The General Problem of the Transformation of the Equations of Dynamics

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INTRODUCTION

In this paper we present essentially the detailed demonstrations of the results announced by A. Lichnerowicz¹ concerning the problem of the transformation of dynamical systems with constraints dependent on time but where the expressions for the kinetic energies and the generalized forces do not depend explicitly on time. The last restriction must be imposed from the very form of the problem.

Briefly, the old problem of the transformation of the equations of dynamics can be stated as follows: *Given two dynamical systems D and E which can be referred to the same set of generalized coordinates, what are the necessary and sufficient conditions for all trajectories of E to be trajectories of D (independently of their time parameterizations)? In particular, what are the conditions for the trajectories of the two systems to be identical?* This problem was studied without complete success by Painlevé, Levi-Civita, Stäckel² and others. T. Y. Thomas³ has completely solved the problem in the case of dynamical systems with constraints independent of time.

Our approach has been similar in general to that of Thomas. Starting with the expressions for the kinetic energies of the two systems and associating with each

¹ C. R. Acad. Sc., **222**, 1946, p. 649.

² PAINLEVÉ, Journ. de Math., 4^e série, **10**, 1894, p. 42; LEVI-CIVITA, Ann. di Matem., 2^e série, **24**, 1896, p. 10.

³ Journ. of Math. and Phy., **25**, 1946, p. 191.

system a Riemann space, we are able to deduce the equations of motion by means of Lagrange's equations. In these resulting equations there appears a bivector $a_{\rho\sigma}$ which is, in the case of dynamical systems, the curl of a covariant vector. We shall generalize the initial problem in adopting as the differential system of paths of D and E , a differential system of the same form but with an arbitrary bivector $a_{\rho\sigma}$. This is the problem of equivalence relative to the two systems of paths defined above which we have completely solved. We are thus led to distinguish two cases: a singular case, referred to as the special case, and a nonsingular case, which is called the general case. The study of the special case is made in section IV and it is found that this case corresponds exactly to the problem investigated by Thomas. The main result of this paper relative to the general case is contained in the Theorems I and II. We note that in the general case if any path of E is a path of D the inverse is also true. This result is contrary to the corresponding result in the special case. This fact is strictly connected with the introduction of a linear first integral in Theorem I, while in the special case a quadratic first integral appears.

We have excluded dynamical systems of two degrees of freedom because such systems require special treatment. In order to permit an easier comparison the notation employed is as similar as possible to that used by Thomas.

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I. THE EQUATIONS OF MOTION

1. Preliminaries. Let us consider two dynamical systems D and E referred to generalized coordinates admitting $n(>2)$ degrees of freedom, where the kinetic energies of the two systems are represented by

$$(1.1) \quad 2T = g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + 2a_\alpha \frac{dx^\alpha}{dt} + 2A, \quad (D),$$

$$(1.2) \quad 2\Theta = h_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} + 2b_\alpha \frac{dx^\alpha}{d\tau} + 2B. \quad (E).$$

The different coefficients depend only on the coordinates x^α and not on time. We introduce the two corresponding Riemann spaces defined by the metrics

$$(1.3) \quad (D) \, ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad (E) \, d\sigma^2 = h_{\alpha\beta} dx^\alpha dx^\beta,$$

whose determinants are g and h respectively.

2. Derivation of the equations of motion. By means of Lagrange's equations,

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}^\gamma} \right) - \frac{\partial T}{\partial x^\gamma} = P_\gamma,$$

we derive the equations of motion of the system D . From (1.1) we obtain

$$\frac{\partial T}{\partial \dot{x}^\gamma} = g_{\gamma\mu} \frac{dx^\mu}{dt} + a_\gamma, \quad \frac{\partial T}{\partial x^\gamma} = \frac{1}{2} \partial_\gamma g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + \partial_\gamma a_\rho \frac{dx^\rho}{dt} + \partial_\gamma A$$

where

$$\partial_\gamma = \frac{\partial}{\partial x^\gamma}.$$

Thus Lagrange's equations can be written explicitly as

$$g_{\gamma\mu} \frac{d^2 x^\mu}{dt^2} + (\partial_\nu g_{\gamma\mu} - \frac{1}{2} \partial_\gamma g_{\mu\nu}) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + (\partial_\rho a_\gamma - \partial_\gamma a_\rho) \frac{dx^\rho}{dt} - \partial_\gamma A = P_\gamma,$$

and by introducing Christoffel symbols of the first kind we have

$$g_{\gamma\mu} \frac{d^2 x^\mu}{dt^2} + [\mu\nu, \gamma] \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + a_{\rho\gamma} \frac{dx^\rho}{dt} = Q_\gamma,$$

where

$$a_{\rho\gamma} = (\partial_\rho a_\gamma - \partial_\gamma a_\rho), \quad Q_\gamma = P_\gamma + \partial_\gamma A.$$

Forming the inner product with $g^{\alpha\gamma}$ we are led to the following equations of motion of the system D :

$$(2.1) \quad \frac{d^2 x^\alpha}{dt^2} + \Gamma_{\mu\nu}^\alpha \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + a_\rho^\alpha \frac{dx^\rho}{dt} = Q^\alpha.$$

In a similar manner the equations of motion of the dynamical system E are found to be

$$(2.2) \quad \frac{d^2 x^\alpha}{d\tau^2} + \Lambda_{\mu\nu}^\alpha \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + b_\rho^\alpha \frac{dx^\rho}{d\tau} = R^\alpha.$$

The $\Gamma_{\mu\nu}^\alpha$ and $\Lambda_{\mu\nu}^\alpha$ are Christoffel symbols of the second kind relative to (1.3), Q and R are generalized force vectors depending only the coordinates x^α , and

$$a_\rho^\alpha = g^{\alpha\sigma} a_{\rho\sigma}, \quad b_\rho^\alpha = h^{\alpha\sigma} b_{\rho\sigma}.$$

In the case considered above the $a_{\rho\sigma}$ and $b_{\rho\sigma}$ are seen to be the components of the curl of a vector. However, in that which follows the problem will be general-

ized such that we shall consider all dynamical systems which can be written in the form given by (2.1) and (2.2) where the $a_{\rho\sigma}$ and $b_{\rho\sigma}$ are *any* skew-symmetric tensors depending only on the x^α .

3. Certain basic assumptions. If one of the force vectors vanishes it is easy to show that the other one vanishes also. We shall consider, therefore, only regions in which neither of the force vectors vanish. We necessarily assume that the time parameters t and τ are restricted to *real* values. In addition, the transformations $\tau \rightarrow t$ and $t \rightarrow \tau$ are assumed to be continuous, as are also their first and second derivatives.

II. PARAMETER TRANSFORMATIONS

4. The transformation $\tau \rightarrow t$. The dynamical system D being given, we assume that all trajectories of E are trajectories of D . If C is any trajectory of E then (2.2) is satisfied when C is referred to the parameter τ and (2.1) is satisfied when C is referred to the parameter t . Under the transformation $\tau \rightarrow t$ the equations (2.2) become

$$(4.1) \quad \frac{d^2 x^\alpha}{dt^2} + \Lambda_{\mu\nu}^\alpha \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + b_\rho^\alpha \frac{dx^\rho}{dt} \frac{d\tau}{dt} = R^\alpha \left(\frac{d\tau}{dt} \right)^2 - \frac{d^2 t}{d\tau^2} \left(\frac{d\tau}{dt} \right)^2 \frac{dx^\alpha}{dt}.$$

5. Formation of an equation in $dt/d\tau$. Subtracting (2.1) from (4.1) we have

$$(5.1) \quad \Phi_{\mu\nu}^\alpha \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = R^\alpha \left(\frac{d\tau}{dt} \right)^2 - Q^\alpha - \frac{d^2 t}{d\tau^2} \left(\frac{d\tau}{dt} \right)^2 \frac{dx^\alpha}{dt} + \left(a_\rho^\alpha - b_\rho^\alpha \frac{d\tau}{dt} \right) \frac{dx^\rho}{dt},$$

where

$$\Phi_{\mu\nu}^\alpha = \Lambda_{\mu\nu}^\alpha - \Gamma_{\mu\nu}^\alpha.$$

Now multiply (5.1) by dx^β/dt and then interchange the indices α and β and subtract to obtain:

$$(5.2) \quad \begin{aligned} \left(\Phi_{\mu\nu}^\alpha \frac{dx^\mu}{dt} - \Phi_{\mu\nu}^\beta \frac{dx^\mu}{dt} \right) \frac{dx^\nu}{dt} &= \left(R^\alpha \frac{dx^\beta}{dt} - R^\beta \frac{dx^\alpha}{dt} \right) \left(\frac{d\tau}{dt} \right)^2 \\ &- \left(Q^\alpha \frac{dx^\beta}{dt} - Q^\beta \frac{dx^\alpha}{dt} \right) + \left(a_\rho^\alpha \frac{dx^\beta}{dt} - a_\rho^\beta \frac{dx^\alpha}{dt} \right) \frac{dx^\rho}{dt} \\ &- \left(b_\rho^\alpha \frac{dx^\beta}{dt} - b_\rho^\beta \frac{dx^\alpha}{dt} \right) \frac{dx^\rho}{dt} \frac{d\tau}{dt}. \end{aligned}$$

We now multiply (5.2) by $(dt/d\tau)^3$, which gives

$$(5.3) \quad \begin{aligned} (\Phi_{\mu\nu}^\alpha \dot{x}^\beta - \Phi_{\mu\nu}^\beta \dot{x}^\alpha) \dot{x}^\mu \dot{x}^\nu &= (R^\alpha \dot{x}^\beta - R^\beta \dot{x}^\alpha) - (Q^\alpha \dot{x}^\beta - Q^\beta \dot{x}^\alpha) \left(\frac{dt}{d\tau} \right)^2 \\ &+ (a_\rho^\alpha \dot{x}^\beta - a_\rho^\beta \dot{x}^\alpha) \dot{x}^\rho \frac{dt}{d\tau} - (b_\rho^\alpha \dot{x}^\beta - b_\rho^\beta \dot{x}^\alpha) \dot{x}^\rho \end{aligned}$$

where

$$\dot{x}^\alpha = \frac{dx^\alpha}{d\tau}.$$

Equation (5.3) can now be written as an equation of the second degree in $dt/d\tau$ where we have put

$$(5.4) \quad A^{\alpha\beta} \left(\frac{dt}{d\tau} \right)^2 + B^{\alpha\beta} \left(\frac{dt}{d\tau} \right) + C^{\alpha\beta} = 0$$

where we have put

$$A^{\alpha\beta} = Q^\alpha \dot{x}^\beta - Q^\beta \dot{x}^\alpha,$$

$$B^{\alpha\beta} = -(a_\rho^\alpha \dot{x}^\beta - a_\rho^\beta \dot{x}^\alpha) \dot{x}^\rho,$$

$$C^{\alpha\beta} = -(R^\alpha \dot{x}^\beta - R^\beta \dot{x}^\alpha) + (b_\rho^\alpha \dot{x}^\beta - b_\rho^\beta \dot{x}^\alpha) \dot{x}^\rho + (\Phi_{\mu\nu}^\alpha \dot{x}^\beta - \Phi_{\mu\nu}^\beta \dot{x}^\alpha) \dot{x}^\mu \dot{x}^\nu.$$

6. The conditions that two systems of the form (5.4) admit a common solution. Let us form a second equation in $dt/d\tau$ similar to (5.4):

$$(6.1) \quad A^{\gamma\delta} \left(\frac{dt}{d\tau} \right)^2 + B^{\gamma\delta} \left(\frac{dt}{d\tau} \right) + C^{\gamma\delta} = 0.$$

The condition that (5.4) and (6.1) have a common solution $dt/d\tau$ is that

$$(6.2) \quad (A^{\alpha\beta} C^{\gamma\delta} - C^{\alpha\beta} A^{\gamma\delta})^2 - (B^{\alpha\beta} C^{\gamma\delta} - C^{\alpha\beta} B^{\gamma\delta})(A^{\alpha\beta} B^{\gamma\delta} - B^{\alpha\beta} A^{\gamma\delta}) = 0,$$

an algebraic equation with respect to the $\dot{x}^\rho$ which must be satisfied identically. Making the substitution

$$C^{\alpha\beta} = C_1^{\alpha\beta} + C_2^{\alpha\beta} + C_3^{\alpha\beta},$$

where

$$C_1^{\alpha\beta} = -(R^\alpha \dot{x}^\beta - R^\beta \dot{x}^\alpha),$$

$$C_2^{\alpha\beta} = (b_\rho^\alpha \dot{x}^\beta - b_\rho^\beta \dot{x}^\alpha) \dot{x}^\rho,$$

$$C_3^{\alpha\beta} = (\Phi_{\mu\nu}^\alpha \dot{x}^\beta - \Phi_{\mu\nu}^\beta \dot{x}^\alpha) \dot{x}^\mu \dot{x}^\nu,$$

we can rewrite (6.2) as

$$(6.3) \quad [A^{\alpha\beta}(C_1^{\gamma\delta} + C_2^{\gamma\delta} + C_3^{\gamma\delta}) - A^{\gamma\delta}(C_1^{\alpha\beta} + C_2^{\alpha\beta} + C_3^{\alpha\beta})]^2 \\ - [B^{\alpha\beta}(C_1^{\gamma\delta} + C_2^{\gamma\delta} + C_3^{\gamma\delta}) - B^{\gamma\delta}(C_1^{\alpha\beta} + C_2^{\alpha\beta} + C_3^{\alpha\beta})] \\ \cdot [A^{\alpha\beta} B^{\gamma\delta} - B^{\alpha\beta} A^{\gamma\delta}] = 0.$$

Thus we have obtained a system of algebraic equations (6.3) in the $\dot{x}^\alpha$, from which will be determined certain conditions involving the quantities there occurring.

III. ALGEBRAIC RELATIONS

7. Preliminaries. An inspection of (6.3) reveals that it contains terms of the 4th, 5th, 6th, 7th, and 8th degrees in $\dot{x}^\alpha$, where the $\dot{x}^\alpha$ can be assigned arbitrary

initial values. It follows that the terms of each respective degree in $\dot{x}^\alpha$ must vanish separately. Therefore, (6.3) will break up into several independent equations which will enable us to determine explicit algebraic relations involving the various quantities.

8. Terms of the fourth degree. From (6.3) we see that the terms of the 4th degree in $\dot{x}^\alpha$ yield

$$(8.1) \quad A^{\alpha\beta}C_1^{\gamma\delta} - A^{\gamma\delta}C_1^{\alpha\beta} = 0.$$

But this expression is identical to one in the problem studied by Thomas⁴. Referring to his work, we see that there exists a function $\psi(x^\alpha)$ such that:

$$(8.2) \quad R^\alpha = \psi Q^\alpha.$$

Replacing (8.2) in the expression for $C_1^{\alpha\beta}$ we obtain

$$(8.3) \quad C_1^{\alpha\beta} = -\psi A^{\alpha\beta}.$$

Then upon introducing (8.3) into (6.3) one obtains

$$(8.4) \quad [A^{\alpha\beta}(C_2^{\gamma\delta} + C_3^{\gamma\delta}) - A^{\gamma\delta}(C_2^{\alpha\beta} + C_3^{\alpha\beta})]^2 - [A^{\alpha\beta}B^{\gamma\delta} - B^{\alpha\beta}A^{\gamma\delta}] \cdot [B^{\alpha\beta}(C_3^{\gamma\delta} + C_2^{\gamma\delta} - \psi A^{\gamma\delta}) - B^{\gamma\delta}(C_3^{\alpha\beta} + C_2^{\alpha\beta} - \psi A^{\alpha\beta})] = 0,$$

which is our new form of the fundamental equation (6.3).

9. Terms of the sixth degree. From (8.4) the terms of the 6th degree in $\dot{x}^\alpha$ yield

$$(A^{\alpha\beta}C_2^{\gamma\delta} - C_2^{\alpha\beta}A^{\gamma\delta})^2 - \psi(A^{\alpha\beta}B^{\gamma\delta} - B^{\alpha\beta}A^{\gamma\delta})^2 = 0.$$

It results from the above equation that ψ must be positive, and thus by letting $\psi = \chi^2$ we reduce the terms of the 6th degree to

$$(9.1) \quad (A^{\alpha\beta}C_2^{\gamma\delta} - C_2^{\alpha\beta}A^{\gamma\delta}) = \chi(A^{\alpha\beta}B^{\gamma\delta} - B^{\alpha\beta}A^{\gamma\delta}).$$

10. The special and general cases. By definition we shall call the *special case* the problem where

$$(10.1) \quad A^{\alpha\beta}B^{\gamma\delta} - B^{\alpha\beta}A^{\gamma\delta} = 0.$$

The *general case*, then, will be the problem where $A^{\alpha\beta}B^{\gamma\delta} - B^{\alpha\beta}A^{\gamma\delta}$ is not identically zero.

11. The terms of the seventh degree. We have as the terms of the seventh degree in $\dot{x}^\alpha$:

$$2(A^{\alpha\beta}C_3^{\gamma\delta} - C_3^{\alpha\beta}A^{\gamma\delta})(A^{\alpha\beta}C_2^{\gamma\delta} - C_2^{\alpha\beta}A^{\gamma\delta}) - (A^{\alpha\beta}B^{\gamma\delta} - B^{\alpha\beta}A^{\gamma\delta})(B^{\alpha\beta}C_2^{\gamma\delta} - B^{\gamma\delta}C_2^{\alpha\beta}) = 0.$$

⁴ Loc. cit. p. 194.

In the general case because of (9.1) these terms of the 7th degree reduce to

$$(11.1) \quad B^{\alpha\beta}C_2^{\gamma\delta} - C_2^{\alpha\beta}B^{\gamma\delta} = 2\chi(A^{\alpha\beta}C_3^{\gamma\delta} - C_3^{\alpha\beta}A^{\gamma\delta}).$$

12. The terms of the eighth degree. From (8.4) we have as the terms of the 8th degree in $\dot{x}^\alpha$:

$$(12.1) \quad (A^{\alpha\beta}C_3^{\gamma\delta} - C_3^{\alpha\beta}A^{\gamma\delta})^2 - (A^{\alpha\beta}B^{\gamma\delta} - B^{\alpha\beta}A^{\gamma\delta})(B^{\alpha\beta}C_3^{\gamma\delta} - C_3^{\alpha\beta}B^{\gamma\delta}) = 0.$$

IV. STUDY OF THE SPECIAL CASE

13. Introduction. We now adopt the hypothesis of the special case:

$$(10.1) \quad A^{\alpha\beta}B^{\gamma\delta} - B^{\alpha\beta}A^{\gamma\delta} = 0,$$

a hypothesis relative to the dynamical system D only. We will have the following relations obtained from (9.1) and (12.1) respectively:

$$(13.1) \quad A^{\alpha\beta}C_2^{\gamma\delta} - C_2^{\alpha\beta}A^{\gamma\delta} = 0,$$

$$(13.2) \quad A^{\alpha\beta}C_3^{\gamma\delta} - C_3^{\alpha\beta}A^{\gamma\delta} = 0.$$

The equation (8.1) and its solution (8.2) remain valid in both the special and general cases.

14. Relations resulting from (10.1) and (13.1). Let us consider (10.1) written in its explicit form:

$$(14.1) \quad (Q^\alpha \dot{x}^\beta - Q^\beta \dot{x}^\alpha)(a_\rho^\gamma \dot{x}^\delta - a_\rho^\delta \dot{x}^\gamma) \dot{x}^\rho - (a_\rho^\alpha \dot{x}^\beta - a_\rho^\beta \dot{x}^\alpha)(Q^\gamma \dot{x}^\delta - Q^\delta \dot{x}^\gamma) \dot{x}^\rho = 0.$$

A straightforward though rather long calculation appearing in Appendix I shows that (14.1) is equivalent to

$$(14.2) \quad (n-1)(Q^\alpha a_\rho^\gamma - Q^\gamma a_\rho^\alpha) + (Q^\sigma a_\sigma^\alpha \delta_\rho^\gamma - Q^\sigma a_\sigma^\gamma \delta_\rho^\alpha) = 0.$$

Multiplying (14.2) by $Q_\alpha = g_{\alpha\beta}Q^\beta$ and summing on the repeated indices one obtains

$$(14.3) \quad (n-1)Q_\alpha Q^\alpha a_\rho^\gamma = (n-1)Q_\alpha Q^\gamma a_\rho^\alpha + Q_\rho Q^\sigma a_\sigma^\gamma - Q_\alpha Q^\sigma a_\sigma^\alpha \delta_\rho^\gamma.$$

The last term of the right member is null because of the skew-symmetry of the tensor $a_{\rho\sigma}$. Now multiply (14.3) by Q^ρ and again sum on the repeated indices, which gives

$$(14.4) \quad (n-1)Q_\alpha Q^\alpha Q^\rho a_\rho^\gamma = (n-1)Q^\gamma Q_\alpha Q^\rho a_\rho^\alpha + Q^\rho Q_\rho Q^\sigma a_\sigma^\gamma.$$

Noting that the first term of the second member is null because of the skew-symmetry of the tensor $a_{\rho\sigma}$, we reduce equation (14.4) to

$$(n-2)Q_\alpha Q^\alpha Q^\rho a_\rho^\gamma = 0,$$

and for $n > 2$ one has

$$(14.5) \quad Q^\rho a_\rho^\gamma = 0 \quad \text{and} \quad Q_\alpha a_\sigma^\alpha = 0.$$

Replacing (14.5) in the right member of (14.3) we find that

$$(14.6) \quad a_p^\gamma = 0.$$

An inspection of (13.1) reveals that it is identical to (10.1) except that the a_p^γ have been replaced by the b_p^γ . Hence it follows that a calculation similar to the one made in the appendix with (10.1) will lead to the following result

$$(14.7) \quad b_p^\gamma = 0.$$

15. Equivalence of the special case to the problem studied by Thomas. The relations (14.6) and (14.7) imply that (10.1) and (13.1) vanish identically. There remain, therefore, only (8.1) and (13.2) in the study of the special case. But this is obviously none other than the problem solved by Thomas, to whose paper we refer the reader for a complete discussion. From the preceding study we have the following theorem:

Theorem. *Given a dynamical system D for which $a_{p\sigma} = 0$, there does not exist a dynamical system E for which $b_{p\sigma} \neq 0$ such that any trajectory of E is a trajectory of D or conversely.*

V. THE GENERAL CASE

16. Statement of the problem. This section and those following are devoted to the study of the case where the condition (10.1) no longer holds. In this section are found the conditions imposed by the algebraic equations of the 6th, 7th, and 8th degrees in x^α .

17. Conditions obtained from the sixth degree terms. In the equation arising from the terms of the 6th degree,

$$(9.1) \quad A^{\alpha\beta}C_2^{\gamma\delta} - C_2^{\beta\delta}A^{\gamma\delta} = \chi(A^{\alpha\beta}B^{\gamma\delta} - B^{\alpha\beta}A^{\gamma\delta}),$$

let us make the substitution⁵

$$(17.1) \quad d_p^\alpha = b_p^\alpha + \chi a_p^\alpha.$$

It follows that

$$(17.2) \quad D^{\alpha\beta} = (d_p^\alpha \dot{x}^\beta - d_p^\beta \dot{x}^\alpha) \dot{x}^p = C_2^{\alpha\beta} - \chi B^{\alpha\beta}.$$

Hence (9.1) can be written as

$$(17.3) \quad A^{\alpha\beta}D^{\gamma\delta} - D^{\alpha\beta}A^{\gamma\delta} = 0.$$

Observing that (17.3) is of the same form as (13.1), we write immediately an equation analogous to (14.2):

$$(17.4) \quad (n-1)(Q^\alpha d_p^\gamma - Q^\gamma d_p^\alpha) + (Q^\sigma d_\sigma^\alpha \delta_p^\gamma - Q^\sigma d_\sigma^\gamma \delta_p^\alpha) = 0.$$

⁵ From (17.1) we see that $d_p^\gamma = 0$, but since the a_p^α and b_p^α are based on different metrics we can not, for instance, lower an index of d_p^α to form a covariant skew-symmetric tensor.

Multiply (17.4) by $Q_\gamma = g_{\beta\gamma}Q^\beta$, divide by $(n-1)Q_\gamma Q^\gamma$ and solve for d_ρ^α . There results an equation of the form:

$$(17.5) \quad d_\rho^\alpha = f_\rho^\alpha Q^\alpha + \frac{Q_\rho Q^\sigma d_\sigma^\alpha}{(n-1)Q_\gamma Q^\gamma} + r\delta_\rho^\alpha,$$

where r is a scalar and f_ρ is a convenient vector. Now multiply (17.5) by Q^ρ . For $n > 2$ we obtain

$$(17.6) \quad Q^\rho d_\rho^\alpha = HQ^\alpha.$$

We do not make use of the explicit form of H . When (17.6) is introduced into the second term of the right member of (17.5) one arrives at an equation of the form

$$(17.7) \quad d_\rho^\alpha = K_\rho Q^\alpha + r\delta_\rho^\alpha,$$

whence and from (17.1) we have

$$(17.8) \quad b_\rho^\alpha = -\chi a_\rho^\alpha + K_\rho Q^\alpha + r\delta_\rho^\alpha.$$

Substituting the values of b_ρ^α given by (17.8) into the left member of (9.1) causes the resulting expression to vanish identically. Hence (17.8) is equivalent to (9.1). Introducing (17.7) into (17.2), we note that

$$(17.9) \quad C_2^{\alpha\beta} = \chi B^{\alpha\beta} + K_\rho x^\rho A^{\alpha\beta}.$$

18. Conditions resulting from the seventh degree terms. In equation

$$(11.1) \quad B^{\alpha\beta}C_2^{\gamma\delta} - C_2^{\alpha\beta}B^{\gamma\delta} = 2\chi(A^{\alpha\beta}C_3^{\gamma\delta} - C_3^{\alpha\beta}A^{\gamma\delta}),$$

which expresses the conditions imposed by the terms of the 7th degree in x^α , we substitute (17.9) and obtain after rearranging certain terms:

$$(18.1) \quad A^{\alpha\beta} \left(C_3^{\gamma\delta} + \frac{1}{2\chi} B^{\gamma\delta} K_\mu x^\mu \right) - A^{\gamma\delta} \left(C_3^{\alpha\beta} + \frac{1}{2\chi} B^{\alpha\beta} K_\mu x^\mu \right) = 0.$$

Writing (18.1) as

$$(18.2) \quad A^{\alpha\beta} \left[C_3^{\gamma\delta} - \frac{1}{2\chi} (K_\mu a_\nu^\gamma x^\delta - K_\mu a_\nu^\delta x^\gamma) x^\mu x^\nu \right] - A^{\gamma\delta} \left[C_3^{\alpha\beta} - \frac{1}{2\chi} (K_\mu a_\nu^\alpha x^\beta - K_\mu a_\nu^\beta x^\alpha) x^\mu x^\nu \right] = 0,$$

we make (18.2) symmetric in the indices μ and ν and skew-symmetric in α and β and in γ and δ , obtaining

$$(18.3) \quad A^{\alpha\beta} \left[C_3^{\gamma\delta} - \frac{1}{4\chi} \{ (K_\mu a_\nu^\gamma + K_\nu a_\mu^\gamma) x^\delta - (K_\mu a_\nu^\delta + K_\nu a_\mu^\delta) x^\gamma \} x^\mu x^\nu \right] - A^{\gamma\delta} \left[C_3^{\alpha\beta} - \frac{1}{4\chi} \{ (K_\mu a_\nu^\alpha + K_\nu a_\mu^\alpha) x^\beta - (K_\mu a_\nu^\beta + K_\nu a_\mu^\beta) x^\alpha \} x^\mu x^\nu \right] = 0.$$

Putting

$$(18.4) \quad \Psi_{\mu\nu}^{\alpha} = \Phi_{\mu\nu}^{\alpha} - \frac{1}{4\chi} (K_{\mu} a_{\nu}^{\alpha} + K_{\nu} a_{\mu}^{\alpha})$$

and

$$(18.5) \quad E^{\alpha\beta} = (\Psi_{\mu\nu}^{\alpha} \dot{x}^{\beta} - \Psi_{\mu\nu}^{\beta} \dot{x}^{\alpha}) \dot{x}^{\mu} \dot{x}^{\nu} = C_3^{\alpha\beta} + \frac{1}{2\chi} B^{\alpha\beta} K_{\mu} \dot{x}^{\mu},$$

we may write (18.1) as

$$(18.6) \quad A^{\alpha\beta} E^{\gamma\delta} - E^{\alpha\beta} A^{\gamma\delta} = 0.$$

Thomas has shown⁶ that the general solution of an equation of this form is

$$(18.7) \quad \Psi_{\mu\nu}^{\alpha} = \delta_{\mu}^{\alpha} \phi_{\nu} + \delta_{\nu}^{\alpha} \phi_{\mu} + F_{\mu\nu} Q^{\alpha},$$

where the ϕ_{ν} and the $F_{\mu\nu}$ ($= F_{\nu\mu}$) are arbitrary tensors. Hence by (18.4) we have

$$(18.8) \quad \Phi_{\mu\nu}^{\alpha} = \delta_{\mu}^{\alpha} \phi_{\nu} + \delta_{\nu}^{\alpha} \phi_{\mu} + F_{\mu\nu} Q^{\alpha} + \frac{1}{4\chi} (K_{\mu} a_{\nu}^{\alpha} + K_{\nu} a_{\mu}^{\alpha}).$$

Substituting (18.4) and (18.8) into (18.5) we obtain

$$(18.9) \quad C_3^{\alpha\beta} = F_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} A^{\alpha\beta} - \frac{1}{2\chi} K_{\rho} \dot{x}^{\rho} B^{\alpha\beta}.$$

One observes that $C_3^{\alpha\beta}$ is linear with respect to $A^{\alpha\beta}$ and $B^{\alpha\beta}$.

19. Conditions imposed by the eighth degree terms. We replace the first term of the equation of the 8th degree terms

$$(12.1) \quad (A^{\alpha\beta} C_3^{\gamma\delta} - C_3^{\alpha\beta} A^{\gamma\delta})^2 - (A^{\alpha\beta} B^{\gamma\delta} - B^{\alpha\beta} A^{\gamma\delta})(B^{\alpha\beta} C_3^{\gamma\delta} - C_3^{\alpha\beta} B^{\gamma\delta}) = 0,$$

by means of (18.1), and upon substituting (18.9) into the second term of (12.1) we see that

$$(19.1) \quad \frac{1}{4\chi^2} K_{\mu} K_{\nu} \dot{x}^{\mu} \dot{x}^{\nu} (A^{\alpha\beta} B^{\gamma\delta} - B^{\alpha\beta} A^{\gamma\delta})^2 + (A^{\alpha\beta} B^{\gamma\delta} - B^{\alpha\beta} A^{\gamma\delta})^2 F_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = 0.$$

Since by hypothesis

$$A^{\alpha\beta} B^{\gamma\delta} - B^{\alpha\beta} A^{\gamma\delta} \neq 0,$$

equation (19.1) implies that

$$F_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = -\frac{1}{4\chi^2} K_{\mu} K_{\nu} \dot{x}^{\mu} \dot{x}^{\nu},$$

or that

$$(19.2) \quad F_{\mu\nu} = -\frac{1}{4\chi^2} K_{\mu} K_{\nu}.$$

⁶ Loc. cit. p. 196.

Substituting (19.2) into (18.8) we obtain:

$$(19.3) \quad \Phi_{\mu\nu}^{\alpha} = \delta_{\mu}^{\alpha} \phi_{\nu} + \delta_{\nu}^{\alpha} \phi_{\mu} - \frac{1}{4\chi^2} K_{\mu} K_{\nu} Q^{\alpha} + \frac{1}{4\chi} (K_{\mu} a_{\nu}^{\alpha} + K_{\nu} a_{\mu}^{\alpha}).$$

20. Summary of the algebraic relations in the general case. The 4th degree terms led to the condition that the force vectors are proportional,

$$(8.2) \quad R^{\alpha} = \chi^2 Q^{\alpha}, \quad \chi^2 > 0.$$

From the terms of the 6th degree we arrived at the condition

$$(17.8) \quad b_{\rho}^{\alpha} = -\chi a_{\rho}^{\alpha} + K_{\rho} Q^{\alpha} + r \delta_{\rho}^{\alpha}.$$

We obtained from the 7th degree terms together with the additional condition imposed by the 8th degree terms that

$$(19.3) \quad \Phi_{\mu\nu}^{\alpha} = \delta_{\mu}^{\alpha} \phi_{\nu} + \delta_{\nu}^{\alpha} \phi_{\mu} - \frac{1}{4\chi^2} K_{\mu} K_{\nu} Q^{\alpha} + \frac{1}{4\chi} (K_{\mu} a_{\nu}^{\alpha} + K_{\nu} a_{\mu}^{\alpha}).$$

Equations (8.2) and (17.8) cause (8.1) and (9.1) to vanish identically, while (19.3) makes (11.1) and (12.1) vanish identically; hence (8.2), (17.8) and (19.3) are strictly equivalent, in the general case, to (8.1), (9.1), (11.1) and (12.1).

VI. PARAMETER TRANSFORMATIONS (*cont.*)

21. Conditions on the transformation $t \leftrightarrow \tau$. The algebraic relations (8.2), (17.8), and (19.3) enable us to determine certain explicit conditions on the parameter transformation $t \leftrightarrow \tau$, which we shall now study.⁷ Substituting for R^{α} from (8.2), for b_{ρ}^{α} from (17.8), and for $\Phi_{\mu\nu}^{\alpha}$ from (19.3) in equation (5.1), *viz.*,

$$(5.1) \quad \Phi_{\mu\nu}^{\alpha} \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt} = R^{\alpha} \left(\frac{d\tau}{dt} \right)^2 - Q^{\alpha} - \frac{d^2 t}{d\tau^2} \left(\frac{d\tau}{dt} \right)^2 \frac{dx^{\alpha}}{dt} + \left(a_{\rho}^{\alpha} - b_{\rho}^{\alpha} \frac{d\tau}{dt} \right) \frac{dx^{\rho}}{dt},$$

we can put the resulting expression into the form

$$(21.1) \quad \left[\chi^2 \left(\frac{d\tau}{dt} \right)^2 - 1 + \frac{1}{4\chi^2} K_{\mu} K_{\nu} \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt} - K_{\rho} \frac{dx^{\rho}}{dt} \frac{d\tau}{dt} \right] Q^{\alpha} \\ = \left[\frac{d^2 t}{d\tau^2} \left(\frac{d\tau}{dt} \right)^2 + 2\phi_{\mu} \frac{dx^{\mu}}{dt} + r \frac{d\tau}{dt} \right] \frac{dx^{\alpha}}{dt} + \left[\frac{1}{2\chi} K_{\mu} \frac{dx^{\mu}}{dt} - 1 - \chi \frac{d\tau}{dt} \right] a_{\rho}^{\alpha} \frac{dx^{\rho}}{dt}.$$

In Appendix II we show that the relation (21.1) implies the following three equations:

$$(21.2) \quad \chi^2 \left(\frac{d\tau}{dt} \right)^2 - 1 + \frac{1}{4\chi^2} K_{\mu} K_{\nu} \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt} - K_{\rho} \frac{dx^{\rho}}{dt} \frac{d\tau}{dt} = 0,$$

$$(21.3) \quad \frac{d^2 t}{d\tau^2} \left(\frac{d\tau}{dt} \right)^2 + 2\phi_{\mu} \frac{dx^{\mu}}{dt} + r \frac{d\tau}{dt} = 0,$$

$$(21.4) \quad \frac{1}{2\chi} K_{\mu} \frac{dx^{\mu}}{dt} - 1 - \chi \frac{d\tau}{dt} = 0.$$

⁷ Loc. cit. p. 196.

Equations equivalent to those above are

$$(21.5) \quad \left(\frac{dt}{d\tau}\right)^2 = \chi^2 + \frac{1}{4\chi^2} K_\mu K_\nu \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - K_\rho \frac{dx^\rho}{d\tau},$$

$$(21.6) \quad \frac{d^2 t}{d\tau^2} + 2\phi_\mu \frac{dx^\mu}{d\tau} \frac{dt}{d\tau} + r \frac{dt}{d\tau} = 0,$$

$$(21.7) \quad \frac{dt}{d\tau} = \frac{1}{2\chi} K_\mu \frac{dx^\mu}{d\tau} - \chi.$$

We remark in particular that (21.5) is obtained from (21.7) by squaring both members of the latter equation. This fact, as we see later, leads to some very interesting consequences.

22. Integration of (21.6). In the next section it is shown that $r = 0$ and that ϕ_μ is the gradient of a scalar function ϕ . Anticipating these results and making the substitution,

$$(22.1) \quad 4\phi = \log \omega,$$

we may integrate equation (21.6) obtaining

$$(22.2) \quad \frac{dt}{d\tau} = \frac{1}{k} e^{-2\phi} = \frac{1}{k\omega^{\frac{1}{2}}},$$

where k is a constant of integration that may depend on the trajectory.

VII. DIFFERENTIAL RELATIONS

23. Conditions arising from differentiating (21.7). Taking the derivative of (21.7) with respect to τ we have

$$\frac{d^2 t}{d\tau^2} = \frac{1}{2\chi} K_\mu \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2\chi} \partial_\gamma K_\mu \frac{dx^\mu}{d\tau} \frac{dx^\gamma}{d\tau} - \frac{\partial_\gamma \chi}{2\chi^2} K_\mu \frac{dx^\mu}{d\tau} \frac{dx^\gamma}{d\tau} - \partial_\gamma \chi \frac{dx^\gamma}{d\tau}.$$

Eliminate $d^2 t/d\tau^2$ by (21.6), $d^2 x^\mu/d\tau^2$ by (2.2), and eliminate $dt/d\tau$, which then appears in the left member, by (21.7). The resulting expression can be written in the form⁸

$$(23.1) \quad \left[\frac{1}{2\chi} \nabla_\gamma K_\mu - \frac{\partial_\gamma \chi}{2\chi^2} K_\mu + \frac{1}{\chi} \phi_\mu K_\gamma \right] \frac{dx^\mu}{d\tau} \frac{dx^\gamma}{d\tau} + \left[-\frac{1}{2\chi} K_\mu b_\gamma^\mu - \partial_\gamma \chi - 2\chi \phi_\gamma + \frac{r}{2\chi} K_\gamma \right] \frac{dx^\gamma}{d\tau} + \frac{1}{2\chi} K_\mu R^\mu - r\chi = 0.$$

⁸ ∇_γ is the symbol of covariant differentiation based on the $\Lambda_{\mu\nu}^\alpha$.

Since we have terms of different degrees in the x^ρ , where the x^ρ can have arbitrary initial values, we are led to the following set of equations:

$$(23.2) \quad \frac{1}{2\chi} K_\mu R^\mu - r\chi = 0,$$

$$(23.3) \quad 2\phi_\gamma + \partial_\gamma \log \chi + \frac{1}{2\chi^2} K_\mu b_\gamma^\mu - \frac{r}{2\chi^2} K_\gamma = 0,$$

$$(23.4) \quad \Sigma [\nabla_\gamma K_\mu - K_\mu \partial_\gamma \log \chi + 2\phi_\mu K_\gamma] = 0,$$

where the summation sign is used to indicate that the expression concerned must be made symmetric in the indices γ and μ .

24. Certain consequences of the relations derived in §23. We obtain from (17.8) by contracting on α and ρ the relation

$$r = -\frac{K_\rho Q^\rho}{n}.$$

Substituting this relation into (23.2), we obtain⁹

$$\frac{1}{2} K_\rho Q^\rho = r = 0.$$

As a result of these conditions we rewrite (23.2), (23.3) and (23.4) as

$$(24.1) \quad K_\rho Q^\rho = r = 0,$$

$$(24.2) \quad 2\phi_\gamma + \partial_\gamma \log \chi + \frac{1}{2\chi^2} K_\mu b_\gamma^\mu = 0,$$

$$(24.3) \quad \Sigma [\nabla_\gamma K_\mu - K_\mu \partial_\gamma \log \chi + 2\phi_\mu K_\gamma] = 0.$$

One notes that the vector K is always orthogonal to the vector Q . Since $r = 0$, we can restate (17.8) as

$$(24.4) \quad b_\rho^\alpha = -\chi a_\rho^\alpha + K_\rho Q^\alpha.$$

25. Significance of the $K_\sigma a_\alpha^\sigma$. Putting $\alpha = \nu$ in (19.3) and summing on the repeated indices, we have

$$(25.1) \quad \Phi_{\mu\sigma}^\sigma = (n+1)\phi_\mu + \frac{1}{4\chi} K_\sigma a_\mu^\sigma$$

Replacing the b_γ^μ in (24.2) by (24.4), we obtain an equation which together with (25.1) permits us to write

$$(25.2) \quad 2\phi_\alpha = \frac{2\Phi_{\alpha\sigma}^\sigma}{(n+1)} - \frac{K_\sigma a_\alpha^\sigma}{2(n+1)\chi} = \frac{K_\sigma a_\alpha^\sigma}{2\chi} - \partial_\alpha \log \chi.$$

⁹ That $r = 0$ justifies our statement in §22 used in the integration of (21.6).

It follows directly from (25.2) that

$$(25.3) \quad \frac{1}{2\chi} K_{\sigma} a_{\alpha}^{\sigma} = \frac{2}{(n+2)} \Phi_{\alpha\sigma}^{\sigma} + \frac{(n+1)}{(n+2)} \partial_{\alpha} \log \chi,$$

but

$$\Phi_{\alpha\sigma}^{\sigma} = \frac{1}{2} \partial_{\alpha} \log \frac{h}{g},$$

where $h = \det |h_{\alpha\beta}|$ and $g = \det |g_{\alpha\beta}|$. Hence (25.3) becomes

$$(25.4) \quad \frac{1}{2\chi} K_{\sigma} a_{\alpha}^{\sigma} = \partial_{\alpha} \left[\frac{1}{(n+2)} \log \frac{h}{g} + \frac{(n+1)}{(n+2)} \log \chi \right].$$

We have thus shown that the $K_{\sigma} a_{\alpha}^{\sigma}$ are the partial derivatives of the expression in brackets except for a constant multiplicative factor.

26. Significance of the ϕ_{μ} . By (25.2) together with (25.4) we write:

$$\frac{1}{2\chi} K_{\sigma} a_{\alpha}^{\sigma} = \partial_{\alpha} \log \chi + 2\phi_{\alpha} = \partial_{\alpha} \left[\frac{1}{(n+2)} \log \frac{h}{g} + \frac{(n+1)}{(n+2)} \log \chi \right].$$

It follows that

$$(26.1) \quad \phi_{\alpha} = \partial_{\alpha} \left[\frac{1}{2(n+2)} \log \frac{h}{\chi g} \right],$$

where $\phi_{\alpha} = \partial_{\alpha} \phi$ with

$$(26.2) \quad \phi = \frac{1}{2(n+2)} \log \frac{h}{\chi g} + \text{constant}.$$

Therefore, the ϕ_{μ} are found to be the components of the *gradient of a scalar function* ϕ . This fact was used in §22 in the integration of (21.6). It is permissible to make the additive constant zero in (26.2). Doing this and using the substitution $4\phi = \log \omega$ as in §22 we obtain:

$$(26.3) \quad \omega = \left(\frac{h}{\chi g} \right)^{2/(n+2)}.$$

27. Differential relations. When ω is substituted into (25.4) by means of (26.3), we obtain:

$$(27.1) \quad \frac{1}{2\chi} K_{\sigma} a_{\alpha}^{\sigma} = \partial_{\alpha} \log (\chi \omega^{\frac{1}{2}}).$$

Also introducing ω into (24.3) we obtain:

$$\nabla_{\gamma} K_{\mu} + \nabla_{\mu} K_{\gamma} + K_{\mu} \partial_{\gamma} \log \frac{\omega^{\frac{1}{2}}}{\chi} + K_{\gamma} \partial_{\mu} \log \frac{\omega^{\frac{1}{2}}}{\chi} = 0.$$

However, $\omega^\dagger/\chi$ is a scalar function; hence its covariant derivatives reduce to ordinary derivatives, and the above equation can be put in the form

$$(27.2) \quad \nabla_\gamma \left(\frac{\omega^\dagger}{\chi} K_\alpha \right) + \nabla_\alpha \left(\frac{\omega^\dagger}{\chi} K_\gamma \right) = 0.$$

VIII. THE LINEAR FIRST INTEGRAL

28. Preliminaries. From the equations (21.7) and (22.2) it follows that:

$$(28.1) \quad \frac{\omega^\dagger}{2\chi} K_\rho \frac{dx^\rho}{d\tau} - \chi \omega^\dagger = \frac{1}{k},$$

where k is a real constant. This equation is satisfied along any trajectory of E which is subject to the conditions under which (21.5), (21.6) and (21.7) were derived. In the study of the special case only an equation quadratic in the velocities appeared. We would find a quadratic equation here if (21.5) were treated as we treated (21.7) in §23, but this quadratic is always the square of (28.1). Referring (28.1) to the parameter t by means of (22.2), we obtain

$$(28.2) \quad \frac{1}{2\chi^\dagger \omega^\dagger} K_\rho \frac{dx^\rho}{dt} - \frac{1}{\chi \omega^\dagger} = k$$

as the corresponding equation for the dynamical system D . This suggests that (28.2) must define a first integral of the system D and that the relations (24.1), (27.1) and (27.2) simply imply that E and D admit respectively the first integrals (28.1) and (28.2).

29. The first integral of the system E . We first determine the conditions in order that

$$(29.1) \quad H_\rho \frac{dx^\rho}{d\tau} + U = \text{constant}$$

be a first integral of E . Taking the derivative of (29.1) with respect to τ and substituting for $d^2x^\rho/d\tau^2$ from (2.2), we can put the resulting equations into the form

$$\nabla_\gamma H_\rho \frac{dx^\rho}{d\tau} \frac{dx^\gamma}{d\tau} + (\partial_\gamma U - H_\sigma b_\gamma^\sigma) \frac{dx^\gamma}{d\tau} + H_\gamma R^\gamma = 0,$$

where the covariant derivatives are based on the Λ_μ^α . These equations must be identities with respect to the $dx^\rho/d\tau$; hence we are led to the following conditions:

$$(29.2) \quad H_\rho R^\rho = 0,$$

$$(29.3) \quad \partial_\gamma U - H_\sigma b_\gamma^\sigma = 0,$$

$$(29.4) \quad \nabla_\gamma H_\rho + \nabla_\rho H_\gamma = 0.$$

These conditions are both necessary and sufficient for (29.1) to be a first integral of E . We now let

$$(29.5) \quad U = -\chi\omega^{\frac{1}{2}},$$

$$(29.6) \quad H_\rho = \frac{\omega^{\frac{1}{2}}}{2\chi} K_\rho.$$

It follows immediately that (24.1), (27.1), and (27.2) are but another form of (29.2), (29.3), and (29.4). Therefore, these equations mean that

$$(29.7) \quad \frac{\omega^{\frac{1}{2}}}{2\chi} K_\rho \frac{dx^\rho}{d\tau} - \chi\omega^{\frac{1}{2}} = \text{constant}$$

where

$$\omega^{\frac{1}{2}} = \left(\frac{h}{\chi g} \right)^{1/(n+2)}$$

is a first integral of the system E .

30. The first integral of the system D . Corresponding to the conditions (29.2) (29.3), and (29.4) we will have for the system D and the expected first integral (28.2),

$$(30.1) \quad \frac{1}{2\chi^2\omega^{\frac{1}{2}}} K_\rho Q^\rho = 0,$$

$$(30.2) \quad \partial_\gamma \left(\frac{-1}{\chi\omega^{\frac{1}{2}}} \right) - \frac{1}{2\chi^2\omega^{\frac{1}{2}}} K_\rho a_\gamma^\rho = 0,$$

$$(30.3) \quad \bar{\nabla}_\gamma \left(\frac{1}{2\chi^2\omega^{\frac{1}{2}}} K_\mu \right) + \bar{\nabla}_\mu \left(\frac{1}{2\chi^2\omega^{\frac{1}{2}}} K_\gamma \right) = 0,$$

where the covariant derivatives are this time based on the $\Gamma_{\mu\nu}^\alpha$. Equation (30.1) is the same as (24.1), equation (30.2) is another form of (27.1), and equation (30.3) is seen to be satisfied on account of (19.3) together with (22.1), (27.1) and (27.2). Thus

$$(30.4) \quad \frac{1}{2\chi\omega^{\frac{1}{2}}} K_\rho \frac{dx^\rho}{dt} - \frac{1}{\chi\omega^{\frac{1}{2}}} = \text{constant}$$

is a first integral of the dynamical system D .

IX. GENERAL CORRESPONDENCE THEOREMS

31. Conditions for all trajectories of E to be trajectories of D . We now prove the following theorem.

Theorem I. *All trajectories of the dynamical system E will be trajectories of the dynamical system D if, and only if, there are a scalar χ and two covariant vectors K_ρ and ϕ_ρ such that the following conditions are satisfied:*

- (a) $R^\alpha = \chi^2 Q^\alpha, \quad \chi^2 > 0;$
- (b) $b_\rho^\alpha = -\chi a_\rho^\alpha + K_\rho Q^\alpha;$
- (c) $\Phi_{\mu\nu}^\alpha = \delta_\mu^\alpha \phi_\nu + \delta_\nu^\alpha \phi_\mu - \frac{1}{4\chi^2} K_\mu K_\nu Q^\alpha + \frac{1}{4\chi} (K_\mu a_\nu^\alpha + K_\nu a_\mu^\alpha);$
- (d) $\frac{1}{2\chi^2} \left(\frac{\chi g}{h} \right)^{1/(n+2)} K_\rho \frac{dx^\rho}{dt} - \frac{1}{\chi} \left(\frac{\chi g}{h} \right)^{1/(n+2)} = \text{constant}$

is a first integral of D . The necessity of these conditions has been proved. It remains to show their sufficiency.

Since (d) is a first integral of D , we have (30.1), (30.2) and (30.3). Hence

$$(24.1) \quad K_\rho Q^\rho = 0$$

and

$$(27.1) \quad \frac{1}{2\chi} K_\sigma a_\alpha^\sigma = \partial_\alpha \log \chi \omega^{\frac{1}{2}}$$

hold, since they are equivalent to (30.1) and (30.2) respectively. Now contract on the indices α and ν in (c), obtaining

$$\Phi_{\mu\sigma}^\sigma = (n+1)\phi_\mu + \frac{1}{4\chi} K_\sigma a_\mu^\sigma = \frac{1}{2} \partial_\mu \log \frac{h}{g}.$$

Eliminating the quantity h/g by means of (26.3) and the $K_\sigma a_\mu^\sigma$ by (27.1), we are led to the relation

$$4\phi_\mu = \partial_\mu \log \omega.$$

Expand the terms in (30.3) to obtain:

$$\frac{1}{\chi^2 \omega^{\frac{1}{2}}} \partial_\gamma K_\mu + \frac{1}{\chi^2 \omega^{\frac{1}{2}}} \partial_\mu K_\gamma - \frac{1}{\chi^2 \omega^{\frac{1}{2}}} K_\sigma \Gamma_{\mu\gamma}^\sigma + K_\mu \partial_\gamma \left(\frac{1}{\chi^2 \omega^{\frac{1}{2}}} \right) + K_\gamma \partial_\mu \left(\frac{1}{\chi^2 \omega^{\frac{1}{2}}} \right) = 0.$$

Eliminating the $\Gamma_{\mu\gamma}^\sigma$ by (c) and using (22.1) and (27.1) we deduce

$$(27.2) \quad \nabla_\gamma \left(\frac{\omega^{\frac{1}{2}}}{\chi} K_\mu \right) + \nabla_\mu \left(\frac{\omega^{\frac{1}{2}}}{\chi} K_\gamma \right) = 0.$$

But from (24.1), (27.1), and (27.2) it follows that

$$(29.7) \quad \frac{\omega^{\frac{1}{2}}}{2\chi} K_\rho \frac{dx^\rho}{d\tau} - \chi \omega^{\frac{1}{2}} = \text{constant}$$

is a first integral of E .

Now should C be some trajectory of E , so that it is necessarily a solution of

$$(2.2) \quad \frac{d^2 x^\alpha}{d\tau^2} + \Lambda_\mu^\alpha \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + b_\rho^\alpha \frac{dx^\rho}{d\tau} = R^\alpha,$$

then (29.7) will hold along C for a suitable selection of the real constant k . We use this value of k to define the parameter transformation $\tau \leftrightarrow t$ by means of (22.2). From (2.2) equation (4.1) is now deduced along C . Eliminate the Λ_μ^α by (c), the R^α by (a), the b_ρ^α by (b); then with the aid of (22.1), (22.2), and (28.1) the equation (4.1) reduces to (2.1). Therefore, C is also a trajectory of the system D , which completes the proof of the sufficiency of the conditions¹⁰ (a), (b), (c), and (d).

32. Conditions that all trajectories of D and E be identical. We first impose the condition that all trajectories of D are trajectories of E and we designate by (a'), (b'), (c'), and (d') the conditions corresponding to (a), (b), (c), and (d) respectively under the change in roles of D and E . We shall now study the manner in which the intervening quantities transform under this change in the roles of the two systems. We have

$$(a') \quad Q^\alpha = \chi'^2 R^\alpha,$$

$$(b') \quad a_\rho^\alpha = -\chi' b_\rho^\alpha + K'_\rho R^\alpha,$$

$$(c') \quad \Gamma_{\mu\nu}^\alpha - \Lambda_{\mu\nu}^\alpha = \delta_\mu^\alpha \phi'_\nu + \delta_\nu^\alpha \phi'_\mu - \frac{1}{4\chi'^2} K'_\mu K'_\nu R^\alpha + \frac{1}{4\chi'} (K'_\mu b_\nu^\alpha + K'_\nu b_\mu^\alpha),$$

and

$$(d') \quad \frac{1}{2\chi'^2} \left(\frac{\chi' g}{h} \right)^{1/(n+2)} K'_\rho \frac{dx^\rho}{d\tau} - \frac{1}{\chi'} \left(\frac{\chi' g}{h} \right)^{1/(n+2)} = \text{constant}$$

is a first integral of the system E .

From (a') we have the obvious transformation:

$$(32.1) \quad \chi' = \frac{1}{\chi},$$

which allows the passage from (a') to (a). It follows that (b') reduces to (b) by the transformation (32.1) and:

$$(32.2) \quad K'_\rho = \frac{K_\rho}{\chi^3}.$$

¹⁰ In the results previously announced (LICHNEROWICZ, loc. cit.) it was stated that there were two distinct first integrals of the system D . This resulted from an error in calculations. We have shown in this paper that actually there exists only the first integral (d) of the system D (it admits as special cases the two first integrals given in the original publication).

We see that one can pass from (c') to (c) using the transformations (32.1), (32.2) and

$$(32.3) \quad \phi'_\mu = -\phi_\mu.$$

The relation (32.2) also corresponds to the obvious transformation

$$(32.4) \quad \omega \rightarrow \frac{1}{\omega}.$$

But the substitutions (32.1), (32.2), (32.3) and (32.4) permit us to pass from (d') to

$$\frac{\omega^\dagger}{2\chi} K_\rho \frac{dx^\rho}{d\tau} - \chi \omega^\dagger = \text{constant},$$

which is merely another form of (d), and conversely.

We have thus obtained the conditions for the trajectories of the two systems to be identical.

Theorem II. *All trajectories of the dynamical system E will be trajectories of the dynamical system D and conversely if, and only if, the conditions of Theorem I are satisfied.*

For convenience we repeat the theorem stated in §15.

Theorem. *Given a dynamical system D for which $a_{\rho\sigma} = 0$, there does not exist a dynamical system E for which $b_{\rho\sigma} \neq 0$ such that any trajectory of E is a trajectory of D or conversely.*

APPENDIX I

We give here the calculation by which (14.2) is deduced from (14.1). Since (14.1) can be written as

$$\begin{aligned} [Q^\alpha a_\rho^\gamma \delta_\lambda^\beta \delta_\mu^\delta - Q^\alpha a_\rho^\delta \delta_\lambda^\beta \delta_\mu^\gamma - Q^\beta a_\rho^\gamma \delta_\lambda^\alpha \delta_\mu^\delta + Q^\beta a_\rho^\delta \delta_\lambda^\alpha \delta_\mu^\gamma - Q^\gamma a_\rho^\alpha \delta_\lambda^\beta \delta_\mu^\delta \\ + Q^\gamma a_\rho^\beta \delta_\lambda^\alpha \delta_\mu^\delta + Q^\delta a_\rho^\alpha \delta_\lambda^\gamma \delta_\mu^\beta - Q^\delta a_\rho^\beta \delta_\lambda^\gamma \delta_\mu^\alpha] \dot{x}^\lambda \dot{x}^\mu = 0, \end{aligned}$$

it follows that when the expression in brackets is made symmetric in the three indices λ , μ , and ρ , it must vanish. Forming the resulting equation, putting $\beta = \lambda$ and $\delta = \mu$, and then summing on the repeated indices, we arrive at the following expression:

$$\begin{aligned} n^2 Q^\alpha a_\rho^\gamma + Q^\alpha a_\rho^\gamma + Q^\alpha a_\rho^\gamma + n Q^\alpha a_\rho^\gamma + n Q^\alpha a_\rho^\gamma + n Q^\alpha a_\rho^\gamma \\ - n Q^\alpha a_\rho^\gamma - Q^\alpha a_\rho^\gamma - Q^\alpha a_\rho^\gamma - Q^\alpha a_\sigma^\gamma \delta_\rho^\gamma - n Q^\alpha a_\sigma^\gamma \delta_\rho^\gamma - Q^\alpha a_\sigma^\gamma \delta_\rho^\gamma \\ - n Q^\alpha a_\sigma^\gamma - Q^\alpha a_\rho^\gamma - Q^\alpha a_\rho^\gamma - Q^\sigma a_\sigma^\gamma \delta_\rho^\alpha - n Q^\sigma a_\sigma^\gamma \delta_\rho^\alpha - Q^\sigma a_\sigma^\gamma \delta_\rho^\alpha \\ + Q^\sigma a_\sigma^\gamma \delta_\rho^\alpha + Q^\gamma a_\rho^\alpha + Q^\alpha a_\rho^\gamma + Q^\sigma a_\sigma^\gamma \delta_\rho^\gamma + Q^\alpha a_\sigma^\gamma \delta_\rho^\gamma + Q^\gamma a_\sigma^\gamma \delta_\rho^\alpha \\ - n^2 Q^\gamma a_\rho^\alpha - Q^\gamma a_\rho^\alpha - Q^\gamma a_\rho^\alpha - n Q^\gamma a_\rho^\alpha - n Q^\gamma a_\rho^\alpha - n Q^\gamma a_\rho^\alpha \\ + n Q^\gamma a_\rho^\alpha + Q^\gamma a_\rho^\alpha + Q^\gamma a_\rho^\alpha + Q^\gamma a_\sigma^\gamma \delta_\rho^\alpha + n Q^\gamma a_\sigma^\gamma \delta_\rho^\alpha + Q^\gamma a_\sigma^\gamma \delta_\rho^\alpha \\ + n Q^\gamma a_\rho^\alpha + Q^\gamma a_\rho^\alpha + Q^\gamma a_\rho^\alpha + Q^\sigma a_\sigma^\gamma \delta_\rho^\gamma + n Q^\sigma a_\sigma^\gamma \delta_\rho^\gamma + Q^\sigma a_\sigma^\gamma \delta_\rho^\gamma \\ - Q^\gamma a_\sigma^\gamma \delta_\rho^\alpha - Q^\alpha a_\rho^\gamma - Q^\gamma a_\rho^\alpha - Q^\alpha a_\sigma^\gamma \delta_\rho^\gamma - Q^\sigma a_\sigma^\gamma \delta_\rho^\alpha - Q^\sigma a_\sigma^\gamma \delta_\rho^\gamma = 0. \end{aligned}$$

Since $a_\sigma^\sigma = 0$ from the skew-symmetry of the tensor $a_{\rho\sigma}$, the above expression reduces to

$$(n^2 + n - 2)(Q^\alpha a_\rho^\gamma - Q^\gamma a_\rho^\alpha) + (n + 2)(Q^\sigma a_\sigma^\alpha \delta_\rho^\gamma - Q^\sigma a_\sigma^\gamma \delta_\rho^\alpha) = 0.$$

Dividing by the factor $(n + 2)$ we obtain equation (14.2).

APPENDIX II

Write (21.1) as a vector equation of the form:

$$(1) \quad \lambda AX = \mu Q + \rho X,$$

where

$$Q = (Q^\alpha), \quad X = \left(\frac{dx^\alpha}{dt} \right), \quad AX = \left(a_\rho^\alpha \frac{dx^\rho}{dt} \right),$$

and where λ , μ , and ρ are the coefficients of $a_\rho^\alpha dx^\rho/dt$, Q^α , and dx^α/dt respectively. From the skew-symmetry of $a_{\rho\sigma}$ it follows that AX is always orthogonal to X . We shall now show that the relation (1) (where λ , μ , ρ , depend on X), if it be satisfied for an arbitrary X , implies that $\lambda = \mu = \rho = 0$.

If $\lambda = 0$, it is clear that the relation $\mu Q + \rho X = 0$ implies $\mu = \rho = 0$ for any X . Let us suppose then that $\lambda \neq 0$ and show that this will lead to a contradiction. Then there exist two scalars α and β such that

$$AX = \alpha Q + \beta X.$$

a) We choose for X a vector Y orthogonal to Q . In particular this gives

$$(2) \quad AY = \gamma Q + \delta Y.$$

Taking the scalar product of the two members of (2) with Y , we conclude that $\delta = 0$, and in this case we have:

$$AY = \gamma Q.$$

i.e., AY is collinear with Q .

b) X , being an arbitrary vector, can be decomposed into the sum of a vector parallel to Q and a vector orthogonal to Q : $X = \sigma Q + Y$.

It follows that

$$(3) \quad AX = \sigma AQ + \gamma Q,$$

which gives us

$$(4) \quad \beta X = \sigma AQ + (\gamma - \alpha)Q.$$

If $AQ \neq 0$, β can never vanish, for otherwise Q and AQ would be collinear, while they are clearly orthogonal. It is sufficient then in the relation (4) to choose X

out of the plane defined by Q and AQ from which it results that $AQ = 0$. From (3) we have

$$AX = \gamma Q$$

for an arbitrary X . But this is impossible, for it implies that $\gamma QX = 0$ for an arbitrary X , therefore $\gamma = 0$, which means that $AX = 0$ for an arbitrary X , which is contrary to our hypothesis ($a_p^\alpha \neq 0$). We have then $\lambda = 0$ and consequently $\mu = \rho = 0$, which justifies the equations given in §21.