Thin Liquid Jets

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1. Introduction. In 1877 Boussinesq [1] obtained equations of motion for thin liquid jets (water bells), taking into account surface tension and body forces. Using these equations, he was able to give a theoretical explanation of certain phenomena observed by Savart [2]. He considered only the case where the streamlines are meridians on a surface of revolution. His theory predicts that, in the absence of body forces, the streamlines will be catenaries. This paper also contains a stability investigation.

We shall generalize Boussinesq's equations to describe bells not axially symmetric. We derive a rather limited class of solutions for the case where the body force vector is constant and a much larger class for the case when the body force vanishes identically. An important subclass of these latter solutions can be obtained by a simple geometric construction. We derive a substitution principle, valid when the body force vanishes identically, this principle enabling one to construct an infinite number of solutions once one is known. A few facts concerning characteristic curves are established.

Consider the stationary motion of a liquid jet in a homogeneous medium, M , which is at rest. In the jet let p denote pressure, ρ the density, f_i the components of the body force per unit mass, and u_i the velocity components. Assuming the liquid non-viscous and incompressible, the equations of motion are

$$(1.1) \quad p_{,i} + \rho u_{i,j} u^j = \rho f_i,$$

$$(1.2) \quad u^i_{,i} = 0,$$

where ρ is a given constant. Throughout this paper, commas denote covariant differentiation and repeated indices are summed in accordance with the usual convention. At the boundary, B , between the jet and M we assume that

$$(1.3) \quad p = p_0 - AC,$$

where p_0 is the pressure in M , A is a positive constant, and C is the mean curvature of B . Here C is defined as in [3], the unit normal to B being directed into M . The theory presented here could be generalized by replacing (1.3) by the more complicated surface conditions of Boussinesq [4] or Duhem [5]. It is clear that one must solve a free boundary problem to determine such a flow. As we shall see, this difficulty is avoided in Boussinesq's treatment.

We deal only with the motion of a thin liquid layer bounded by free surfaces. Volterra [6] wrote a memoir on jets in which he exhibits a number of solutions to this problem, though here again the flow is assumed to be axially symmetric. Later Prs [7] obtained equations of motion for flows not axially symmetric but his treatment is quite different from that given here. These authors neglect surface tension, though body forces are taken into account.

Now, Magnus [8] observed that, under suitable conditions, a vein of water issuing from an orifice in the form of an equilateral triangle first resolves itself into three thin sheets whose planes are perpendicular to the sides of the triangle. In many cases the jet then contracts until it assumes a form whose cross section is similar to that of the orifice, whereupon the phenomenon repeats itself. Similar results hold whenever the orifice is in the form of a regular polygon. Bidone [9] and Buff [10] suggested that this contraction is caused by surface tension, and later work by Rayleigh [11] supports this view. It thus appears that the effect of surface tension is not always negligible in problems such as we consider. For further references to the experimental literature on this subject, see reference [12].

We assume the reader familiar with the theory of surfaces as presented in [3] or in [13].

2. The equations of motion. Equations (1.1) and (1.2) can be written as

$$(2.1) \quad \oint_{\Sigma} (p \nu^i + \rho u^i u_j \nu^j) dS = \int_V \rho f^i dV,$$

$$(2.2) \quad \oint_{\Sigma} u^i \nu_i dS = 0,$$

where Σ is an arbitrary closed surface in the fluid and moving with the fluid, ν is the outward unit normal to Σ , and V is the volume enclosed by Σ . We now proceed to obtain equations of motion for a thin liquid layer bounded by two free surfaces. We assume p_0 has the same value throughout M . This will be true if, for example, M is simply connected.

Let Σ' and Σ'' denote the two free surfaces. Let the normal to Σ' at a point P' intersect Σ'' in a unique point P'' . Let a' be a bounded region on Σ' , a'' the

corresponding region on Σ'' , and a''' the ruled surface, made up of normals to a' , which joins the boundary of a' to that of a'' . Taking $\Sigma = a' + a'' + a'''$ and using (1.3) we obtain

$$\oint_{\Sigma} p \nu_i dS = -A \int_a C \nu_i dS - A \int_{a''} C \nu_i dS + \int_{a'''} (p - p_0) \nu_i dS.$$

To obtain a direct generalization of Boussinesq's equations, we first approximate (2.1) by

$$-2A \int_a C \nu^i dS + \oint_{\sigma} \rho l u^i u_j \nu_2^j ds + \oint_{\sigma} (p - p_0) l \nu_2^i ds = \int_a \rho l f^i dS,$$

where σ is an arbitrary closed curve lying on a stream surface Σ_1 "midway" between Σ' and Σ'' , a is the region of Σ_1 enclosed by σ , C is the mean curvature of Σ_1 , ν_1 is the unit normal to Σ_1 , l is the thickness of the layer measured normal to Σ_1 , and ν_2 is the outward unit normal to σ tangent to Σ_1 . Now, in first approximation, $C(P') = -C(P'')$. Thus, along the straight line joining P' to P'' , $p - p_0$ varies from $-C(P')$ to $C(P')$. It is therefore not unreasonable to suppose that $p \approx p_0$ on Σ_1 so that

$$(2.3) \quad -2A \int_a C \nu_1^i dS + \oint_{\sigma} \rho l u^i u_j \nu_2^j ds = \int_a \rho l f^i dS.$$

Similarly, from (2.2)

$$(2.4) \quad \oint_{\sigma} l u^i \nu_{2i} ds = 0.$$

Let Σ_1 be given parametrically by

$$x^i = x^i(y^1, y^2),$$

where y^1 and y^2 are possible surface coordinates. In order to make use of formulae readily available in texts on differential geometry, we assume the x^i are rectangular Cartesian coordinates.* Then, on Σ_1 , the metric tensor is given by

$$g_{\alpha\beta} = \delta_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta},$$

the second fundamental tensor by

$$d_{\alpha\beta} = \nu_{1i} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta},$$

the Christoffel symbols by

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha\sigma} \left[\frac{\partial g_{\sigma\beta}}{\partial y^\gamma} + \frac{\partial g_{\sigma\gamma}}{\partial y^\beta} - \frac{\partial g_{\beta\gamma}}{\partial y^\sigma} \right],$$

* To avoid confusion, we use $\left\{ \begin{smallmatrix} \text{Latin} \\ \text{Greek} \end{smallmatrix} \right\}$ indices for quantities referred to the coordinates $\left\{ \begin{smallmatrix} x^i \\ y^\alpha \end{smallmatrix} \right\}$.

and the Riemann tensor by

$$R_{\beta\gamma\delta}^{\alpha} = \frac{\partial}{\partial y^{\gamma}} \Gamma_{\beta\delta}^{\alpha} - \frac{\partial}{\partial y^{\delta}} \Gamma_{\beta\gamma}^{\alpha} + \Gamma_{\beta\delta}^{\sigma} \Gamma_{\sigma\gamma}^{\alpha} - \Gamma_{\beta\gamma}^{\sigma} \Gamma_{\sigma\delta}^{\alpha}.$$

We then have

$$(2.5) \quad C = d_{\alpha}^{\alpha},$$

$$(2.6) \quad \nu_{1i} = \frac{1}{\sqrt{g}} \epsilon_{ijk} \frac{\partial x^j}{\partial y^1} \frac{\partial x^k}{\partial y^2},$$

$$(2.7) \quad d_{\alpha\beta,\gamma} = d_{\alpha\gamma,\beta},$$

$$(2.8) \quad R_{1212} = d_{11} d_{22} - d_{12}^2.$$

Here g is the determinant of $g_{\alpha\beta}$ and ϵ_{ijk} is that completely skew symmetric tensor for which $\epsilon_{123} = 1$. Equations (2.7) are the Mainardi-Codazzi relations, (2.8) is Gauss's equation.

On Σ_1 , the streamlines are given by relations of the form

$$y^{\alpha} = y^{\alpha}(y_0^1, y_0^2, t) = y^{\alpha}(y_0, t),$$

where t is time and $y_0^{\alpha} = y^{\alpha}$ at $t = 0$, or alternately by

$$x^i = x^i(x_0, t) = x^i[y^{\alpha}(y_0, t)],$$

where $x_0^i = x^i = x^i(y_0)$ at $t = 0$. In the following, dots will indicate partial differentiation with respect to t holding y_0^{α} constant. Then

$$u^i = \dot{x}^i = \frac{\partial x^i}{\partial y^{\alpha}} \dot{y}^{\alpha}.$$

Since the motion is steady, l and $\dot{y}^{\alpha}$ do not depend explicitly on the time when expressed in terms of y^1 and y^2 . Also, since Σ_1 is a stream surface, $u^i \nu_{1i} = 0$. Now, from (2.4)

$$\oint_{\sigma} l u^i \nu_{2i} ds = \oint_{\sigma} l \dot{y}^{\alpha} \nu_{2\alpha} ds = \int_a (l \dot{y}^{\alpha})_{,\alpha} dS = 0,$$

or, since a is arbitrary,

$$(2.9) \quad (l \dot{y}^{\alpha})_{,\alpha} = 0.$$

Similarly

$$\begin{aligned} \oint_{\sigma} l u^i u_j \nu_2^j ds &= \oint_{\sigma} l \frac{\partial x^i}{\partial y^{\alpha}} \dot{y}^{\alpha} \dot{y}^{\beta} \nu_{2\beta} ds, \\ &= \int_a \left(l \frac{\partial x^i}{\partial y^{\alpha}} \dot{y}^{\alpha} \dot{y}^{\beta} \right)_{,\beta} dS, \\ &= \int_a l \dot{y}^{\beta} \left(\frac{\partial x^i}{\partial y^{\alpha}} \dot{y}^{\alpha} \right)_{,\beta} dS. \end{aligned}$$

But

$$\left(\frac{\partial x^i}{\partial y^\alpha}\right)_{,\beta} = d_{\alpha\beta} \nu_1^i,$$

so that (2.3) becomes

$$\int_a \left\{ [-2AC + \rho l \dot{y}^\alpha \dot{y}^\beta d_{\alpha\beta}] \nu_1^i + \rho l \frac{\partial x^i}{\partial y^\alpha} \dot{y}_{,\beta}^\alpha \dot{y}^\beta \right\} dS = \int_a \rho l f^i da,$$

from which

$$(2.10) \quad \dot{y}_{,\beta}^\alpha \dot{y}^\beta = f^\alpha = g^{\alpha\beta} \frac{\partial x^i}{\partial y^\beta} f_i,$$

$$(2.11) \quad d_{\alpha\beta} \dot{y}^\alpha \dot{y}^\beta = f_i \nu_1^i + \frac{2AC}{\rho l}.$$

Equation (2.11) can be written in the form

$$(2.12) \quad kv^2 = f_i \nu_1^i + \frac{2AC}{\rho l},$$

where k is the normal curvature of Σ_1 in the direction determined by $\dot{y}^\alpha$ and v is the speed. Equations (2.10) and (2.11) together are equivalent to

$$(2.13) \quad \frac{D\dot{x}^i}{dt} = f^i + \frac{2AC}{\rho l} \nu_1^i,$$

where $\frac{D\dot{x}^i}{dt}$ is the absolute acceleration.

Now let

$$u_\alpha = g_{\alpha\beta} \dot{y}^\beta,$$

$$t_{\alpha\beta} = lu_\alpha u_\beta - \frac{2A}{\rho} g_{\alpha\beta}.$$

In terms of these quantities, (2.10) and (2.11) become

$$(2.14) \quad t_{\beta,\alpha}^\alpha = lf_\beta,$$

$$(2.15) \quad d_{\alpha\beta} t^{\alpha\beta} = lf_i \nu_1^i.$$

Here the term $lu_\alpha u_\beta$ is closely related to the momentum transfer tensor $\rho u_i u_j$. One might generalize the equations of motion given here by supposing that (2.9), (2.14), and (2.15) hold quite generally, but that $t_{\alpha\beta}$ is given, for example, by

$$t_{\alpha\beta} = lu_\alpha u_\beta + (\lambda_0 + \lambda_1 u_{,\sigma}^\sigma) g_{\alpha\beta} + \lambda_2 (u_{\alpha,\beta} + u_{\beta,\alpha}),$$

where λ_0 , λ_1 , and λ_2 are material constants, this generalization being suggested to me by results obtained by Boussinesq [4].

We shall not attempt to give a precise meaning to the term "thin layer"

3. An initial value problem. We consider the following initial value problem. Prescribe

- (I) the body force vector $f_i = f_i(x^1, x^2, x^3)$,
- (II) a curve Γ , given parametrically by $x^i = x_0^i(u)$,
- (III) functions $l_0(u)$ and $\dot{x}_0^i(u)$, defined on Γ ,
- (IV) functions $y^\alpha = y^\alpha(u, t)$.

One must then determine functions $x^i = x^i(y^1, y^2)$ and $l = l(y^1, y^2)$ such that

$$(I') \quad \text{at } t = 0, \quad x^i = x_0^i(u), \quad \frac{\partial x^i}{\partial y^\alpha} \dot{y}^\alpha = \dot{x}_0^i(u), \quad l = l_0(u)$$

and such that

(II') for $t > 0$ equations (2.9), (2.10), and (2.11) hold when $g_{\alpha\beta}$, $d_{\alpha\beta}$, C , etc., are defined as in Section 2.

Here dots denote differentiation with respect to t , holding u constant. Intuitively, we agree to focus our attention on those fluid particles which lie on Γ at $t = 0$. We assume that the function $\dot{x}_0^i(u)$ and $y^\alpha(u, t)$ are of class C'' , that $\dot{x}_0^i(u)$ is of class C' , and that the coordinate transformation IV is one to one, these conditions holding for $u_0 \leq u \leq u_1$, $0 \leq t < t_0$. In the remainder of this section we also assume $f_i \equiv 0$.

We can write (2.9) in a form better adapted to this problem by noting that

$$\begin{aligned} \sqrt{g} (l \dot{y}^\alpha)_{,\alpha} &= \frac{\partial}{\partial y^\alpha} (l \sqrt{g} \dot{y}^\alpha) \\ &= \frac{\partial (l \sqrt{g})}{\partial y^\alpha} \dot{y}^\alpha + l \sqrt{g} \frac{\partial \dot{y}^\alpha}{\partial y^\alpha} \\ &= \dot{l} \sqrt{g} + \frac{\sqrt{g}}{J} l \sqrt{g} \frac{\dot{J}}{\sqrt{g}} \\ &= \frac{\sqrt{g}}{J} \dot{l} J = 0, \end{aligned}$$

where J is the absolute scalar

$$J = \sqrt{g} \frac{\partial(y^1, y^2)}{\partial(u, t)}.$$

We now take $y^1 = u$, $y^2 = t$. Then $J = \sqrt{g}$, and

$$(3.1) \quad l \sqrt{g} = \lambda(y^1, y^2) = \lambda(y^1, 0).$$

Substituting (3.1) in (2.11) we obtain

$$(3.2) \quad d_{22} = \frac{2A \sqrt{g}}{\rho \lambda - 2A \sqrt{g} g^{22}} [g^{11} d_{11} + 2g^{12} d_{12}],$$

provided $\rho\lambda \neq 2A \sqrt{g} g^{22}$. Also, from (2.5), (2.7), (2.8), and (2.10)

$$\begin{aligned}
 \frac{\partial g_{22}}{\partial y^2} &= 0, \\
 \frac{\partial g_{12}}{\partial y^2} &= \frac{1}{2} \frac{\partial g_{22}}{\partial y_1}, \\
 \frac{\partial d_{11}}{\partial y^2} &= \frac{\partial d_{12}}{\partial y^1} + d_{1\alpha} \Gamma_{12}^\alpha - d_{2\alpha} \Gamma_{11}^\alpha, \\
 \frac{\partial d_{12}}{\partial y^2} &= \frac{\partial d_{22}}{\partial y^1} - d_{2\alpha} \Gamma_{12}^\alpha + d_{1\alpha} \Gamma_{22}^\alpha, \\
 \frac{\partial^2 g_{11}}{\partial y^{22}} &= 2g_{\alpha\beta} \Gamma_{12}^\alpha \Gamma_{12}^\beta - 2(d_{11} d_{22} - d_{12}^2),
 \end{aligned}
 \tag{3.3}$$

where $\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha\sigma} \left[\frac{\partial g_{\sigma\beta}}{\partial y^\gamma} + \frac{\partial g_{\sigma\gamma}}{\partial y^\beta} - \frac{\partial g_{\beta\gamma}}{\partial y^\sigma} \right]$. Suppose for the moment that these

equations, together with the given initial data, determine $g_{\alpha\beta}$ and $d_{\alpha\beta}$ uniquely and that $g = g_{11}g_{22} - g_{12}^2 > 0$, $g_{11} > 0$, $g_{22} > 0$. There will then exist a unique surface Σ_1 , given parametrically by $x^i = x^i(y^1, y^2) = x^i(u, t)$ for which $g_{\alpha\beta}$ and $d_{\alpha\beta}$ are, respectively, the first and second fundamental tensors, and for which (I') is satisfied. Also, by (3.1), l is determined. The functions $x^i(u, t)$ and $l(u, t)$ thus determined solve the initial value problem.

Assume now that the vectors $\dot{x}_0^i$ and $d\dot{x}_0/du$ are linearly independent on Γ . Then, at $t = 0$, $g_{\alpha\beta}$, ν_{1i} , d_{11} , d_{12} , and $\partial g_{11}/\partial y^2$ can be determined as follows:

$$\begin{aligned}
 g_{11} &= \delta_{ij} \frac{dx_0^i}{du} \frac{dx_0^j}{du}, \\
 g_{12} &= \delta_{ij} \frac{dx_0^i}{du} \dot{x}_0^j, \\
 g_{22} &= \delta_{ij} \dot{x}_0^i \dot{x}_0^j, \\
 \nu_{1i} &= \frac{1}{\sqrt{g}} \epsilon_{ijk} \frac{dx_0^j}{du} \dot{x}_0^k, \\
 d_{11} &= \nu_{1i} \frac{d^2 x_0^i}{dy^2}, \\
 d_{12} &= \nu_{1i} \frac{d\dot{x}_0^i}{du}, \\
 \frac{\partial g_{11}}{\partial y^2} &= 2\delta_{ij} \frac{dx_0^i}{du} \frac{d\dot{x}_0^j}{du},
 \end{aligned}$$

where $g_{11} > 0$, $g_{22} > 0$, $g = g_{11}g_{22} - g_{12}^2 > 0$ because of the assumed linear independence. Equations (3.1) and (3.2) then determine d_{22} , unless

$$(3.4) \quad \rho l_0 = 2Ag^{22} = \frac{2Ag_{11}}{g}.$$

If we further assume that (3.4) holds at no point of Γ , the initial value problem is well put. That is, when $d_{\alpha\beta}$, $g_{\alpha\beta}$, and $\partial g_{11}/\partial y^2$ are analytic functions of y^1 at $y^2 = 0$, equations (3.2) and (3.3), together with this given initial data, determine a unique set of analytic functions $d_{\alpha\beta}$ and $g_{\alpha\beta}$, at least in the neighborhood of $y^2 = t = 0$ [14], so the initial value problem has a unique solution.

Now let ω be the angle determined by the tangent vector to Γ and the velocity vector. In terms of these quantities, (3.4) can be written in the invariant form

$$(3.5) \quad \sin^2 \omega = \frac{2A}{\rho l v^2},$$

the right hand side being evaluated at $t = 0$. We now make the following

Definition: A curve lying on the surface Σ_1 will be called *characteristic* if and only if it intersects the streamlines at an angle ω satisfying (3.5).

Assuming $(2A)/(\rho l v^2) \neq 1$, we see that through each point of Σ_1 pass two characteristics, these curves being real if $(2A)/(\rho l v^2) < 1$, imaginary if $(2A)/(\rho l v^2) > 1$. The streamline through this point bisects the angle between these directions. When $(2A)/(\rho l v^2) = 1$, these two characteristic curves coincide.

For physical reasons, one should take $l_0 > 0$, though this is not essential in the above discussion.

The above definition of characteristic curves applies whether or not the body force is zero.

4. Properties of the solutions of the equations of motion. Consider the case where the body force vanishes identically. Let $\varphi = \varphi(y^1, y^2)$ be any continuously differentiable non-vanishing function which is constant along streamlines, $\dot{\varphi} = \varphi_{,\alpha} \dot{y}^\alpha = 0$. Consider the substitution

$$l = \varphi^2 \bar{l}$$

$$\bar{t} = \int \varphi[y^1(y_0, t), y^2(y_0, t)] dt.$$

Then, from equation (2.9),

$$(l\dot{y}^\alpha)_{,\alpha} = \left(\frac{l}{\varphi} \frac{dy^\alpha}{d\bar{t}} \right)_{,\alpha} = \frac{1}{\varphi} \left(l \frac{dy^\alpha}{d\bar{t}} \right)_{,\alpha} = 0.$$

By equation (2.10)

$$\dot{y}_{,\beta}^\alpha \dot{y}^\beta = \varphi \left(\varphi \frac{dy^\alpha}{d\bar{t}} \right)_{,\beta} \frac{dy^\beta}{d\bar{t}} = \varphi^2 \left(\frac{dy^\alpha}{d\bar{t}} \right)_{,\beta} \frac{dy^\beta}{d\bar{t}} = 0,$$

while (2.11) becomes

$$d_{\alpha\beta} \dot{y}^\alpha \dot{y}^\beta - \frac{2A}{\rho l} C = \varphi^2 \left[d_{\alpha\beta} \frac{dy^\alpha}{dt} \frac{dy^\beta}{dt} - \frac{2AC}{\rho l} \right] = 0.$$

Thus the equations of motion are invariant under this substitution, which multiplies the thickness of the layer by φ^2 and the velocity vector by $1/\varphi$, leaving the surface Σ_1 and the streamline pattern on Σ_1 invariant. This substitution is, in some respects, similar to the Munk-Prim substitution [15] in gas dynamics. Also, in this case, the speed is constant along streamlines, which are geodesics on Σ_1 . This is an immediate consequence of (2.10).

Next assume the body force derivable from a potential, $f_i = -\Omega_{,i}$. Then, if v denotes the speed,

$$\begin{aligned} \frac{\dot{v}^2}{2} &= \frac{\dot{g}_{\alpha\beta} \dot{y}^\alpha \dot{y}^\beta}{2} \\ &= g_{\alpha\beta} \dot{y}_{,\sigma}^\alpha \dot{y}^\sigma \dot{y}^\beta \\ &= f_\beta \dot{y}^\beta = -\Omega_{,i} \frac{\partial x^i}{\partial y^\beta} \dot{y}^\beta = -\dot{\Omega}, \end{aligned}$$

or

$$(4.1) \quad v^2 + 2\Omega = h,$$

where h is constant along the streamlines.

The dynamical similarity parameters of this theory are the Weber number

$$\mathfrak{W} = \rho L V^2 / A$$

and the Froude number

$$\mathfrak{F} = V^2 / (L' F)$$

where L is a typical length, L' a typical distance, V a typical speed, and F a typical acceleration associated with the body force. Ordinarily, F is the gravitational constant. If we define a local Weber number by taking $L = \frac{1}{2}l$, $V = v$, equation (3.5) becomes simply

$$(4.2) \quad \sin^2 \omega = 1/\mathfrak{W}.$$

The characteristics are thus real if $\mathfrak{W} > 1$, imaginary if $\mathfrak{W} < 1$.

5. Pseudo-plane flow. Here we derive a solution of an initial value problem of the type discussed in Section 3. Prescribe constants a , $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$, and l_0 , with $l_0 > 0$, $\dot{x}_0^2 + \dot{y}_0^2 > 0$, $a \neq 0$. Let Γ be a straight line, given parametrically by $x = 0$, $y = 0$, $z = u$, let $y^1 = u$, $y^2 = t$, and assume the body force derivable from the potential $\Omega = -ax$. We look for a solution of the form

$$x = x(t), \quad y = y(t), \quad z = u + f(t), \quad l = l(t)$$

The surface Σ_1 will then be a cylinder whose rulings, the lines $t = \text{constant}$, are parallel to the z -axis. We call this pseudo-plane flow because the velocity vector and l will be constant along these rulings. Noting that $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ we have, in obvious notation,

$$\begin{aligned} g_{uu} &= 1, & g_{ut} &= \dot{z}, & g_{tt} &= v^2, \\ g^{uu} &= \frac{v^2}{\dot{x}^2 + \dot{y}^2}, & g^{ut} &= \frac{-\dot{z}}{\dot{x}^2 + \dot{y}^2}, & g^{tt} &= \frac{1}{\dot{x}^2 + \dot{y}^2}, \\ \nu_{1x} &= \frac{-\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, & \nu_{1y} &= \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, & \nu_{1z} &= 0, \\ d_{uu} &= 0, & d_{ut} &= 0, & d_{tt} &= \frac{(\dot{x}\ddot{y} - \dot{y}\ddot{x})}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \\ C &= \frac{(\dot{x}\ddot{y} - \dot{y}\ddot{x})}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \\ f_u &= 0, & f_t &= a\dot{x}, & f, \nu_1^i &= \frac{-a\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}. \end{aligned}$$

The equations of motion then become

$$\begin{aligned} \frac{\dot{z}}{v^2} &= 2a\dot{x}, & \ddot{z} &= 0, & \overline{l\sqrt{\dot{x}^2 + \dot{y}^2}} &= 0, \\ \frac{(\dot{y}\ddot{x} - \dot{x}\ddot{y})}{\sqrt{\dot{x}^2 + \dot{y}^2}} &= \frac{a\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + \frac{2A}{\rho l} (\dot{y}\ddot{x} - \dot{x}\ddot{y}) \frac{v^2}{(\dot{x}^2 + \dot{y}^2)^{3/2}}, \end{aligned}$$

so that

$$(5.1) \quad \dot{x}^2 + \dot{y}^2 = \dot{x}_0^2 + \dot{y}_0^2 + 2ax,$$

$$(5.2) \quad \dot{z} = f = \dot{z}_0,$$

$$(5.3) \quad l\sqrt{\dot{x}^2 + \dot{y}^2} = l_0\sqrt{\dot{x}_0^2 + \dot{y}_0^2} = \alpha = \text{constant},$$

$$(5.4) \quad (\dot{y}\ddot{x} - \dot{x}\ddot{y}) [1 - \beta(\dot{x}^2 + \dot{y}^2)^{-1}] = a\dot{y},$$

where $\beta = (2A)/(\rho\alpha)$. Now let

$$\dot{x} = q \cos \theta, \quad \dot{y} = q \sin \theta, \quad q = \sqrt{\dot{x}^2 + \dot{y}^2}, \quad q_0 = \sqrt{\dot{x}_0^2 + \dot{y}_0^2}.$$

Then (5.1) and (5.2) require that, so long as $q \neq 0$,

$$\dot{q} = a \cos \theta, \quad \dot{\theta}(q - \beta) = -a \sin \theta.$$

Let $\gamma = (q_0 - \beta) \sin \theta_0$, $\theta_0 = \tan^{-1} \dot{y}_0/\dot{x}_0$. Then, if $\gamma \neq 0$, the above system can be integrated and we have

$$(5.5) \quad \cot \theta = \cot \theta_0 + at/\gamma,$$

$$(5.6) \quad q = \beta + \gamma \csc \theta.$$

If $\gamma = 0$ and if θ is finite at $t = 0$, we must have $\sin \theta_0 = 0$. Then

$$x = \frac{1}{2}at^2 + \dot{x}_0 t, \quad y = 0, \quad z = u + \dot{z}_0 t, \quad l = \frac{l_0 |\dot{x}_0|}{|at + \dot{x}_0|}.$$

We now exclude this case. The equations of motion then yield

$$(5.7) \quad x = \frac{1}{2a} (q^2 - q_0^2) = \frac{\gamma}{2a} (\csc \theta - \csc \theta_0) [\beta + \gamma (\csc \theta + \csc \theta_0)],$$

$$(5.8) \quad y = \frac{\gamma^2}{a} (\cot \theta - \cot \theta_0) - \frac{\beta\gamma}{a} \log \frac{\tan \frac{1}{2} \theta}{\tan \frac{1}{2} \theta_0},$$

$$(5.9) \quad z = u + \dot{z}_0 t,$$

as equations of the streamlines, on which $u = \text{constant}$. The thickness, l , is given by

$$(5.10) \quad l = l_0 q_0 / q.$$

Let us investigate the details of the motion. For this purpose we can, without loss of generality, assume $a > 0$, $\sin \theta_0 > 0$. As long as $q \neq 0$, equations (5.5) and (5.6) show that $\sin \theta$ and $q - \beta$ can not vanish. Consider first the case $q_0 > \beta$. From (5.6), q can never vanish, so (5.7), (5.8), and (5.9) hold throughout the motion. From (5.5)

$$\lim_{t \rightarrow \infty} \theta = 0,$$

while (5.7) and (5.8) yield

$$\lim_{t \rightarrow \infty} \frac{y^2}{x} = \lim_{\theta \rightarrow 0} \frac{2\gamma}{a} \frac{\left[\gamma(\cos \theta - \cot \theta_0 \sin \theta) - \beta \sin \theta \log \frac{\tan \frac{1}{2} \theta}{\tan \frac{1}{2} \theta_0} \right]^2}{(1 - \sin \theta \csc \theta_0)[\beta \sin \theta + \gamma(1 + \sin \theta \csc \theta_0)]} = \frac{2\gamma^2}{a}.$$

The sections of Σ_1 by planes $z = \text{constant}$ resemble parabolas with axes parallel to the body force vector. If surface tension be neglected ($\beta = 0$), they are strictly such parabolas.

Consider now the case $q_0 < \beta$. In this case it is easily verified that θ will increase until, at $t = t^*$, $\sin \theta = (\beta - q_0)/\beta$ when $q = 0$. If (5.5) and (5.6) were to continue to hold for $t > t^*$, q would become negative, which is impossible. By (5.3), l increases without bound as $t \rightarrow t^*$, so there is doubt as to the validity of our theory. Furthermore, our surface coordinate system is singular at $t = t^*$. One can, however, continue the solution in a reasonable fashion, this continuation being given by

$$x = \frac{a}{2} (t - t^*)^2 + x(t^*), \quad y = y(t^*), \quad z = u + \dot{z}_0 t, \quad l = \frac{\alpha}{a(t - t^*)}.$$

for $t > t^*$.

The above considerations indicate that flows for which $\beta/q < 1$ are qualitatively different from those for which $\beta/q > 1$. I conjecture that flows for which

$\beta/q > 1$ are unstable. If so, the peculiar behavior predicted in this case may not be observable. Now

$$\begin{aligned}\frac{\beta}{q} &= \frac{2A}{\rho\alpha q} = \frac{2A}{\rho l q^2} \\ &= \frac{2Ak}{l \cos^2 \varphi},\end{aligned}$$

where k is the normal curvature of Σ_1 in the direction of the velocity vector, φ the angle between the velocity vector and the line of curvature obtained by intersecting Σ_1 with a plane $z = \text{constant}$. Using Euler's formula, we obtain

$$\frac{\beta}{q} = \frac{\frac{2AC}{l}}{\rho v^2 k}.$$

Here $2AC/l$ is the magnitude of the force due to surface tension, $\rho v^2 k$ the projection of the centripetal force acting on a fluid particle on the normal to Σ_1 .

In case $a = 0$, it is easy to show that Σ_1 is plane, the streamlines being straight lines, unless $q_0 = \beta$. When $q_0 = \beta$, the initial curve is characteristic, the surface Σ_1 and the streamlines being indeterminate. If $a \neq 0$ and the initial curve is characteristic, no solution will exist unless $y_0 = 0$.

6. An infinite family of solutions. In this section we obtain an infinite family of solutions to the equations of motion, using an inverse method. Neglecting body forces, we find all solutions for which the streamlines are lines of curvature.

We again introduce u and t as surface coordinates. We first simplify the equations of motion by using the substitution principle. The general solution to the problem considered can then be obtained by applying this substitution to the general solution of the simplified equations. There is no loss of generality in taking $g_{ut} = 0$ at $t = 0$. Equations (3.5) and the substitution principle then permit us to take $g_{ut} \equiv 0$, $g_{tt} = v^2 = 1$, provided we exclude stagnation points. In other words, u and t are geodesic parameters, t being arc length along the streamlines, which are geodesics. It now follows from (3.3) that, for a proper choice of the parameter u , $l \sqrt{g_{uu}} = 2A/\rho$. For the streamlines to be lines of curvature, we must have $d_{ut} = 0$. Equations (3.3) then yield

$$(6.1) \quad \frac{\partial d_{tt}}{\partial u} = 0,$$

$$(6.2) \quad \frac{\partial}{\partial t} \frac{d_{uu}}{\alpha} = d_{tt} \frac{\partial \alpha}{\partial t},$$

$$(6.3) \quad d_{uu} d_{tt} = -\alpha \frac{\partial^2 \alpha}{\partial t^2},$$

$$(6.4) \quad d_{uu} = \alpha(1 - \alpha) d_{tt},$$

where $\alpha = \sqrt{g_{uu}} > 0$. These equations are satisfied if $d_{uu} = d_{tt} = 0$. Σ_1 is then a plane and the streamline pattern consists of an arbitrary family of straight lines. We now exclude this case.

Before proceeding further, we observe that the streamlines must be plane curves since they are simultaneously geodesics and lines of curvature. Also, it follows from well-known theorems of differential geometry that d_{tt} is, to within a sign, the curvature of the streamlines.

Now, equations (6.1) through (6.4) are also satisfied if $\alpha = 1$, $d_{uu} = 0$, $d_{tt} = k(t)$. Noting that

$$\alpha = 1 = \frac{2A}{\rho l} = \frac{2A}{\rho l v^2},$$

we see that the orthogonal trajectories of the streamlines are characteristic curves. From the equations

$$g_{uu} = 1, \quad g_{ut} = 0, \quad g_{tt} = 1, \quad d_{uu} = 0, \quad d_{ut} = 0, \quad d_{tt} = k(t),$$

one can conclude that Σ_1 is a cylinder representable in the form $x = x(t)$, $y = y(t)$, $z = u$, the functions $x(t)$ and $y = y(t)$ being arbitrary except for the restriction $\dot{x}^2 + \dot{y}^2 = 1$. The thickness of the layer is given by $l = 2A/\rho$. We now exclude this case. Then

$$\frac{\partial}{\partial t} \frac{d_{uu}}{\alpha} = \frac{\partial \alpha}{\partial t} d_{tt} = \frac{\partial \alpha}{\partial t} \frac{d_{uu}}{(1 - \alpha)},$$

so $d_{uu} = \frac{\alpha}{\alpha - 1} f_1(u)$. Also

$$\left(\frac{d_{uu}}{\alpha}\right) \frac{\partial}{\partial t} \left(\frac{d_{uu}}{\alpha}\right) = \frac{1}{\alpha} \frac{\partial \alpha}{\partial t} d_{uu} d_{tt} = -\frac{\partial \alpha}{\partial t} \frac{\partial^2 \alpha}{\partial t^2},$$

or

$$\left(\frac{d_{uu}}{\alpha}\right)^2 = \frac{f_1^2(u)}{(\alpha - 1)^2} = f_2(u) - \left(\frac{\partial \alpha}{\partial t}\right)^2,$$

where $f_2(u) > 0$. Solving this differential equation for α we obtain

$$\alpha = 1 \pm [f_1^2/f_2 + (\sqrt{f_2} t + f_3(u))^2]^{\frac{1}{2}}$$

while

$$d_{tt} = \frac{d_{uu}}{\alpha(1 - \alpha)} = -\frac{f_1}{(\alpha - 1)^2}.$$

Using (6.1), we then find that

$$\begin{aligned}\alpha &= 1 + f(u) [a^2 + (t + b)^2]^{\frac{1}{2}}, \\ g_{uu} &= \alpha^2, & g_{ut} &= 0, & g_{tt} &= 1 \\ d_{uu} &= \alpha a f^2 / (\alpha - 1), & d_{ut} &= 0, & d_{tt} &= -a f^2 / (\alpha - 1)^2,\end{aligned}$$

where a and b are constants. We must have $a f(u) \neq 0$, $\alpha > 0$. If f , a , and b are prescribed, Σ_1 is determined to within a rigid motion. With proper choice of axes, Σ_1 is given parametrically by

$$\begin{aligned}(6.5) \quad x &= -a \cosh \frac{z}{a} \sin \theta + \int_0^u \cos \theta \, du, \\ y &= a \cosh \frac{z}{a} \cos \theta + \int_0^u \sin \theta \, du,\end{aligned}$$

where $\theta = \theta(u) = -\frac{a}{|a|} \int_0^u f(u) \, du + 3\pi/2$, z being given as a function of time by

$$(6.6) \quad z = \log \left[\frac{t+b}{a} + \sqrt{1 + \left(\frac{t+b}{a} \right)^2} \right]^a.$$

The thickness l is given by

$$(6.7) \quad l = \frac{2A}{\rho \left[1 - a \theta' \cosh \frac{z}{a} \right]},$$

where $\theta' = d\theta/du$. We leave it to the reader to verify this solution. The curvature, k , of the streamlines is given by

$$k = |d_{tt}| = \frac{1}{\left[1 + \left(\frac{t+b}{a} \right)^2 \right]},$$

which is the intrinsic equation of a catenary. Since k is independent of u , a rigid motion takes any given streamline into any other. We note that, at $u = 0$, $x = a \cosh (z/a)$, $y = 0$. Furthermore, the catenary $u = u_1$ lies in a plane with unit normal $[\cos \theta(u_1), \sin \theta(u_1), 0]$, and its directrix is the line

$$x = \int_0^{u_1} \cos \theta \, du, \quad y = \int_0^{u_1} \sin \theta \, du$$

parallel to the z -axis. Thus the directrices are rulings on the cylinder generated by the curve

$$x = \int_0^u \cos \theta \, du, \quad y = \int_0^u \sin \theta \, du, \quad z = 0,$$

the unit tangent to this curve being normal to the plane of the catenary. Such flows can thus be obtained by the following geometric construction: *draw the catenary $x = a \cosh (z/a)$ in the plane $y = 0$, its directrix being the line $x = y = 0$. Let the catenary move in such a way that its directrix traces out the cylinder*

$$x = \int_0^u \cos \theta \, du, \quad y = \int_0^u \sin \theta \, du,$$

the plane of the catenary moving perpendicular to the z -axis so as to remain normal to this cylinder. The catenary then sweeps out the surface Σ_1 . This construction also produces a solution if $\theta' \equiv 0$, in which case $\alpha = 1$.

Now θ must be such that

$$\alpha = 1 + |a| f(u) \left[1 + \left(\frac{t+b}{a} \right)^2 \right]^{\frac{1}{2}} = 1 - a\theta' \cosh (z/a) > 0,$$

at least in the neighborhood of $t = 0$. For this to be true for all $t > 0$, we must have a $\theta' < 0$, so that

$$1/\mathfrak{W} = \frac{2A}{\rho b^2} = \alpha > 1.$$

If, on the contrary, $a\theta' > 0$, α will decrease with time, approaching zero at $t = t^*(u)$. It is easily verified that the locus of points where $t = t^*$ is, in general, a line segment parallel to the z -axis. In the particular case treated by Boussinesq, the streamlines converge to a single point. It seems that we can do no more than speculate concerning the nature of the flow past this point, so we leave this discussion, after noting that here

$$1/\mathfrak{W} = \alpha < 1.$$

In the geometric construction mentioned above, the cylinder may be chosen arbitrarily so long as the curvature of its generating curve exists and is bounded. The constant a must then be chosen so that $\alpha > 0$ at $t = 0$.

It would be desirable to obtain solutions similar to those given by equations (6.5), (6.6), and (6.7), but taking gravity into account.

Boussinesq's stability studies seem to indicate that the above flows may be unstable if $\mathfrak{W} < 1$.

7. Axially symmetric flow. Suppose Σ_1 is a surface of revolution, given parametrically in cylindrical coordinates by

$$r = r(t), \quad \theta = u + f(t), \quad z = z(t),$$

where t is the time and $0 = u < 2\pi$. As initial conditions, take $r(0) = r_0$, $f(0) = z(0) = 0$, $\dot{r}(0) = \dot{r}_0$, $\dot{f}(0) = \dot{f}_0$, $\dot{z}(0) = \dot{z}_0$, $l(u, 0) = l_0$, where r_0 , $\dot{r}_0$, $\dot{f}_0$, $\dot{z}_0$, and l_0

are given constants with $r_0 > 0$, $l_0 > 0$, $\dot{r}_0^2 + \dot{z}_0^2 > 0$. We assume the body force vanishes and take $y^1 = u$, $y^2 = t$. Letting $v = \sqrt{\dot{r}^2 + r^2 \dot{f}^2 + \dot{z}^2}$ denote the speed, we have

$$\begin{aligned} g_{uu} &= r^2, & g_{ut} &= r^2 \dot{f}, & g_{tt} &= v^2, \\ g^{uu} &= \frac{v^2}{r^2(\dot{r}^2 + \dot{z}^2)}, & g^{ut} &= \frac{-\dot{f}}{\dot{r}^2 + \dot{z}^2}, & g^{tt} &= \frac{1}{\dot{r}^2 + \dot{z}^2}, \\ d_{uu} &= -\frac{r\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}}, & d_{ut} &= \frac{r\dot{z}\dot{f}}{\sqrt{\dot{r}^2 + \dot{z}^2}}, & d_{tt} &= \frac{r\dot{z}\dot{f}^2 + \dot{r}\dot{z} - \dot{r}\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}}, \\ C &= -\frac{(\dot{r}^2 + \dot{z}^2)\dot{z} + r(\dot{z}\dot{r} - \dot{z}\dot{r})}{r(\dot{r}^2 + \dot{z}^2)^{\frac{3}{2}}} \end{aligned}$$

The equations of motion become

$$\begin{aligned} \frac{\dot{v}}{v} &= 0, & \frac{\dot{r}}{r^2} \dot{f} &= 0, & \frac{\dot{r}}{lr\sqrt{\dot{r}^2 + \dot{z}^2}} &= 0, \\ \frac{r\dot{z}\dot{f}^2 + \dot{r}\dot{z} - \dot{r}\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}} &= \frac{2A}{\rho lr} \left[\frac{\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}} + \frac{r(\dot{r}\dot{z} - \dot{z}\dot{r})}{(\dot{r}^2 + \dot{z}^2)^{\frac{3}{2}}} \right]. \end{aligned}$$

Thus,

$$(7.1) \quad v^2 = r_0^2 + r_0^2 \dot{f}_0^2 + \dot{z}_0^2 = v_0^2,$$

$$(7.2) \quad r^2 \dot{f} = r_0^2 \dot{f}_0 = \alpha,$$

$$(7.3) \quad lr\sqrt{\dot{r}^2 + \dot{z}^2} = l_0 r_0 \sqrt{\dot{r}_0^2 + \dot{z}_0^2} = \beta,$$

where α , β , and v_0 are constants. It follows from these equations that

$$\dot{r}[r\dot{z}\dot{f}^2 + \dot{r}\dot{z} - \dot{z}\dot{r}] = \dot{z}(\dot{r}^2 + \dot{z}^2)$$

Thus, except perhaps when $\dot{r} = 0$,

$$\dot{z} = \gamma \frac{r\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}},$$

where $\gamma = (2A)/(\rho\beta)$, or

$$(7.4) \quad \dot{z}[1 - \gamma r/\sqrt{\dot{r}^2 + \dot{z}^2}] = \delta,$$

where δ is constant. Then, unless $\dot{r} \equiv 0$, we have

$$(7.5) \quad \dot{z} = \frac{\delta \sqrt{v_0^2 - \alpha^2/r^2}}{[\sqrt{v_0^2 - \alpha^2/r^2} - \gamma r]},$$

$$(7.6) \quad \dot{r} = \pm \sqrt{v_0^2 - \alpha^2/r^2} \{1 - \delta^2[\sqrt{v_0^2 - \alpha^2/r^2} - \gamma r]^{-2}\}^{\frac{1}{2}},$$

$$(7.7) \quad \dot{f} = \alpha/r^2.$$

$$(7.8) \quad l = \frac{\beta}{r \sqrt{v_0^2 - \alpha^2/r^2}}.$$

The system (7.5) and (7.6) determines the meridians. In the rest of this section, we assume that $\dot{r} = 0$ only at isolated points along a meridian. We further assume that $\dot{r}_0 > 0$, $\dot{z}_0 > 0$, $\alpha > 0$, $2A < \rho l_0(\dot{r}_0^2 + \dot{z}_0^2)$. Now let $a = v_0/\delta$, $b = \alpha/\delta > 0$, $c = \gamma/\delta > 0$, and $\varphi(r) = \sqrt{a^2 - b^2/r^2} - cr$. Under the above assumptions, it follows that $\varphi(r) \geq 1$, $\dot{z} > 0$, $r > b/a$, throughout the motion. Here the meridians satisfy the differential equation

$$(7.9) \quad \left(\frac{dr}{dz}\right)^2 = \varphi^2 - 1.$$

Let us now consider the curves $C_1: y = \sqrt{a^2 - b^2/r^2}$ and $C_2: y = cr + 1$ in the (r, y) plane (see Figure 1). Under the above assumptions C_1 and C_2 have two

Figure 1

points in common, (r_1, y_1) and (r_2, y_2) . The condition $\varphi(r) \geq 1$ implies that dynamically possible values of r satisfy the inequality

$$(7.10) \quad r_1 \leq r \leq r_2,$$

while $\varphi(r_1) = \varphi(r_2) = 1$. Also, $r_1 < r_0 < r_2$. Now draw the line segment L joining $(r_0, \sqrt{a^2 - b^2/r_0^2})$ to (r_2, y_2) . From the geometrically evident fact that L lies below C_1 and above C_2 , one can deduce the inequality

$$\varphi(r) > \lambda(r_2 - r) + 1 \quad \text{for } r_0 < r < r_2,$$

where

$$\lambda = \frac{[\varphi(r_0) - 1]}{(r_2 - r_0)} > 0.$$

Using (7.9), we then find that

$$\frac{dr}{dz} > \{[\lambda(r_2 - r) + 1]^2 - 1\}^{\frac{1}{2}} \quad \text{for } r_0 \leq r < r_2.$$

Having this inequality, it is not difficult to show that for some finite $z = z^*$, $r = r_2$; $dr/dz = 0$. Now, from (7.10), dr/dz must become negative for $z > z^*$. A slight variation of the above argument shows that we eventually have $r = r_1$, after which dr/dz again becomes positive, the motion being periodic. In the case considered by Boussinesq, $b = 0$, the curve C_1 degenerates to the line $y = a$, and the point (r_1, y_1) does not exist. The motion is then not periodic. The solution corresponding to $c = 0$ is contained in Volterra's paper. Here C_2 is parallel to the r -axis, so (r_2, y_2) does not exist, and again the motion is not periodic. Other solutions of (7.4) through (7.6) corresponding to initial conditions not satisfying the restrictions imposed here can be discussed in a similar way.

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