, \quad k^{34} = k_{12}, \quad k^{23} = k_{14}, \quad k^{14} = k_{23}, \quad k^{31} = k_{24}, \quad k^{24} = k_{31}.$$

A plane wave in the electromagnetic theory of light may be given by two periodic functions $a \equiv a(z - t)$ and $b \equiv b(z - t)$, as follows:

$$(6.3a) \quad \begin{aligned} k^{14} &= a, & k^{23} &= -b, \\ k^{24} &= b, & k^{31} &= a, \\ k^{34} &= 0, & k^{12} &= 0. \end{aligned}$$

Here we have $k = 0$ and $D = 0$, so that our electromagnetic field belongs to the class (5.5c). We may very easily find the vectors ${}^a u_\lambda$ involved in (5.5c): the line whose homogeneous coördinates are given in I_3 by (6.3) joins the points whose homogeneous coördinates are $(0, 0, -1, 1)$ and $(a, b, 0, 0)$. The first point is the tangential point e^ν of $k^{\lambda\mu}$ on H while the second point defines the vector

$${}^1 u^\nu \left(\frac{-a}{\sqrt{a^2 + b^2}}, \frac{-b}{\sqrt{a^2 + b^2}}, 0, 0 \right).$$

The components of ${}^1 u_\lambda$ are

$${}^1 u_\lambda \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}, 0, 0 \right)$$

while the vectors ${}^3 u^\nu$ and ${}^4 u^\nu$ may be taken as

$$\begin{aligned} {}^3 u^\nu &(0, 0, -1, 0), \\ {}^4 u^\nu &(0, 0, 0, 1), \end{aligned}$$

* By this statement we mean only that the ξ -component and the corresponding component in the table are simultaneously either equal to zero or different from zero.

so that

$${}^3u_\lambda = {}^3u_\lambda (0, 0, 1, 0),$$

$${}^4u_\lambda = {}^4u_\lambda (0, 0, 0, 1),$$

and $\epsilon = +1$. As far as the vector ${}^2u^\nu$ is concerned, we know that it is a unit vector perpendicular to ${}^1u^\nu$, ${}^3u^\nu$, ${}^4u^\nu$, so that its components are

$${}^2u^\nu \left(\frac{b}{\sqrt{a^2 + b^2}}, \frac{-a}{\sqrt{a^2 + b^2}}, 0, 0 \right),$$

and the covariant components are

$${}^2u_\lambda \left(\frac{-b}{\sqrt{a^2 + b^2}}, \frac{a}{\sqrt{a^2 + b^2}}, 0, 0 \right).$$

Substituting these values into (5.5c) we obtain for example

$$k_{23} = k^{14} = \frac{\gamma a}{\sqrt{a^2 + b^2}}.$$

Comparing this result with the first equation of (6.3a) we obtain $\gamma = \sqrt{a^2 + b^2}$. Hence the field given by (6.3a) may be written

$$(6.3b) \quad k_{\lambda\mu} = 2\sqrt{a^2 + b^2} {}^2u_{[\lambda} ({}^3u_{\mu]} + {}^4u_{\mu]})$$

If a/γ and b/γ are constants and if we associate with the field a coördinate system ${}^*x, {}^*y, {}^*z, {}^*t$ such that

$$x = {}^*x \frac{a}{\gamma} - {}^*y \frac{b}{\gamma}, \quad y = {}^*x \frac{b}{\gamma} + {}^*y \frac{a}{\gamma}, \quad z = {}^*z, \quad t = {}^*t,$$

we obtain for the new components ${}^*k_{\lambda\mu}$ of the field

$$(6.3c) \quad {}^*k_{23} = {}^*k_{24} = \gamma, \quad \text{the remaining } k_{\lambda\mu} = 0$$

(cf. the last row of the table).

The electromagnetic field of an electron moving with the velocity V may be given by

$$(6.4) \quad \begin{aligned} -k_{23} &= \frac{1}{\sqrt{1 - V^2}} X, & -k_{31} &= \frac{1}{\sqrt{1 - V^2}} Y, & -k_{12} &= Z, \\ k_{14} &= -\frac{V}{\sqrt{1 - V^2}} Y, & k_{24} &= \frac{V}{\sqrt{1 - V^2}} X, & k_{34} &= 0, \end{aligned}$$

where (X, Y, Z) is the electric field of the electron at rest. Here we have

$$k = 0, \quad D = (X^2 + Y^2 + Z^2)^2 > 0,$$

so that this field belongs to one of the subclasses (5.5b₁, b₂). The equation (1.4c) reduces here to

$$\lambda^2 (\lambda^2 + E^2) = 0 \quad (E^2 = X^2 + Y^2 + Z^2),$$

so that only the roots $\lambda = 0$ are real. Consequently the line $k_{\lambda\mu}$ intersects H in real points ${}_3e''{}_4e''$, so that this field belongs to the subclass (5.5b₁). It is not difficult to find the vectors ${}_1u_\lambda$ and ${}_2u_\lambda$ involved in (5.5b₁).—It may be shown easily in a similar manner that the electromagnetic field of a moving magnetic dipole belongs to the subclass (5.5b₂).

Finally, any electromagnetic field whose electric and magnetic vectors are both non-vanishing and non-perpendicular belongs to the class (5.5a). For in this case we have according to the equations of the footnote (p. 558)

$$k D \neq 0.$$

5) We end these physical conclusions with a hint for the investigation of the principle C (cf. the Introduction). If we denote by $k_{\lambda\mu}$ the components of an electromagnetic field, then the Maxwell equations in the special theory of relativity may be written in the form

$$(6.5) \quad a) h^{\alpha\beta} \frac{\partial}{\partial x^\alpha} k_{\beta\lambda} = 0, \quad b) 3 \frac{\partial}{\partial x^{[\omega}} k_{\mu\lambda]} = I^\nu \epsilon_{\nu\omega\mu\lambda},$$

where I' is the current electricity vector (density) and $\epsilon_{\nu\omega\mu\lambda}$ is equal to $\begin{cases} +1 \\ -1 \end{cases}$ if

$\nu\omega\mu\lambda$ is an $\begin{cases} \text{even} \\ \text{odd} \end{cases}$ permutation of 1234 and is equal to zero otherwise*. Both

equations (6.5) may be united in one system

$$(6.6) \quad {}^0\Delta k_{\mu\lambda} = \left(\frac{\partial}{\partial x^\alpha} I_\beta \right) h^{\alpha\omega} h^{\beta\nu} \epsilon_{\nu\omega\mu\lambda}, \quad \left({}^0\Delta = h^{\lambda\mu} \frac{\partial^2}{\partial x^\lambda \partial x^\mu} \right).$$

Moreover, we have from (6.5)

$$(6.7) \quad h^{\lambda\mu} \frac{\partial}{\partial x^\lambda} I_\mu = 0.$$

In dealing with problem C) we have to impose some conditions on the Γ 's. In the case of the special (unified) theory of relativity these conditions reduce to some system which will constitute the basis of the corresponding theory. According to the previous equations it is reasonable to expect that this system will have the form (6.6) of Maxwell's equations with (6.7). We have here a hint for conditions to be looked for in the general case.

* $\epsilon_{\nu\omega\mu\lambda}$ is obviously a four vector (density):