*Steady Rotational Flow of Ideal Gases**

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Chapter I. INTRODUCTION

This paper concerns the steady rotational flow of non-viscous and thermally non-conducting gases subject to no extraneous force field. For the most part attention is restricted to gases having constant specific heats. However, some of the results are valid for more general classes of fluids. Uniformity of total flow energy (stagnation enthalpy) or of entropy throughout the flow is not assumed.

The present work is intended to be a comprehensive treatment of the status (in 1949) of rotational flow theory from the point of view of the establishment of general properties of such flows and the discovery and study of families of exact solutions to the equations governing them. The considerable progress made is due largely to the fortuitous combination of two factors. The first was the discovery early in 1947 of the Substitution Principle and the Canonical Equations of Motion (Chapter II) which permitted a great simplification of the differential equations governing rotational flow. The second was the exploitation of this simplified formulation by the use of inverse methods for discovery of families of exact flow solutions. That is, the solutions sought were those having certain geometric or kinematic properties, rather than those satisfying prescribed boundary conditions. This approach was not practicable using the older formulations of the equations.

Chapter II is devoted to the basic Substitution Principle and the associated reformulation of the equations of motion and to related topics of a general nature. Chapters III, IV and V contain the application of these fundamental relations to the successively more complex cases of plane flow fields, axially-symmetric flow fields, and truly spatial flow fields.

Chapter II. SOME FUNDAMENTAL RELATIONS

Section A. The Substitution Principle. Steady flow of an inviscid, thermally nonconducting fluid subject to no extraneous force is governed by the following equations:

$$\operatorname{div} \rho \mathbf{V} = 0, \quad \rho \mathbf{V} \cdot \operatorname{grad} \mathbf{V} + \operatorname{grad} p = 0, \quad (1), (2)$$

$$\rho = \rho(p, s), \quad \mathbf{V} \cdot \operatorname{grad} s = 0, \quad (3), (4)$$

wherein $\mathbf{V}$ denotes the *velocity vector*, ρ the *density*, p the *pressure*, and s the *specific entropy*.

Through scalar multiplication of (2) by $\mathbf{V}$, taking into account (4), we obtain

$$V^2 + 2h(p, s) = a^2 = 2h(p_0, s), \quad (5)$$

wherein h denotes the *specific enthalpy* defined here by

$$h(p, s) \equiv \int_0^p \frac{d\tilde{p}}{\rho(\tilde{p}, s)}, \quad (6)$$

a the limiting or *ultimate velocity* magnitude, and p_0 the *stagnation pressure*. a is a constant along any individual streamline (i.e.,

$$\mathbf{V} \cdot \operatorname{grad} a = 0); \quad (7)$$

it is the hypothetical velocity magnitude attainable by expansion of the gas on a given streamline to zero pressure. p_0 is the hypothetical pressure attained by the gas on a given streamline when it is brought to rest isentropically. From the second equality of (5) and from (4) and (7) it is evident that in a region free from shocks p_0 is also a quantity that is a constant along any individual streamline, i.e.,

$$\mathbf{V} \cdot \operatorname{grad} p_0 = 0. \quad (8)$$

The quantity $h(p_0, s)$ representing the hypothetical enthalpy of the gas on a given streamline under rest or "stagnation" conditions is called the *stagnation enthalpy* or *total head* and will be designated sometimes for convenience by h_0 . Either the stagnation enthalpy or the ultimate velocity can be regarded as a measure of the total specific energy content of the gas on a given streamline. At a shock discontinuity the total specific energy content—and hence a and h_0 —for the flow along a given streamline are unchanged; however the specific entropy—and hence, by (5), the stagnation pressure—is changed discontinuously. The same

is true of the *stagnation density*, defined by

$$\rho_0 \equiv \rho(p_0, s). \quad (9)$$

The basic relations (1)–(4) constitute a set of six equations in six dependent variables: the three components of the velocity vector and the three thermodynamic quantities p , ρ , and s . By use of the state equation (3) one of the thermodynamic variables can be eliminated, giving five equations in five dependent variables. For an arbitrary state equation further reduction of the number of equations and unknowns can be attained only by specializing assumptions, such as uniformity of the specific entropy and/or the stagnation enthalpy throughout the flow. However, for a broad class of state equations, including that of a perfect gas, we will now show that one more of the thermodynamic variables is redundant and hence can be eliminated by a suitable choice of the dependent variables to yield a set of four equations in four unknowns.

If the vector $\mathbf{q}$ defined by

$$\mathbf{q} \equiv \sqrt{\rho} \mathbf{V} \quad (10)$$

is introduced into (1) and (2), these equations may be written

$$\text{div } \mathbf{q} + \frac{1}{2} (\mathbf{q} \cdot \text{grad } \ln \rho) = 0, \quad (11)$$

$$\mathbf{q} \cdot \text{grad } \mathbf{q} - \frac{1}{2} (\mathbf{q} \cdot \text{grad } \ln \rho) \mathbf{q} + \text{grad } p = 0. \quad (12)$$

Since (3) implies the relation

$$\text{grad } \ln \rho = \left(\frac{\partial \ln \rho}{\partial p} \right)_s \text{grad } p + \left(\frac{\partial \ln \rho}{\partial s} \right)_p \text{grad } s, \quad (13)$$

(4) can be written in the form

$$\mathbf{q} \cdot \text{grad } \ln \rho = \left(\frac{\partial \ln \rho}{\partial p} \right)_s \mathbf{q} \cdot \text{grad } p. \quad (14)$$

It is now immediately evident that, *provided* $(\partial \ln \rho / \partial p)_s$ is a function of p alone, (11), (12), and (14) will be unaffected by any change of dependent variables leaving $\mathbf{q}$, $\mathbf{q} \cdot \text{grad } \ln \rho$, and p unchanged.

From this we conclude that for a gas having an equation of state of the form

$$\rho = P(p)S(s) \quad (15)$$

any flow field satisfying (1)–(4) is a member of an infinite group of flow fields sharing the same streamlines, the same pressure field, and—as will be demonstrated below—the same Mach number field, the members of the group being related by transformations of the type

$$p^* = p; \quad \rho^* = m^2 \rho; \quad \mathbf{V}^* = \frac{1}{m} \mathbf{V}; \quad s^* = S^{-1}[m^2 S(s)], \quad (16)$$

where

$$\mathbf{V} \cdot \text{grad } m = 0. \quad (17)$$

That is, m is any scalar function constant along each individual streamline.

The invariance of the Mach number M defined by

$$M \equiv \frac{V}{c} \equiv \frac{V}{\sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}} \quad (c \text{ denotes the velocity of sound}) \quad (18)$$

under this group of transformations follows at once from the invariance of p and q and from the following identity:

$$M^2 = \frac{V^2}{\left(\frac{\partial p}{\partial \rho}\right)_s} = \frac{\rho V^2}{\left(\frac{\partial p}{\partial \ln \rho}\right)_s} = q^2 \frac{d}{dp} \ln P(p). \quad (19)$$

The substitution (16) is also valid for a flow field containing shocks, as is immediately evident if we write the familiar conditions for conservation of mass, momentum, and energy across a shock front in the following form:

$$\begin{aligned} \sqrt{\rho_1} q_1 \sin \alpha_1 &= \sqrt{\rho_2} q_2 \sin \alpha_2, & p_1 + q_1^2 \sin^2 \alpha_1 &= p_2 + q_2^2 \sin^2 \alpha_2, \\ \frac{q_1^2}{\rho_1} + 2 \int_0^{p_1} \frac{d\tilde{p}}{\rho(\tilde{p}, s_1)} &= \frac{q_2^2}{\rho_2} + 2 \int_0^{p_2} \frac{d\tilde{p}}{\rho(\tilde{p}, s_2)}, \end{aligned}$$

where subscripts 1 and 2 designate quantities on a given streamline before and after the shock and α denotes the angle between shock front and streamline.

For the *product* equation of state (15), the expression (6) for the enthalpy can be written

$$h(p, s) = \frac{H(p)}{S(s)}, \quad H(p) \equiv \int_0^p \frac{d\tilde{p}}{P(\tilde{p})}. \quad (20), (21)$$

The Bernoulli equation (5) then becomes

$$V^2 + 2 \frac{H(p)}{S(s)} = a^2 = 2 \frac{H(p_0)}{S(s)}, \quad (22)$$

or

$$H(p) = H(p_0) \left(1 - \frac{V^2}{a^2}\right). \quad (23)$$

For one of the substitute flows of (16), we have

$$a^{*2} = \frac{a^2}{m^2}, \quad S(s^*) = m^2 S(s). \quad (24), (25)$$

Hence, in view of (7) and (17), the function m can always be so chosen that the ultimate velocity—and hence the stagnation enthalpy—of the substitute flow is uniform throughout the field of flow. Or, alternatively, m can be so chosen that throughout a shock-free region of the flow the entropy is uniform. Thus a flow having both a non-uniform entropy distribution and a non-uniform stagnation enthalpy distribution can always be replaced by a substitute flow having a uniform stagnation enthalpy, or (in a shock-free region) a uniform entropy, with the same geometric properties, Mach number and pressure distributions as the original flow. This demonstrates that the geometric and dynamic variety of flows which are neither isoenergetic nor isentropic is no greater than that of flows having only one of these properties. However, it follows at once from (24) and (25) that the function m can be so chosen that *both* a^{*2} and s^* are constants *only* if for the original flow

$$a^2 S(s) = \text{constant.} \quad (26)$$

From (22) it is seen that this condition implies that $p_0 = \text{constant}$. This special role of flow fields having a uniform stagnation pressure throughout will be further clarified in Sections B and C.

This establishment of the multiplicity* of flows having the same streamline patterns and pressure fields reveals a redundancy of dependent variables in the set of equations (1)–(4) beyond that removable by use of the state equation (3). Thus the introduction of a modified velocity vector which will be invariant under the group of substitutions (16) will lead to a reformulation of the basic equations in terms of this modified velocity vector and the pressure only. Each solution to these reduced equations then generates the entire group of substitution flows associated with its streamline pattern and pressure field. This reformulation is carried out in Section B.

Section B. The Canonical Equations.** We now set about the reformulation discussed in the closing paragraphs of Section A by choosing a suitable modified velocity vector which will be an invariant under the group of substitutions (16). Equation (24) immediately suggests the choice $m = a$ yielding the flow field having unit ultimate velocity in each group. This is tantamount to the introduction of the *reduced velocity* vector $\mathbf{W}$ through the definition

* Established by MAX M. MUNK and the author [8] for the case of a perfect gas, *i.e.*, a gas having constant specific heats and hence a state equation of the form

$$\rho = K p^{1/\gamma} e^{-s/c_p}, \quad (27)$$

where K is a dimensional constant, γ denotes the ratio of specific heats, and c_p the specific heat at constant pressure; generalized in [14].

** Equations (40) and (41) and the compatibility condition (35) were first derived by L. Crocco [2] for the special case of isoenergetic flows of a perfect gas, *i.e.*, flows of uniform stagnation enthalpy and hence uniform ultimate velocity. The existence of such a reduced set of equations for non-isoenergetic flows of a perfect gas was inferred by MUNK and PRIM [8] as a consequence of the Substitution Principle and later the specific form of the equations was reported independently by them [9] and by B. HICKS and his collaborators [6]. The following more general analysis is taken from [14].

$$\mathbf{W} \equiv \frac{\mathbf{V}}{a}. \quad (28)$$

Since the substitutions (16) multiply $\mathbf{V}$ and a by the same scalar function, $\mathbf{W}$ is the same for all members of the family of flows associated with a given streamline pattern and pressure distribution.

Substitution of (28) into (2) yields (by virtue of (7))

$$\mathbf{W} \cdot \text{grad } \mathbf{W} + \frac{1}{\rho a^2} \text{grad } p = 0. \quad (29)$$

But, from (5)

$$a^2 = 2h(p_0, s) = W^2 a^2 + 2h(p, s) = \frac{2h(p, s)}{1 - W^2}. \quad (30)$$

Substitution of (30) into (29) gives

$$\mathbf{W} \cdot \text{grad } \mathbf{W} + \frac{1}{2}(1 - W^2) \frac{\text{grad } p}{\rho(p, s)h(p, s)} = 0. \quad (31)$$

Now for the product equation of state (15) the product of ρ and h is given by

$$\rho(p, s)h(p, s) = P(p)H(p) = \frac{H(p)}{\frac{dH(p)}{dp}} \quad (32)$$

—a function of pressure alone. (It can further be shown that *only* for the product equation of state is ρh independent of s .) Substitution of (32) into (31) now yields the dynamical equation in the canonical form:

$$\mathbf{W} \cdot \text{grad } \mathbf{W} + \frac{1}{2}(1 - W^2) \text{grad } \ln H(p) = 0. \quad (33)$$

Substitution of (28) into (1) yields (making use of (7))

$$\text{div } [\rho(p, s) \mathbf{W}] = 0,$$

which for the product equation of state becomes

$$\text{div } [P(p) \mathbf{W}] = 0. \quad (34)$$

Equations (33) and (34) constitute a set of four equations in four dependent variables—the three components of $\mathbf{W}$ and p . Given a solution of this set of equations, one can compute the stagnation pressure from (23) by $H(p_0) = H(p)/(1 - W^2)$. Then, by (22), the ultimate velocity a can be assigned arbitrarily on each streamline and the entropy determined from $S(s) = 2H(p_0)/a^2$. Or, alternatively, the entropy can be assigned on each streamline in a shock-free

region and the ultimate velocity determined from $a^2 = 2H(p_0)/S(s)$. The actual velocity is then given by $\mathbf{V} = a\mathbf{W}$, the density by $\rho = P(p)S(s)$.

As would be expected, the shock conditions mentioned in the previous section can be expressed in terms of $\mathbf{W}$, p , and geometric quantities. This reformulation is readily accomplished by use of the relations

$$\rho = P(p)S(s) = \frac{2P(p)H(p_0)}{a^2} = \frac{2P(p)H(p)}{a^2(1 - W^2)}, \quad \mathbf{V} = a\mathbf{W}.$$

The resulting form of the equations is:

$$\begin{aligned} \frac{P(p_1)H(p_1)W_1}{a_1(1 - W_1^2)} \sin \alpha_1 &= \frac{P(p_2)H(p_2)W_2}{a_2(1 - W_2^2)} \sin \alpha_2, \\ p_1 + \frac{2P(p_1)H(p_1)W_1^2}{1 - W_1^2} \sin^2 \alpha_1 &= p_2 + \frac{2P(p_2)H(p_2)W_2^2}{1 - W_2^2} \sin^2 \alpha_2, \\ a_1 &= a_2. \end{aligned}$$

The manner of appearance of the pressure in (33) leads to a compatibility condition

$$\text{curl} \left[\frac{\mathbf{W} \cdot \text{grad } \mathbf{W}}{1 - W^2} \right] = 0, \quad \text{curl} \left[\frac{\mathbf{W} \times \text{curl } \mathbf{W}}{1 - W^2} \right] = 0, \quad (35)$$

involving $\mathbf{W}$ alone. Only for $\mathbf{W}$ fields satisfying (35) can a consistent scalar pressure field be obtained by integration from (33). (It should be pointed out that in general the pressure function determined from (33) will be single-valued only in suitably restricted regions of the flow field.)

For a general state equation, the function of $\mathbf{W}$ appearing in the compatibility condition (35) is given from (31) by

$$\text{curl} \left[\frac{\mathbf{W} \times \text{curl } \mathbf{W}}{1 - W^2} \right] = \frac{\text{grad } p \times \text{grad } \rho h}{2\rho^2 h^2}.$$

This expression will be zero for an arbitrary flow of the gas concerned if, and only if, the product ρh (the enthalpy per unit volume) is a function of pressure alone. As has been noted above, this implies a product equation of state. Hence a compatibility condition for the dynamic equation (31) free of thermodynamic quantities exists only for gases having a product equation of state.

We now inquire which members of the family of product equations of state (15) permit the elimination of the pressure from the continuity equation (34). To answer this question we observe that (34) can be written in the form

$$\text{div } \mathbf{W} + \frac{d \ln P(p)}{d \ln H(p)} \mathbf{W} \cdot \text{grad } \ln H(p) = 0. \quad (36)$$

But, by virtue of (8) and (23),

$$\mathbf{W} \cdot \text{grad} \ln H(p) = \mathbf{W} \cdot \text{grad} \ln (1 - W^2), \quad (37)$$

so that (36) is independent of pressure if, and only if,

$$\frac{d \ln P(p)}{d \ln H(p)} = \text{constant}.$$

It is easily shown from (21) that this implies that the function $P(p)$ is of the form

$$P(p) = Cp^k, \quad (38)$$

where C and k are constants and $k < 1$. Hence the most general product equation of state permitting the elimination of the pressure from the continuity equation is the *power-law* equation of state

$$\rho = p^k S(s) \quad (39)$$

with $k < 1$.

For this power-law state equation the dynamic equation (33) becomes

$$\mathbf{W} \cdot \text{grad} \mathbf{W} + \frac{1-k}{2} (1 - W^2) \text{grad} \ln p = 0, \quad (40)$$

and the continuity equation (34) becomes after use of (37)

$$\text{div} [(1 - W^2)^{k/(1-k)} \mathbf{W}] = 0. \quad (41)$$

The compatibility equation (35) and the continuity equation (41) now constitute a set of necessary and sufficient conditions for a vector field $\mathbf{W}$ to serve as the reduced velocity field of a flow of the type under consideration.

Given a $\mathbf{W}$ field satisfying (35) and (41), a consistent pressure field can be found by integration from (40). The stagnation pressure field is then given from (23) by

$$p_0 = p(1 - W^2)^{-1/(1-k)}. \quad (42)$$

Either the ultimate velocity or the specific entropy can then be assigned arbitrarily on each streamline (the latter only in a shock-free region, of course) and the other determined by

$$a^2 S(s) = \frac{2}{1-k} p_0^{1-k}. \quad (43)$$

The actual velocity field is then given by $\mathbf{V} = a\mathbf{W}$, the density field by $\rho = p^k S(s)$.

It is fortunate that the state equation (27) for a gas having constant specific heats—the case of greatest practical interest—is of the power-law type (39) so that the relatively simple formulation (40) and (41) applies, with $k = 1/\gamma$.

Section C. General Thermodynamic and Vorticity Considerations. If the velocity field $\mathbf{V}$ of (2) is a Beltrami field, that is if

$$\mathbf{V} \times \text{curl } \mathbf{V} = 0,$$

(2) becomes

$$\text{grad } \frac{1}{2} V^2 + \frac{1}{\rho} \text{grad } p = 0. \quad (44)$$

Taking the curl of (44) gives the condition

$$\text{grad } \rho \times \text{grad } p = 0 \quad (45)$$

and by (3) the condition

$$\text{grad } s \times \text{grad } p = 0 \quad (46)$$

(except in the degenerate case where (3) does not involve s and (45) is automatically satisfied). If the pressure and density are not constant along each streamline (46) requires that the specific entropy be uniform throughout the flow ($\text{grad } s = 0$). If $\text{grad } s \neq 0$, p and ρ must be constant along each streamline and furthermore both $\text{grad } p$ and $\text{grad } \rho$ must either be parallel to $\text{grad } s$ or must vanish.

Again excepting the case $\mathbf{V} \cdot \text{grad } p = 0$, the uniformity of the specific entropy leads by (5) and (44) to

$$\text{grad } a^2 = \text{grad } 2h_0 = \text{grad } V^2 + 2 \frac{\text{grad } p}{\rho} = 0.$$

That is, a Beltrami flow for which the surfaces of constant p , ρ , and s do not coincide and for which $\text{grad } \rho \neq 0$ and $\text{grad } p \neq 0$ has a uniform ultimate velocity and stagnation enthalpy, as well as a uniform specific entropy. Also, since $h_0 \equiv h(p_0, s)$ uniformity of h_0 and s implies uniformity of p_0 .

It should be noted that irrotational flows are included as special cases in the above statements concerning Beltrami flows. It should further be noticed that *the only doubly-laminar Beltrami fields are irrotational fields*. (A doubly-laminar vector field is a field normal to a one-parameter family of surfaces, hence a field of the form $\mathbf{F} = \alpha \text{grad } \beta$ where α and β are suitable scalar functions. Included in this class of fields are the important cases of *plane* and *axially-symmetric* vector fields.)

We now derive a formula due to Vazsonyi [21] which shows immediately that uniformity of specific entropy and stagnation enthalpy is also a sufficient condition that a flow have a Beltrami velocity field. From (2) we have

$$\mathbf{V} \times \text{curl } \mathbf{V} = \text{grad } \frac{1}{2} V^2 + \frac{1}{\rho} \text{grad } p. \quad (47)$$

The second law of thermodynamics yields the relation

$$T \text{grad } s = \text{grad } (h - p/\rho) + p \text{grad } \frac{1}{\rho},$$

where T denotes the absolute temperature, or

$$\frac{1}{\rho} \text{grad } p = \text{grad } h - T \text{grad } s. \quad (48)$$

Substituting (48) into (47) we obtain

$$\mathbf{V} \times \text{curl } \mathbf{V} = \text{grad } (\tfrac{1}{2} V^2 + h) - T \text{grad } s,$$

or, from (5), Vazsanyi's formula:

$$\mathbf{V} \times \text{curl } \mathbf{V} = \text{grad } h_0 - T \text{grad } s. \quad (49)$$

From this formula it follows at once that uniformity of h_0 and s implies that $\mathbf{V} \times \text{curl } \mathbf{V} = 0$, but that uniformity of one of these quantities alone does not.

We now restrict our attention to gases having a product equation of state (15) and derive a relation which further clarifies the compatibility condition (35). Substitution of (23) into (33) yields at once:

$$\frac{\mathbf{W} \times \text{curl } \mathbf{W}}{1 - W^2} = \frac{1}{2} \text{grad } \ln H(p_0). \quad (50)$$

From this relation it follows that the reduced velocity field is a Beltrami field if, and only if, the stagnation pressure is uniform. In the case of plane, axially-symmetric, and other doubly-laminar flow fields this means that the reduced velocity field is irrotational if and only if the stagnation pressure is uniform. The compatibility condition (35) is seen to be merely an integrability condition for the scalar function p_0 .

Now by straightforward vector manipulation making use of the relations $\mathbf{V} = a\mathbf{W}$ and $\mathbf{V} \cdot \text{grad } a = 0$ we obtain a companion equation to (50):

$$\mathbf{V} \times \text{curl } \mathbf{V} = \tfrac{1}{2}(a^2 - V^2) \text{grad } \ln H(p_0) + V^2 \text{grad } \ln a, \quad (51)$$

from which it is immediately evident that a field for which the stagnation pressure is uniform and hence for which $\mathbf{W} \times \text{curl } \mathbf{W} = 0$ has $\mathbf{V} \times \text{curl } \mathbf{V} = 0$ if and only if the ultimate velocity (and hence the stagnation enthalpy) is also uniform. On the other hand we have already established the fact that, except for the degenerate flows in which the surfaces of constant p , ρ , and s coincide or for which $\text{grad } p$ or $\text{grad } \rho = 0$, the condition $\mathbf{V} \times \text{curl } \mathbf{V} = 0$ implies that the stagnation pressure is uniform, and hence by (50) that $\mathbf{W} \times \text{curl } \mathbf{W} = 0$.

It will be convenient to introduce the terminology *Beltrami flows* for flow fields in which $\mathbf{V} \times \text{curl } \mathbf{V} = 0$ and *generalized Beltrami flows* for flow fields in which $\mathbf{W} \times \text{curl } \mathbf{W} = 0$. The conclusions of the preceding paragraph can now be expressed more succinctly as follows:

A *Beltrami flow* is either a *generalized Beltrami flow* or a flow in which the surfaces of constant density, pressure and entropy coincide or a flow of uniform pressure or a flow of uniform density.

A *generalized Beltrami flow* is a *Beltrami flow* if and only if it has a uniform ultimate velocity.

For doubly-laminar flow fields the preceding statements can be strengthened by the substitution of "*irrotational flow*" ($\text{curl } \mathbf{V} = 0$) for "*Beltrami flow*" and of "*generalized irrotational flow*" ($\text{curl } \mathbf{W} = 0$) for "*generalized Beltrami flow*."

Thus we see that $\mathbf{V} \times \text{curl } \mathbf{V} = 0$ generally implies that also $\mathbf{W} \times \text{curl } \mathbf{W} = 0$, but that the converse is true for, and only for, flows of uniform stagnation enthalpy. Further, in the case of doubly-laminar fields $\text{curl } \mathbf{V} = 0$ generally implies that also $\text{curl } \mathbf{W} = 0$, but $\text{curl } \mathbf{W} = 0$ implies that $\text{curl } \mathbf{V} = 0$ if and only if the flow is isoenergetic. This suggests the application of the terminology *weakly rotational* for a flow in which $\text{curl } \mathbf{V} \neq 0$ but $\text{curl } \mathbf{W} = 0$, and *strongly rotational* for a flow in which $\text{curl } \mathbf{V} \neq 0$ and $\text{curl } \mathbf{W} \neq 0$. A *weakly rotational flow* has associated with it by the substitution group (16) irrotational flows having the same streamline pattern, pressure field, and Mach number field. On the other hand, a *strongly rotational field* is in general distinct from all irrotational fields in its streamline pattern and pressure field.

Any surfaces (in a non-Beltrami flow) whose normals satisfy the condition

$$\mathbf{N} \times (\mathbf{V} \times \text{curl } \mathbf{V}) = 0$$

are called *Bernoullian surfaces*. Analogously we shall call any surfaces (in a flow that is not a generalized Beltrami flow) whose normals satisfy the condition

$$\mathbf{N} \times (\mathbf{W} \times \text{curl } \mathbf{W}) = 0$$

generalized Bernoullian surfaces. It must be noted that such surfaces do not generally exist.

From (50) we see that in flows of gases having a product equation of state generalized Bernoullian surfaces do exist and are surfaces of constant stagnation pressure.

From (51) we see that in a generalized Beltrami flow the surfaces of constant ultimate velocity are Bernoullian surfaces and that the acceleration vector $\mathbf{A}$ is given by

$$\mathbf{A} = \text{grad } \frac{1}{2} V^2 - \mathbf{V} \times \text{curl } \mathbf{V} = V^2 \text{grad } \ln W.$$

Hence $\mathbf{A}$ is normal to the surfaces of constant reduced velocity magnitude. It should be noted that $\text{curl } \mathbf{A} = 0$, *i.e.*, the motion is circulation preserving, only if $a = \text{const.}$ or $V = V(a)$. In the latter case, the velocity magnitude is necessarily constant along each streamline.

Section D. Joining of Given Rotational Flow Fields to Irrotational Ones Through Shocks. Of great practical importance are rotational flow fields produced by the passage of an initially irrotational flow through a shock front. For example, supersonic air flow past a missile or projectile is characterized by the presence of a strong, curved bow shock wave, and the uniform undisturbed flow in

front of this shock becomes rotational behind the shock because of the entropy gradients introduced by the variation in strength of the shock along its length. Because of this familiar role of shock fronts as generators of rotation in initially irrotational flow fields, it is of interest to inquire whether any given solution of the rotational flow equations can be joined to some irrotational flow across a suitably shaped shock front.

In particular we shall consider plane flow fields of a perfect gas with the state equation (27) and satisfying the canonical equations of motion

$$\mathbf{W} \cdot \text{grad } \mathbf{W} + \frac{\gamma - 1}{2\gamma} (1 - W^2) \text{grad } \ln p = 0, \quad (52)$$

$$\text{div} [(1 - W^2)^{1/(\gamma-1)} \mathbf{W}] = 0. \quad (53)$$

For such a perfect gas (27), (23), and (22) easily yield the useful relations:

$$p = p_0 (1 - W^2)^{\gamma/(\gamma-1)}, \quad \rho = \rho_0 (1 - W^2)^{1/(\gamma-1)}, \quad a^2 = \frac{2\gamma}{\gamma - 1} \frac{p_0}{\rho_0}. \quad (54), (55), (56)$$

And, from (27), the squared velocity of sound is given by

$$c^2 \equiv \left(\frac{\partial p}{\partial \rho} \right)_s = \frac{\gamma p}{\rho}. \quad (57)$$

Using (54)–(57), we relate the Mach number squared to the reduced velocity magnitude by

$$M^2 = \frac{2}{\gamma - 1} \frac{W^2}{1 - W^2}. \quad (58)$$

From (58) it is readily seen that the Mach number and the magnitude of the reduced velocity may be regarded as substantially equivalent variables. (This is, however, the case only for gases having a power-law equation of state.) In dealing with the relations connecting flow quantities on the two sides of a shock front, it is often convenient to express formulae in terms of the Mach number rather than the reduced velocity.

The following equations express the basic flow quantities in front of a shock in terms of quantities behind the shock:

$$p_1 = p_2 \left(\frac{\gamma - 1}{\gamma + 1} \right) \left[\frac{2\gamma}{\gamma - 1} M_2^2 \sin^2 \alpha_2 - 1 \right], \quad (59)$$

$$\omega_1 = \omega_2 - \alpha_2 + \tan^{-1} \left[\frac{M_2^2 \sin^2 \alpha_2 + \frac{2}{\gamma - 1}}{\left(\frac{\gamma + 1}{\gamma - 1} \right) M_2^2 \sin \alpha_2 \cos \alpha_2} \right], \quad (60)$$

$$M_1^2 = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)^2 \left(M_2^2 + \frac{2}{\gamma-1}\right) M_2^2 \sin^2 \alpha_2}{\left(\frac{2\gamma}{\gamma-1} M_2^2 \sin^2 \alpha_2 - 1\right) \left(M_2^2 \sin^2 \alpha_2 + \frac{2}{\gamma-1}\right)} - \frac{2}{\gamma-1}, \quad (61)$$

$$p_{01} = p_{02} \frac{\left(\frac{\gamma+1}{\gamma-1}\right)^{(\gamma+1)/(\gamma-1)} \left(M_2^2 \sin^2 \alpha_2\right)^{\gamma/(\gamma-1)}}{\left(M_2^2 \sin^2 \alpha_2 + \frac{2}{\gamma-1}\right)^{\gamma/(\gamma-1)} \left(\frac{2\gamma}{\gamma-1} M_2^2 \sin^2 \alpha_2 - 1\right)^{1/(\gamma-1)}}, \quad (62)$$

where a subscript 1 designates quantities immediately in front of the shock and a subscript 2 designates quantities on the same streamline immediately behind the shock, α denotes the acute angle between the streamline and the shock front, and ω denotes the angle between a fixed reference line (say the positive x -axis) and the streamline. When M_2 and ω_2 are known at a point (equivalent to a knowledge of $\mathbf{W}_2$) along with p_2 and p_{02} , (59)–(62) determine a set of values for M_1 , ω_1 , p_1 , and p_{01} for each choice of α_2 (between $-\pi/2$ and $\pi/2$). A parallel set of equations for determining the basic flow quantities behind the shock from those in front is obtained from (59)–(62) by replacing α_2 by α_1 and interchanging p_1 and p_2 , p_{01} and p_{02} , M_1 and M_2 .

We now suppose that a plane flow field $\mathbf{W}$ satisfying (52) and (53) is known and ask whether a shock line can be constructed in this field so that the upstream flow quantities determined along the shock by (59)–(62) will be consistent with a generalized irrotational flow field ($\text{curl } \mathbf{W} = 0$) on that side of the shock.

The key to this situation is equation (50), which shows that the necessary and sufficient condition for a plane flow field to have $\text{curl } \mathbf{W} = 0$ is that it have a uniform stagnation pressure. Hence it is only necessary that the shock angle α_2 be so chosen at each point on the shock line that the function p_{01} given by (62) has a constant value everywhere along the shock front. The application of this principle for the construction of shock fronts is clarified by Fig. 1, in which (62) has been used to plot the stagnation pressure ratio p_{02}/p_{01} against $M_2 |\sin \alpha_2|$ for $\gamma = 1.4$. Now at each point of the given flow field the streamline direction and the quantities M_2 and p_{02} are known, although each differs in general, from point to point. If a constant value P_{01} be chosen for p_{01} , Fig. 1 (*i.e.*, eq. (62)) determines a value of $M_2 |\sin \alpha_2|$ at each point of the given flow field for which $p_{02} \leq P_{01}$. At those points for which the value of $M_2 |\sin \alpha_2|$ so determined is less than or equal to the value of M_2 , a pair of shock directions is determined by $\alpha_2 = \pm \sin^{-1}(M_2 |\sin \alpha_2| / M_2)$. The condition $p_{02} \leq P_{01}$ can always be met in a given flow region by simply choosing P_{01} equal to or greater than the maximum value of p_{02} in that region. As is evident from Fig. 1, the condition $M_2 |\sin \alpha_2| \leq M_2$ can also be met (by choosing P_{01} sufficiently large) for any flow regime in which there is a fixed quantity $\bar{M}_2$ such that $\sqrt{(\gamma-1)/2\gamma} < \bar{M}_2 \leq M_2$ throughout. In a flow region where $M_2 < \sqrt{(\gamma-1)/2\gamma}$ no choice of P_{01} will establish permissible shock directions because $M_1 = \infty$ is required to give

$M_2 = \sqrt{(\gamma - 1)/2\gamma}$ (see curve of $(M_1 \sin \alpha_1)^{-1}$ against $M_2 \sin \alpha_2$ in Fig. 1). With the exception of such low Mach number regions each sufficiently large value of P_{01} establishes two direction fields for shock fronts of the type sought. Hence an infinity of such shocks can be constructed passing through any specified point in the given field at which $M_2 > \sqrt{(\gamma - 1)/2\gamma}$.

When a particular shock line has been selected by the method just discussed, the pre-shock flow quantities are determined along the line by (59)–(62). The pre-shock stagnation pressure will be constant along the line (by construction) and the pre-shock Mach numbers will be supersonic (see Fig. 1), so the irrotational flow in front of the shock can be extended forward using the Prandtl-Busemann characteristic method or variations thereof. In this manner the initially given rotational flow can be joined, for example, to a uniform parallel flow.

FIGURE 1

Instead of finding shock lines which will produce the initially given rotational flow field from an irrotational flow, one may alternatively construct shock lines which will produce an irrotational flow from the given rotational one. In this case the given flow quantities are associated with the subscript 1 (pre-shock) quantities and suitably small constant values P_{02} assigned to the post-shock stagnation pressure. The two relations plotted in Fig. 1 will suffice to determine direction fields for permissible shocks as before, but the excluded Mach number range (previously $0 \leq M_2 < \sqrt{(\gamma - 1)/(2\gamma)}$) is now wider: $0 \leq M_1 < 1$. (The pre-shock flow speed must always be supersonic.)

The shock construction principles we have discussed for plane flow fields are also valid for axially-symmetric flow fields, and indeed can be readily extended to any doubly-laminar flow field. For general spatial flow fields the enforcement of $\text{grad } p_0 = 0$ on one side of the constructed shock surface guarantees through

(50) a generalized Beltrami flow on that side of the shock, but not necessarily a generalized irrotational flow.

The shock construction described above can also be applied, of course, to a given irrotational flow field to find shock fronts joining two irrotational flows. In this case both p_{01} and p_{02} are constants and the shock construction reduces to the establishment of direction fields by $M_2 |\sin \alpha_2| = \text{const.} > \sqrt{(\gamma - 1)/(2\gamma)}$. As an illustration of a case of this kind which permits the explicit determination of the required shock lines we consider as the given flow field the simple irrotational vortex

$$\mathbf{W} = \theta_1 \frac{1}{r}$$

where θ_1 is the unit tangent vector in a polar coordinate system employing the radius r and the polar angle θ . Using (58), we obtain

$$M_2^2 = \frac{2/(\gamma - 1)}{r^2 - 1},$$

and the condition $M_2 |\sin \alpha_2| = \text{const.} = A$ yields

$$|\sin \alpha_2| = A \sqrt{\frac{\gamma - 1}{2} (r^2 - 1)},$$

whence

$$\frac{1}{r} \frac{dr}{d\theta} = \tan \alpha_2 = \pm \sqrt{\frac{(r^2 - 1)A^2}{\frac{2}{\gamma - 1} - (r^2 - 1)A^2}},$$

giving the following equation for shock lines producing the given vortex flow from another irrotational flow:

$$\theta = \theta_0 \pm \left(\tan^{-1} \sqrt{\frac{b^2 - r^2}{r^2 - 1}} - b \tan \frac{1}{b} \sqrt{\frac{b^2 - r^2}{r^2 - 1}} \right),$$

where $b \equiv \sqrt{1 + 2A^{-2}/(\gamma - 1)}$ and θ_0 is an arbitrary constant angle. This equation defines two shock lines through each point in the annular region $1 < r < b$ for each choice of b in the range $\sqrt{(\gamma + 1)/(\gamma - 1)} < b < (\gamma + 1)/(\gamma - 1)$. (At $r = 1$, $M = \infty$; at $r = b$, $M = \sqrt{(\gamma - 1)/(2\gamma)}$)

Chapter III. PLANE FLOW FIELDS

Section A. Generalized Form of Crocco's Pressure Theorem.* We will now make use of (50) to establish an interesting relation between pressure and vorticity in plane flow fields of gases having a perfect equation of state (15).

The continuity equation (34) implies the existence for plane flow fields of a streamfunction ψ such that

$$\frac{d\psi}{dl} = 0 \quad \text{and} \quad \frac{d\psi}{dn} = P(p)W, \quad (63)$$

where d/dl denotes differentiation parallel to $\mathbf{W}$ and d/dn denotes differentiation normal to $\mathbf{W}$. The streamlines are lines $\psi = \text{const.}$

Now the generalized rotation $\text{curl } \mathbf{W}$ in any plane flow field is always perpendicular to the flow plane and so can be completely characterized by a single scalar function:

$$\Omega \equiv \frac{\text{curl } \mathbf{W}}{b}, \quad (64)$$

where b is the unit vector normal to the flow plane.

For a plane flow field, eq. (50) by virtue of (8) can be written

$$W\Omega = \frac{1}{2}(1 - W^2) \frac{d}{dn} \ln H(p_0);$$

whence, using (23) and (63) we derive

$$\Omega = \frac{1}{2H(p_0)} \frac{d \ln H(p_0)}{d\psi} H(p)P(p),$$

or

$$\Omega = f(\psi)H(p)P(p), \quad f(\psi) \equiv \frac{\frac{dp_0}{d\psi}}{2P(p_0)[H(p_0)]^2} \quad (65)$$

(It will be recalled that the stagnation pressure is a constant along each individual streamline in a region free from shocks). In (65) then we have the theorem that in a region between two successive shocks the vorticity of the reduced velocity field is proportional to $H(p)P(p)$ along each streamline.

* Crocco's original pressure theorem [2] was established for the special case of isoenergetic flows of a perfect gas; the restriction to isoenergetic flow is removed in [15].

For gases having a power law state equation (39), (65) becomes

$$\Omega = \frac{1-k}{2} p_0^{k-1} \frac{d \ln p_0}{d\psi} p_0 \quad (66)$$

That is, the vorticity of the reduced velocity field is proportional to the pressure along each streamline in a region free from shocks.

For the special case of isoenergetic flow fields ($a = \text{const.}$), (65) and (66) reduce to relations involving the vorticity ω of the actual velocity field by the transformation, $\omega = a\Omega$.

Section B. Plane Flows Having a Constant Velocity Magnitude Along Each Streamline.* For many investigations involving plane flow fields of a gas having a product equation of state it is convenient to refer the continuity equation and the compatibility condition, respectively

$$\text{div} [P(p)\mathbf{W}] = 0, \quad \text{curl} \left[\frac{\mathbf{W} \times \text{curl } \mathbf{W}}{1 - W} \right] = 0, \quad (67), (68)$$

to natural coordinates consisting of the streamlines $\xi = \text{const.}$ and their orthogonal trajectories $\eta = \text{const.}$ That is, we employ a cylindrical coordinate system (ξ, η, z) based on a plane orthogonal net (ξ, η) so chosen that the vector $\mathbf{W}$ has the components: $(\mathbf{W})_\xi = 0$, $(\mathbf{W})_\eta = W(\xi, \eta)$, and $(\mathbf{W})_z = 0$. The squared element of arc length in this coordinate system is given by

$$(ds)^2 = g_1^2(\xi, \eta) (d\xi)^2 + g_2^2(\xi, \eta) (d\eta)^2,$$

where the restriction of the (ξ, η) net to a plane requires that g_1 and g_2 satisfy Gauss's Equation

$$\frac{\partial}{\partial \xi} \left[\frac{1}{g_1} \frac{\partial g_2}{\partial \xi} \right] + \frac{\partial}{\partial \eta} \left[\frac{1}{g_2} \frac{\partial g_1}{\partial \eta} \right] = 0. \quad (69)$$

Referred to these coordinates (67) becomes

$$\frac{\partial}{\partial \eta} [P(p)Wg_1] = 0, \quad (70)$$

and (68) becomes

$$\frac{\partial}{\partial \eta} \left[\frac{W^2}{1 - W^2} \frac{\partial}{\partial \xi} \ln g_2 W \right] = 0. \quad (71)$$

Equations (69), (70), and (71) constitute a set of necessary and sufficient conditions on the reduced velocity function $W(\xi, \eta)$ and the two metric functions $g_1(\xi, \eta)$ and $g_2(\xi, \eta)$ [10].

* The special case of a perfect gas is considered in [10].

We shall first employ the formulation (69)–(71) to determine for a gas obeying a product equation of state what plane flows which have a constant reduced velocity magnitude (and hence by (28) a constant actual velocity magnitude and by (63) a constant pressure) along each individual streamline are possible. To set this problem we introduce the condition $W = W(\xi)$ (i.e., $\partial W/\partial \eta = 0$) into (70) and (71) and obtain, making use of (23) to infer $\partial p/\partial \eta = 0$,

$$\frac{\partial g_1}{\partial \eta} = 0, \quad \frac{\partial^2 \ln g_2}{\partial \xi \partial \eta} = 0. \quad (72), (73)$$

We satisfy these two conditions by setting

$$g_1 = g_1(\xi), \quad g_2 = b(\xi)c(\eta), \quad (72a), (73a)$$

and we substitute into (69) to obtain

$$\left[\frac{b'(\xi)}{g_1(\xi)} \right]' c(\eta) = 0,$$

whence ($c(\eta) \neq 0$)

$$b(\xi) = A \int g_1(\xi) d\xi + B, \quad A = \text{const.}, B = \text{const.},$$

giving the following final permissible forms for g_1 and g_2 :

$$g_1 = g_1(\xi), \quad g_2 = \left[A \int g_1(\xi) d\xi + B \right] c(\eta). \quad (74)$$

To determine the geometric implications of the metric limitations (74) we make use of a technique employed by W. Tollmien [20] in a different connection. We introduce the ordinary complex variable $z = x + iy$ and note that

$$g_1 = \sqrt{\left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2} = \left| \frac{\partial z}{\partial \xi} \right|, \quad g_2 = \sqrt{\left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2} = \left| \frac{\partial z}{\partial \eta} \right|.$$

Hence, making use of the orthogonality of the (ξ, η) net we write

$$\frac{\partial z}{\partial \eta} = g_2 e^{i\theta}, \quad \frac{\partial z}{\partial \xi} = i g_1 e^{i\theta}, \quad (75)$$

where $\theta(\xi, \eta)$ is a real function (actually the local angle between the x -axis and the streamline tangent). By cross-differentiating (75) and separating real and imaginary parts we obtain the relations

$$\frac{\partial \theta}{\partial \xi} = \frac{1}{g_2} \frac{\partial g_1}{\partial \eta}, \quad \frac{\partial \theta}{\partial \eta} = - \frac{1}{g_1} \frac{\partial g_2}{\partial \xi}, \quad (76)$$

connecting the angle θ and the metric functions g_1 and g_2 . (It should be noted that Gauss's Equation (69) is obtained by cross-differentiation from (76) as the integrability condition for θ .)

Substitution of the expressions (74) for g_1 and g_2 into the right side of (76) yields

$$\theta = \theta(\eta), \quad \theta'(\eta) = -Ac(\eta). \quad (77)$$

From (75), (77), and (74) we now obtain for $A \neq 0$

$$z = i \left[\int g_1(\xi) d\xi + \frac{B}{A} \right] e^{i\theta(\eta)} + C + iD,$$

or

$$\begin{aligned} x &= C - \left[\int g_1(\xi) d\xi + \frac{B}{A} \right] \sin \theta(\eta), \\ y &= D + \left[\int g_1(\xi) d\xi + \frac{B}{A} \right] \cos \theta(\eta), \end{aligned} \quad (78)$$

where C and D are real constants. From (78) follows

$$(x - C)^2 + (y - D)^2 = \left[\int g_1(\xi) d\xi + \frac{B}{A} \right]^2, \quad \frac{y - D}{x - C} = \cot \theta(\eta),$$

from which it is seen that the lines $\xi = \text{const.}$ form a set of concentric circles. For the limiting case $A = 0$, θ is a constant and the set of circles becomes a family of parallel straight lines. The arbitrary functions $g_1(\xi)$ and $\theta(\eta)$ only establish the parametrization of the circles $\xi = \text{const.}$ and the radii $\eta = \text{const.}$

Thus we have shown that for gases obeying a product equation of state the only plane flow fields in which the velocity magnitude is a constant along each streamline are the general vortex flow and its limiting case of flow in parallel straight lines.

In preparation for the demonstration of other peculiar properties of this geometric family of flows we note that (23) and (65) together lead to an equation for the reduced vorticity Ω of the form

$$\Omega = f_1(\xi)f_2(\xi, W). \quad (79)$$

We also derive from the vector identity

$$\text{curl } \mathbf{V} = \text{curl } a \mathbf{W} = a \text{curl } \mathbf{W} + \text{grad } a \times \mathbf{W} \quad (80)$$

the relation

$$\omega = a\Omega + \frac{W}{g_1} \frac{da}{d\xi}, \quad (81)$$

connecting the reduced vorticity Ω and the actual vorticity ω . We finally note that (70) together with (23) leads to an equation for g_1 of the form

$$g_1 = g_1(\xi, W). \quad (82)$$

From (79) it is immediately apparent that when $\Omega \neq 0$ the reduced vorticity Ω is constant along each streamline if and only if the velocity is constant along each streamline. From (81) and (82) the same result follows for the actual vorticity ω .

From (79) it is also seen that the lines of constant reduced vorticity and reduced velocity coincide if and only if the velocity is constant along each streamline. From (81) and (82) the same result follows for the coincidence of the lines of constant reduced velocity and actual vorticity. The same condition is also evident for the coincidence of the lines of constant actual velocity and either reduced or actual vorticity ($V = a(\xi)W$).

Section C. Plane Flows with Straight Streamlines. Among the familiar flow patterns of plane gas flow are two groups, the straight parallel flows and the source flows, which have straight streamlines. The question presents itself whether these are the only plane flows with straight streamlines. We shall answer this question in the affirmative for flows of a perfect gas.

We shall employ the natural coordinate formulation of Section B. However in place of (67) we use the more special form of the continuity equation (53) valid for perfect gases, leading instead of to (70) to

$$\frac{\partial}{\partial \eta} [g_1 (1 - W^2)^{1/(\gamma-1)} W] = 0. \quad (83)$$

A governing set of equations for the plane flow of perfect gases is then (69), (71), and (83).

For the flow to have straight streamlines, the streamline inclination angle θ must not vary along each streamline. That is, $\partial\theta/\partial\eta = 0$, which condition by (76) implies $\partial g_2/\partial\xi = 0$, or

$$g_2 = g_2(\eta). \quad (84)$$

The conditions (69) and (84) require that

$$\frac{\partial g_1}{\partial \eta} = g_2(\eta)b(\xi) \quad (85)$$

(b a suitably differentiable arbitrary function). Similarly (83) requires

$$g_1 = c(\xi)m(W), \quad (86)$$

while (71) and (84) require

$$\frac{\partial W}{\partial \xi} = d(\xi)n(W), \quad (87)$$

where

$$m(W) \equiv W^{-1}(1 - W^2)^{-1/(\gamma-1)}, \quad n(W) \equiv W^{-1}(1 - W^2).$$

Substitution of (86) into (85) yields

$$\frac{\partial W}{\partial \eta} = \frac{b(\xi)}{c(\xi)} g_2(\eta) [m'(W)]^{-1}. \quad (88)$$

Equating the derivative of (87) with respect to η and the derivative of (88) with respect to ξ and making use of (87) and (88) to eliminate $\partial W/\partial \xi$ and $\partial W/\partial \eta$ from the resulting expression, we obtain

$$d(\xi) g_2(\eta) \left\{ [\ln n(W) m'(W)]' - \left[\ln \frac{b(\xi)}{c(\xi)} \right]' \right\} = 0. \quad (89)$$

Since the degenerate case $g_2 = 0$ is of no significance, (89) implies either

$$[\ln n(W) m'(W)]' - \left[\ln \frac{b(\xi)}{c(\xi)} \right]' = 0 \quad \text{or} \quad d(\xi) = 0.$$

The first condition implies $W = W(\xi)$, and hence, by the analysis of Section B, flow in parallel lines. The second condition, by (87), leads to $W = W(\eta)$ and hence by (86) to

$$g_1 = c(\xi) f(\eta), \quad (90a)$$

where $f(\eta) \equiv m[W(\eta)]$. The geometric implications of the conditions (84) and (90) are immediately inferred from the analysis of the converse conditions (72a) and (73a), namely that the streamlines must be a family of radial lines or of parallel lines.

It is thus established that *the only plane flow fields of a perfect gas having straight streamlines (even locally) are the simple source flows and flow in parallel lines.*

Section D. Plane Flows Having an Isometric Streamline Pattern.* We now inquire what plane flow patterns of a perfect gas which have the property that the streamlines and their orthogonal trajectories form an isometric net are possible. This is equivalent to asking what plane flow patterns can serve either for a flow (rotational or not) of a perfect gas or for an irrotational flow of an ideal incompressible fluid. Familiar flow fields having this property are those of parallel straight flow, the simple vortex flows (flows with concentric circular streamlines), and the simple source flows (flows with radial straight streamlines). We shall show that these are the only flow fields having the property under consideration.

* [16]. The special case of irrotational flow is treated by D. GILBARG [3].

We use the formulation (69), (71), and (83) of the governing equation in terms of natural coordinates. The condition for isometry of the (ξ, η) net is

$$g_1 = g_2 = g. \quad (90b)$$

Substitution of this condition into (71) and (83) leads to

$$\frac{W^2}{1 - W^2} \frac{\partial}{\partial \xi} \ln gW = b(\xi), \quad gW(1 - W^2)^{1/(\gamma-1)} = e^{c(\xi)} \quad (91), (92)$$

wherein $b(\xi)$ and $c(\xi)$ are suitably differentiable arbitrary functions. Equation (71) becomes

$$\frac{\partial^2 \ln g}{\partial \xi^2} + \frac{\partial^2 \ln g}{\partial \eta^2} = 0. \quad (93)$$

Substitution of gW from (92) into (91) gives

$$\frac{\partial W^2}{\partial \xi} = (\gamma - 1) \frac{1 - W^2}{W^2} \{b(\xi) - [b(\xi) + c'(\xi)]W^2\} \equiv m(W^2, \xi). \quad (94)$$

Also, substitution of g from (92) into (93) gives

$$\frac{\partial^2 W^2}{\partial \xi^2} + \frac{\partial^2 W^2}{\partial \eta^2} = f(W^2) \left[\left(\frac{\partial W^2}{\partial \xi} \right)^2 + \left(\frac{\partial W^2}{\partial \eta} \right)^2 \right] + h(W^2)c''(\xi) \quad (95)$$

where

$$f(W^2) \equiv \frac{\frac{\gamma+1}{\gamma-1} W^4 - 2W^2 + 1}{W^2(1 - W^2) \left(1 - \frac{\gamma+1}{\gamma-1} W^2 \right)}, \quad h(W^2) \equiv \frac{2W^2(1 - W^2)}{1 - \frac{\gamma+1}{\gamma-1} W^2}. \quad (96), (97)$$

We now use (94) to eliminate $\partial W^2/\partial \xi$ and $\partial^2 W^2/\partial \xi^2$ from (95) and obtain

$$\frac{\partial^2 W^2}{\partial \eta^2} = f(W^2) \left[m^2 + \left(\frac{\partial W^2}{\partial \eta} \right)^2 \right] - m \frac{\partial m}{\partial W^2} - \frac{\partial m}{\partial \xi} + h(W^2)c''(\xi). \quad (98)$$

By differentiating (94) twice with respect to η and (98) once with respect to ξ , using (94) and (98) to eliminate $\partial W^2/\partial \xi$ and $\partial^2 W^2/\partial \eta^2$ from the resulting expressions and setting them equal, we finally obtain

$$\begin{aligned} \left(\frac{\partial W^2}{\partial \eta} \right)^2 &= \left[m^2 \frac{\partial^2 m}{\partial W^2} + 2m \frac{\partial^2 m}{\partial \xi \partial W^2} + \frac{\partial^2 m}{\partial \xi^2} - f'(W^2)m^3 \right. \\ &\quad - f(W^2)m \left(2 \frac{\partial m}{\partial \xi} + m \frac{\partial m}{\partial W^2} \right) + h(W^2) \frac{\partial m}{\partial W^2} c''(\xi) - mh'(W^2)c''(\xi) \\ &\quad \left. - h(W^2)c'''(\xi) \right] \cdot \left[- \frac{\partial^2 m}{\partial W^2} + f(W^2) \frac{\partial m}{\partial W^2} + f'(W^2)m \right]^{-1} \equiv l^2(W^2, \xi). \end{aligned} \quad (99)$$

Two cases must be considered.

Case I: $m(W^2, \xi) = 0$. From (94) (excluding the degenerate case $W^2 = 1$), $m = 0$ implies $b(\xi) = c'(\xi) = 0$, and $W = W(\eta)$. (92) then gives $g = g(\eta)$ and (76) gives $\theta = \theta(\xi)$. By the analysis of Section C this is consistent only with radial or parallel straight streamlines.

Case II: $m(W^2, \xi) \neq 0$. Eq. (99) now determines $\partial W^2 / \partial \eta$ as a function of W^2 and ξ :

$$\frac{\partial W^2}{\partial \eta} = l(W^2, \xi). \quad (100)$$

Differentiation of (94) with respect to η and of (100) with respect to ξ , elimination of $\partial W^2 / \partial \xi$ and $\partial W^2 / \partial \eta$ from the resulting expressions by use of (94) and (100), and equating the resulting relations leads to

$$\frac{\partial l(W^2, \xi)}{\partial W^2} m(W^2, \xi) + \frac{\partial l(W^2, \xi)}{\partial \xi} = \frac{\partial m(W^2, \xi)}{\partial W^2} l(W^2, \xi), \quad (101)$$

which implies $W = W(\xi)$. The analysis of Section B shows that the only possible flow patterns consistent with this condition are those having concentric circles or parallel straight lines for streamlines.

Hence, *the only possible isometric streamline patterns for plane flows of a perfect gas consist of concentric circles, radial straight lines, and parallel straight lines.*

Section E. Generalized Prandtl-Meyer Corner Flows.* The simple Prandtl-Meyer corner flow solution which plays a basic role in the study of irrotational flow of a perfect gas can be obtained by seeking solutions to the differential equations governing irrotational gas flow having the property that the velocity components in a polar coordinate system are functions of the polar angle alone. We shall generalize this investigation by seeking solutions of the rotational flow equations having a similar property. We shall find that the irrotational Prandtl-Meyer flow solution is a singular member of a doubly-infinite family of flows having the property under consideration.

Specifically we will seek plane reduced velocity fields satisfying (35) and (53) and having the property that $\mathbf{W} = \mathbf{W}(\theta)$ in a plane polar coordinate system (r, θ) . We designate the *radial* component of $\mathbf{W}$ by u and the *tangential* component by v . Thus we have

$$\mathbf{W} = \mathbf{V}_1 u(\theta) + \mathbf{e}_1 v(\theta) \quad (102)$$

where $\mathbf{V}_1$ and $\mathbf{e}_1$ are unit vectors in the radial and tangential directions respectively.

* The family of flows derived here is the first infinite set of strongly rotational gas flow solutions to be exhibited in explicit formulae [17]. This investigation was reported at the VII International Congress of Applied Mechanics, London, 1948.

Substitution of (102) into (35) and (53) yields the following two conditions limiting the functions $u(\theta)$ and $v(\theta)$:

$$\left[\frac{v(v - u')}{1 - u^2 - v^2} \right]' = 0, \quad u + v' + v[\ln(1 - u^2 - v^2)]^{1/(\gamma-1)}' = 0, \quad (103), (104)$$

wherein (as throughout this Section) primes denote differentiation with respect to θ .

We satisfy the condition (103) by setting

$$\frac{v(v - u')}{1 - u^2 - v^2} = B \quad (105)$$

where B is a constant parameter determining, in a sense, the degree of rotationality of the flow. Since the reduced vorticity (readily computed from (102)) is given by

$$\Omega = \frac{v - u'}{r} = \frac{B(1 - u^2 - v^2)}{v} \quad (106)$$

the flow is irrotational if and only if $B = 0$. (We neglect as usual the degenerate case of flow fields with $u^2 + v^2 = W^2 = 1$ throughout as being of no physical interest. Such flow fields have either zero pressure or infinite stagnation pressure everywhere (42)). Solving (105) for u' , eliminating u' from (104), and solving the resulting expression for v' we obtain the following pair of ordinary differential equations restricting the functions $u(\theta)$ and $v(\theta)$:

$$\frac{du}{d\theta} = -Bv \left[\frac{1 - u^2}{v^2} - \frac{B + 1}{B} \right], \quad (107)$$

$$\frac{dv}{d\theta} = -[(\lambda - 1)B + 1]u \left[\frac{\frac{1 - u^2}{v^2} - \frac{(\lambda - 1)B + 1}{(\lambda - 1)B + 1}}{\frac{1 - u^2}{v^2} - \lambda} \right] \quad (108)$$

where $\lambda \equiv (\gamma + 1)/(\gamma - 1)$. For air $\lambda = 6$.

Our problem is now reduced to the study of the solutions of (107) and (108) for $-\infty < B < \infty$ and $0 < u^2 + v^2 < 1$. We may also restrict our attention to positive values of v without loss of generality.

It is convenient to transfer equations (107) and (108) to a polar-hodograph plane in which v is the dependent variable and u the independent variable. We then obtain the non-linear first order equation:

$$\frac{dv}{du} = \frac{(\lambda - 1)B + 1}{B} \frac{u}{v} \left[\frac{\frac{1 - u^2}{v^2} - \frac{(\lambda - 1)B + 1}{(\lambda - 1)B + 1}}{\left(\frac{1 - u^2}{v^2} - \lambda \right) \left(\frac{1 - u^2}{v^2} - \frac{B + 1}{B} \right)} \right]. \quad (109)$$

We now introduce the new independent variable S defined by

$$S = \frac{1 - u^2}{v^2}. \quad (110)$$

Equation (109) becomes

$$\frac{d \ln v^2}{dS} = -\frac{(\lambda - 1)B + 1}{\lambda B + 1} \left[\frac{S - \frac{(\lambda - 1)B + \lambda}{(\lambda - 1)B + 1}}{[S - 1] \left[S - \frac{\lambda(B + 1)}{\lambda B + 1} \right]} \right]. \quad (111)$$

We now set aside for later attention the cases where one or more of the following conditions hold: $\lambda B + 1 = 0$, $S = 1$, and $S = \lambda(B + 1)/(\lambda B + 1)$. Ruling out these cases, we can integrate (111) and obtain

$$v^2 = C(S - 1)^{-1} \left[S - \frac{\lambda(B + 1)}{\lambda B + 1} \right]^{B/(\lambda B + 1)}, \quad (112)$$

where C is an integration constant. (110) and (112) then yield

$$u^2 = 1 - CS(S - 1)^{-1} \left[S - \frac{\lambda(B + 1)}{\lambda B + 1} \right]^{B/(\lambda B + 1)}. \quad (113)$$

Now, making use of (107), (112) and (113), we obtain a quadrature for the determination of θ as a function of the auxiliary variable S :

$$\theta = \frac{1}{2(\lambda B + 1)} \int_{s_0}^S \frac{[\sigma - \lambda] d\sigma}{[\sigma - 1] \left[\sigma - \frac{\lambda(B + 1)}{\lambda B + 1} \right] \left\{ C^{-1}[\sigma - 1] \left[\sigma - \frac{\lambda(B + 1)}{\lambda B + 1} \right]^{-B/(\lambda B + 1)} - \sigma \right\}^{\frac{1}{2}}}. \quad (114)$$

We introduce the streamline radius $r(\theta)$ by the defining relation

$$\frac{dr}{r d\theta} = \frac{u}{v}. \quad (115)$$

Rewriting (115) as

$$\frac{d \ln r}{dS} = \frac{1}{2} \frac{\left(\frac{d \ln v^2}{dS} \right)}{\left(\frac{1}{u} \frac{dv}{d\theta} \right)},$$

we make use of (108) and (111) and obtain

$$r^2 = D(S - 1) \left[S - \frac{\lambda(B + 1)}{\lambda B + 1} \right]^{-\lambda B / (\lambda B + 1)}, \quad (116)$$

where D is an integration constant.

Equations (112), (113), (114), and (116) constitute a complete parametric solution to (10) and (11) except for the cases set aside for special consideration.

The case $S = 1$ is of no interest. From (110) it is seen that this condition implies $W^2 = 1$ throughout the flow. Hence the pressure must be everywhere zero or the stagnation pressure infinite.

The case $\lambda B + 1 = 0$ leads to a formally different set of equations paralleling (112)–(116) for this particular value of B ($B = -1/\lambda$):

$$v^2 = C'(S - 1)^{-1} \exp \left[\frac{S}{\lambda(\lambda - 1)} \right], \quad (112')$$

$$u^2 = 1 - C'S(S - 1)^{-1} \exp \left[\frac{S}{\lambda(\lambda - 1)} \right], \quad (113')$$

$$\theta = \frac{1}{2(\lambda + 1)} \int_{s_0}^S \frac{[\sigma + \lambda] d\sigma}{[\sigma - 1] \left[C'^{-1}(\sigma - 1) \exp \left[-\frac{\sigma}{\lambda(\lambda - 1)} \right] - \sigma \right]}, \quad (114')$$

$$r^2 = D'(S - 1) \exp \left[\frac{S}{\lambda + 1} \right]. \quad (116')$$

The third special case $S = \lambda(B + 1)/(\lambda B + 1)$ leads to particularly interesting results. In this case (111) shows that $dS/dv = 0$. Hence

$$S \equiv \frac{1 - u^2}{v^2} = \frac{\lambda(B + 1)}{\lambda B + 1} \quad (117)$$

constitutes a special solution of the polar-hodograph equation. Substitution of (117) into (107) and (108) permits the direct integration of these equations to obtain u and v as functions of θ :

$$v = v_0 \cos \frac{\theta}{\lambda v_0}, \quad u = \sin \frac{\theta}{\lambda v_0}, \quad (118), (119)$$

where, for convenience, B has been displaced by $v_0(B)$ defined by

$$v_0 \equiv \sqrt{\frac{\lambda B + 1}{\lambda(B + 1)}}. \quad (120)$$

The integration constants have been so chosen that $u = 0$, $v = v_0$ at $\theta = 0$.

It will be noted that v_0 varies from zero to one as B is changed from $-1/\lambda$ to $+\infty$. For $-1 \leq B < -1/\lambda$ there exists no real v_0 and for $B < -1$ v_0 is real but greater than one, and hence physically meaningless.

Upon substitution of (118) and (119) into the differential equation defining the streamlines:

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{u}{v}, \quad (121)$$

we obtain by a simple integration the equation for the streamlines

$$r = R \left(\sec \frac{\theta}{\lambda v_0} \right)^\lambda, \quad (122)$$

where R is a parameter (the value of r at $\theta = 0$) singling out an individual streamline. It is evident that the radii along any two streamlines (v_0 fixed) stand in a constant ratio.

Substituting (118) and (119) into (106) we obtain an equation for the vorticity distribution in these flow fields:

$$\Omega = \frac{v_0^2 - \frac{1}{\lambda} \cos \frac{\theta}{\lambda v_0}}{v_0} \frac{1}{r}. \quad (123)$$

It is evident that $\Omega = 0$ if and only if $v_0^2 = 1/\lambda$ (by (58) $M_0 = 1$). This value of v_0 is associated (120) with the choice $B = 0$, yielding the irrotational Prandtl-Meyer flow.

We obtain also a simple expression for the pressure distribution by integration from (52):

$$p = P r^{(\lambda+1)(v_0^2-1/\lambda)/(1-v_0^2)} \left[\cos \frac{\theta}{\lambda v_0} \right]^{(\lambda^2-1)v_0^2/(1-v_0^2)} \quad (124)$$

where P is an arbitrary constant reference pressure. It will be observed that the nature of the pressure distribution is quite different depending on whether the velocity at $\theta = 0$ is subsonic, sonic, or supersonic (that is, whether $v_0^2 < 1/\lambda$, $= 1/\lambda$, or $> 1/\lambda$). For $v_0^2 < 1/\lambda$, the pressure falls off monotonely with r from a singular point of infinite pressure at $r = 0$. For $v_0^2 > 1/\lambda$, the pressure increases monotonely with r from a zero value at $r = 0$. For $v_0^2 = 1/\lambda$, the pressure is constant along radial lines, a familiar property of this irrotational case.

From (118) and (119) we have

$$W^2 = u^2 + v^2 = 1 - (1 - v_0^2) \cos^2 \frac{\theta}{\lambda v_0}, \quad (125)$$

which with (58) gives

$$M^2 = \frac{(\lambda - 1) \left(v_0^2 + \tan^2 \frac{\theta}{\lambda v_0} \right)}{1 - v_0^2} = M_0^2 \left(1 + \frac{1}{v_0^2} \tan^2 \frac{\theta}{\lambda v_0} \right), \quad (126)$$

where M_0 denotes the Mach number at $\theta = 0$ ($M_0^2 = (\lambda - 1)v_0^2/(1 - v_0^2)$).

It will be observed from the formulae (118)–(126) that each member of this family of flows has “limiting angles”

$$\pm\theta = \theta_{\text{lim}} = \frac{\pi}{2} \lambda v_0, \quad (127)$$

at which the streamlines become radial and the velocity becomes ultimate ($W = 1$). This property of the family is a direct generalization of the same phenomenon in the special case of the irrotational Prandtl-Meyer flow where the limiting angle is $(\pi/2)\sqrt{\lambda}$ (obtainable from (127) by setting $v_0 = 1/\sqrt{\lambda}$). By choosing values of the parameter v_0 in the permissible range of zero to one, flow patterns can be obtained in which the limiting angle has any value from zero to $(\pi/2)\lambda$.

Another familiar property of the Prandtl-Meyer flow is the equality throughout the flow field of the local values of the velocity of sound and the tangential component of flow velocity. This property is also shared, in a somewhat more general form, by the rotational members of this family of flows. The squared velocity of sound, referred to the ultimate velocity as unit velocity, is given by

$$c^2 = \frac{W^2}{M^2} = \frac{1 - W^2}{\lambda - 1}. \quad (128)$$

From (118), (119), and (128) we then have

$$\frac{v}{c} = \left[\frac{(\lambda - 1)v_0^2}{1 - v_0^2} \right]^{\frac{1}{2}} = M_0. \quad (129)$$

That is, throughout the flow field of any particular member of the family of flows under consideration there is a constant ratio between the local velocity of sound and the tangential component of flow velocity, the ratio of the quantities being the initial (*i.e.*, minimum) Mach number. For the irrotational case this Mach number is unity, so the familiar ratio of one is obtained. For the rotational flows of the family, however, this ratio can be anything from zero to infinity.

It will be observed that the pressure at a given radius falls from a maximum value at $\theta = 0$ to zero at the limiting angles (the velocity there is ultimate). On the other hand, the Mach number increases with increasing $|\theta|$ from a minimum value (M_0) at $\theta = 0$ to infinity at the limiting angles. Those flows for which $v_0^2 > 1/\lambda$ ($M_0 > 1$) are supersonic throughout, while those for which

$v_0^2 < 1/\lambda$ contain also a subsonic region $|\theta| < |\arccos \sqrt{\lambda(1 - v_0^2)/(\lambda - 1)}|$. That is, for $v_0^2 \leq 1/\lambda$, $M = 1$ at

$$\theta = \arccos \sqrt{\frac{\lambda(1 - v_0^2)}{\lambda - 1}}. \quad (130)$$

Another familiar and useful property of the irrotational Prandtl-Meyer flow is that the radial lines $\theta = \text{const.}$ are characteristics of the solution. That is, they stand at an angle $\arcsin 1/M$ to each streamline [7]. Therefore this solution can be patched to other solutions along the straight radial lines to build up composite solutions. However, it is readily shown that the radial lines are characteristics *only* in the irrotational case. Letting ν denote the angle between streamline and radius, we have $\nu = \arcsin v/W$. But from (129) $v = cM_0$ for the family being studied, so

$$\nu = \arcsin \frac{cM_0}{W} = \arcsin \frac{M_0}{M},$$

and clearly $\nu = \arcsin 1/M$ if and only if $M_0 = 1$; i.e., $\Omega = 0$.

The complete solutions of (107) and (108) constitute a family involving two essential parameters. (The third parameter merely establishes the orientation of the line $\theta = 0$.) The first parameter is B , which establishes through (106) the degree and sign of rotationality of the solution. After B has been chosen, the second parameter singles out a particular solution trajectory in the u - v plane. It will be convenient to take as this parameter v_0 , the value of v when $u = 0$.

In order to determine and classify the different types of solutions contained in this doubly-infinite family, we shall investigate the topography of the solution trajectories in the u - v plane and the effect of the choice of B on this topography.

Inspection of (109) and (111) shows that the topography of the solution trajectories in the u - v plane is substantially determined by the lines of zero slope $u = 0$ and $S = [(\lambda - 1)B + \lambda]/[(\lambda - 1)B + 1]$; the lines of infinite slope $v = 0$, $S = \lambda$, and $S = (B + 1)/B$; and the special solutions $S = 1$ and $S = \lambda(B + 1)/(\lambda B + 1)$. (In the upper half-plane which we are considering, the lines $S = \text{const.}$ are a family of semi-ellipses through the points $(u = \pm 1, v = 0)$ and $(u = 0, v = 1/\sqrt{S})$. It can be easily verified that these lines are lines of constant ratio of tangential velocity component to sonic velocity ($v^2/c^2 = (\lambda - 1)/(S - 1)$)). The lines $u = 0$, $v = 0$, $S = \lambda$, and $S = 1$ do not change their positions as B varies, but the other three ellipses move across the plane as B is changed from $-\infty$ to $+\infty$. The values of B for which they coincide with one another or with the ellipse $S = \lambda$, or for which they move into or out of the region $v > 0$, $u^2 + v^2 < 1$, determine the ranges of values of B for which the topography of the solution trajectories is essentially the same.

Seven qualitatively different types of pattern of the solution trajectories arise as $-\infty < B \leq -1/(\lambda - 1)$, $-1/(\lambda - 1) < B \leq -1/\lambda$, $-1/\lambda < B < 0$, $B = 0$,

$0 < B < 1/(\lambda - 1)$, $B = 1/(\lambda - 1)$, or $1/(\lambda - 1) < B < \infty$. The patterns characteristic of these seven regimes are indicated by Figures (2) – (8). These figures* are intended to be only qualitatively accurate. They furnish a basis for a broad classification of the solutions and indicate the values to be assigned to the parameters B and v_0 in order to single out a solution of a particular type. The arrowheads on the trajectories indicate the direction of change of the solution with increasing θ . Points on a trajectory indicated by an “X” are points at which $du/d\theta$ and $dv/d\theta$ are indeterminate. Points indicated by an “O” are stationary points of the solution, *i.e.* points at which $du/d\theta = dv/d\theta = 0$.

FIGURE 2. $-\infty < B \leq -(1/(\lambda - 1))$

In Figure 2 ($-\infty < B \leq -1/(\lambda - 1)$) the pattern of solution trajectories is quite simple, as only the fixed lines $u = 0$, $v = 0$, $S = \lambda$, and $S = 1$ influence the pattern. It will be noted that each trajectory is double-branched. The points at which the two branches appear to enter or emerge from the line $S = \lambda$, will be associated with a *limiting line* at which the streamline pattern foldover and recovers the plane.

In Figure 3 ($-1/(\lambda - 1) < B \leq -1/\lambda$) the pattern differs from that of Figure 2 because of the appearance within the semi-circle bounded by $v = 0$ and $S = 1$ of the line $S = [(\lambda - 1)B + \lambda]/[(\lambda - 1)B + 1]$ on which $dv/du = 0$. The general behavior of the solution trajectories is, however, the same as that of the *basic limiting line* solutions in Figure 2.

In Figure 4 ($-1/\lambda < B < 0$) the pattern is further modified by the entry (through $v = 0$) of the *special solution* ellipse $S = \lambda(B + 1)/(\lambda B + 1)$. In the

* The author is indebted to Miss GLADYS FULTON for assistance in computation and figure preparation for this section.

portion of the plane above this ellipse the trajectories are of the same *basic limiting line* type appearing in Figures 2 and 3. Below this ellipse, the solutions are similar in nature to the *special solutions*.

FIGURE 3. $-1/(\lambda - 1) < B \leq -1/\lambda$

FIGURE 4. $-1/\lambda < B < 0$

In Figure 5 ($B = 0$) the three ellipses $S = \lambda$, $S = [(\lambda - 1)B + \lambda]/[(\lambda - 1)B + 1]$ and $S = \lambda(B + 1)/(\lambda B + 1)$ coincide to produce a special solution ellipse (the Prandtl-Meyer flow), and at the same time the ellipse $S = (B + 1)/B$ for which $dv/du = \infty$ more than cancels the effect of the zero-slope line $v = 0$. The

semi-circular trajectories are associated with uniform parallel flows in the physical plane.

FIGURE 5. $B = 0$

FIGURE 6. $0 < B < 1/(\lambda - 1)$

In Figure 6 ($0 < B < 1/(\lambda - 1)$) the *special solution* ellipse $S = \lambda(B + 1)/(\lambda B + 1)$ and the zero-slope ellipse $S = [(\lambda - 1)B + \lambda]/[(\lambda - 1)B + 1]$ have emerged in the order named from the ellipse $S = \lambda$. Also the infinite-slope ellipse $S = (B + 1)/B$ has moved from the line $v = 0$ into the region below $S = \lambda$. The point of intersection of the ellipse $S = (B + 1)/B$ with the v -axis is a

stationary point of the solution, associated with a constant-velocity vortex in the physical plane. Around this point and inside the lower loop of the trajectory through $(u = 0, v = 1/\sqrt{\lambda})$ is a family of closed trajectories without limiting lines. The nature of these *perturbed vortex* solutions is clarified by finding the solutions to (107), (108), and (115) valid very near the vortex point $(u = 0, v = \sqrt{B/(B + 1)})$. The resulting approximate solutions are

$$u = u_0 \sin \sqrt{\frac{2(B + 1)}{1 - (\lambda - 1)B}} \theta, \quad (131)$$

$$v = \sqrt{\frac{B}{B + 1}} + \frac{u_0}{\sqrt{2(B + 1)[1 - (\lambda - 1)B]}} \cos \sqrt{\frac{2(B + 1)}{1 - (\lambda - 1)B}} \theta, \quad (132)$$

$$\frac{r}{r_0} = 1 - \sqrt{\frac{1 - (\lambda - 1)B}{2B}} u_0 \cos \sqrt{\frac{2(B + 1)}{1 - (\lambda - 1)B}} \theta. \quad (133)$$

These relations show that in the vicinity of the vortex point the solution trajectories are ellipses with a ratio between axes changing from $\sqrt{2}$ to 0 as B varies from 0 to $1/(\lambda - 1)$. The streamline pattern in the physical plane consists of a sinusoidal perturbation superimposed on the concentric circles of the basic vortex. The period of this perturbation depends on B and changes from $\sqrt{2}\pi$ to 0 (radians) as B varies from 0 to $1/(\lambda - 1)$. For suitable choices of B these wavy streamlines form a closed pattern. The perturbed vortex flows for which the velocity variations are not extremely small are similar in nature to the limiting case. The streamline radius and velocity variation as functions of θ are distorted from the limiting sinusoidal shapes and the period of these variations for a fixed B is changed somewhat, but the qualitative behavior of the flows is the same. The solutions whose trajectories lie in that part of the u - v plane above the upper branches of the trajectory through $(u = 0, v = 1/\sqrt{\lambda})$ are similar in type to the special solution lying in their midst. The trajectories in the third part of the plane are of a different type, a *mixed* three-branched type with limiting lines. Two of the branches yield solutions similar to the special solutions, while the third is similar to the basic limiting line solutions occurring in Figures 2 and 3. The singular trajectory through $u = 0, v = 1/\sqrt{\lambda}$ is associated with a three-branched solution, two branches of which resemble the special solutions and the third the perturbed vortex solutions.

In Figure 7 ($B = 1/(\lambda - 1)$) the ellipses $S = \lambda$ and $S = (B + 1)/B$ coincide. The point $v = 1/\sqrt{\lambda}$ represents a simple vortex solution, but the family of perturbed vortex solutions has disappeared. The singular cusped trajectory at $v = 1/\sqrt{\lambda}$ is associated with a two-branched *spiral vortex* solution resembling the special solutions, but asymptotically emerging from or entering a circular vortex streamline pattern. Those trajectories for which $v_0 < 1/\sqrt{\lambda}$ are of the mixed type appearing in Figure 5, while those with $v_0 > 1/\sqrt{\lambda}$ are similar to the special solutions.

In Figure 8 ($1/(\lambda - 1) < B < \infty$) the infinite-slope ellipse $S = (B + 1)/B$ has moved into the space between $S = \lambda$ and the zero-slope ellipse

$$S = [(\lambda - 1)B + \lambda]/[(\lambda - 1)B + 1].$$

FIGURE 7. $B = 1/(\lambda - 1)$

FIGURE 8. $1/(\lambda - 1) < B < \infty$

The singular multi-branched solution through the stationary vortex point $u = 0, v = \sqrt{B/(B + 1)}$, has branches of the spiral vortex type and of a *spiral limiting line* type, which has limiting lines resembling those of the basic limiting line solutions, but asymptotically emerges from or enters a circular vortex stream-

line pattern. The trajectories lying within the lower lobe of this singular trajectory are of the basic limiting line type. The remaining trajectories lying below the singular trajectory are of the mixed type appearing in Figures 6 and 7. The trajectories above the singular trajectory are again similar to the special solutions.

The information derived from Figures 2-8 concerning the dependence of the general type of the solution on the parameters B and v_0 is summarized graphically in Figure 9. The lines O , S , and V in this diagram show the change in location with B respectively of the three ellipses $S = [(\lambda - 1)B + \lambda]/[(\lambda - 1)B + 1]$, $S = \lambda(B + 1)/(\lambda B + 1)$, and $S = (B + 1)/B$ whose location establishes the to-

FIGURE 9

pography of the solution trajectories in the u - v plane. Choices of B and v_0 corresponding to points on S yield one of the special solutions given in (118) and (119). Points lying in the regions A_1 and A_2 correspond to flows of the basic limiting line type associated with Figure 2. The solutions associated with regions B_1 and B_2 are qualitatively similar to the special solutions. The line $B = 0$ is a singular one: the point at $v_0 = 1/\sqrt{\lambda}$ is associated with the irrotational Prandtl-Meyer corner flow, the other points on the line correspond to uniform parallel flows at reduced velocity v_0 . The line V is associated with constant-velocity vortex flows ($u = 0, v = v_0$). The point on V at $B = 1/(\lambda - 1)$ is also associated with the singular spiral vortex solution described in the discussion of Figure 7. The points on V and on Y for $B > 1/(\lambda - 1)$ are also associated with the spiral limiting line plus spiral vortex type of solution discussed for Figure 8. Points in the region C which do not fall on V are associated with the perturbed

vortex solutions discussed for Figure 6. On the lines X and $v_0 = 1/\sqrt{\lambda}$ bounding C are points associated with the singular three-branched solution which shares the properties of the perturbed vortex and special types. The points in region D are associated with the three-branched mixed solutions which have branches of the basic limiting line type and of the type resembling the special solutions.

It will be noted that the types of solution arising and their general dependence on the parameters B and v_0 does not differ *qualitatively* from one value of λ to another so long as $\lambda > 1$ (for all ideal gases $\lambda \geq 4$). Hence, solutions computed for a particular value of λ (in particular, $\lambda = 6$, the ideal value for air) are

FIGURE 10. Radial and Tangential Velocities Versus θ
 $B = -\frac{1}{2}$, $v_0 = 0.2$, $\lambda = 6$

qualitatively representative of the possible solutions for all physically significant values of λ . It is worthy of note, however, that the use of the well-exploited fictitious gas with $\gamma = -1$, giving $\lambda = 0$, leads to results which, even qualitatively, are completely different from those obtained for physically significant values of γ . In particular, note the disappearance in this case of the fundamental ellipse $S = \lambda$ from the flow region for which $u^2 + v^2 < 1$.

In Figures 10-23 the velocity distributions and streamline patterns are presented as computed for nine representative examples of the solutions discussed above. In all cases the computations were performed using $\lambda = 6$.

The solution presented in Figures 10 and 11 is an example of the basic limiting line type of solution occurring in regions A_1 and A_2 of Figure 9. The periodic pattern of limiting lines restricts the extent of any one branch of the solution to

FIG. 11

FIG. 13

FIGURE 11. Streamline Pattern
 $B = -\frac{1}{4}$, $v_0 = 0.2$, $\lambda = 6$

FIGURE 13. Streamline Pattern
 $B = \frac{1}{11}$, $v_0 = 0.2754$, $\lambda = 6$

FIGURE 12. Radial and Tangential Velocities Versus θ

$B = \frac{1}{11}$, $v_0 = 0.2754$, $\lambda = 6$

a rather small sector of the flow plane. It will be noted that the limiting lines always occur at points where the tangential velocity component is just sonic.

The solution shown in Figures 12 and 13 is of the mixed type occurring in region *D* of Figure 9. The sector between the two limiting lines is triply covered

FIG. 14

FIGURE 14. Radial and Tangential Velocities

$$B = \frac{1}{r_1}, \quad v_0 = 0.3, \quad \lambda = 6$$

FIGURE 15. Streamline Pattern

$$B = \frac{1}{r_1}, \quad v_0 = 0.3, \quad \lambda = 6$$

FIG. 15

FIG. 16

FIGURE 16. Radial and Tangential Velocities

$$B = \frac{1}{r_1}, \quad v_0 = 0.4, \quad \lambda = 6$$

FIGURE 17. Streamline Pattern

$$B = \frac{1}{r_1}, \quad v_0 = 0.4, \quad \lambda = 6$$

FIG. 17

by solutions—one resembling a branch of the basic limiting line solutions, and the other two resembling the special solutions. That part of the plane outside of this sector contains only the branches of the special solution type.

FIG. 18

FIGURE 18. Radial and Tangential Velocities

$$B = \frac{1}{2}, \quad v_0 = 0.4082, \quad \lambda = 6$$

FIGURE 19. Streamline Pattern

$$B = \frac{1}{2}, \quad v_0 = 0.4082, \quad \lambda = 6$$

FIG. 19

FIG. 20

FIGURE 20. Streamline Pattern

$$B = -\frac{3}{4}, \quad v_0 = 0.0833, \quad \lambda = 6$$

FIG. 21

FIGURE 21. Streamline Pattern

$$B = -\frac{1}{2}, \quad v_0 = 0.2778, \quad \lambda = 6$$

Figures 14 and 15 exhibit an example of the perturbed vortex solutions occurring in region *C* of Figure 9. The parameters in this case have been so chosen that the streamlines form a closed pattern and that the perturbation of the velocities from the basic vortex flow are small. These exact results agree quite closely with the approximate formulae eqs. (131)–(133). Figures 16 and 17 show another perturbed vortex solution for which the streamline pattern is not closed and for which the velocity perturbation is sufficiently large that the computed velocity distributions are noticeably different from those given by eqs. (131)–(133).

In Figures 18 and 19 is represented the singular spiral vortex solution associated with the point $B = 1/(\lambda - 1)$, $v_0 = 1/\sqrt{\lambda}$ in Figure 9. It consists of two

FIG. 22

FIG. 23

FIGURE 22. Streamline Pattern
 $B = -\frac{1}{16}$, $v_0 = 0.3333$, $\lambda = 6$

FIGURE 23. Streamline Pattern
 $B = \frac{1}{6}$, $v_0 = 0.5$, $\lambda = 6$

branches (only one is shown in the figures) of the special solution type spiraling into or out of a simple vortex flow.

Figures 20–23 exhibit the streamline patterns associated with four of the special solutions of eqs. (118)–(130). The velocity distributions for these special solutions (given by eqs. (118) and (119)) are sufficiently simple that graphs of them are not included.

The question naturally arises whether a generalization of the Prandtl-Meyer solution different from that just investigated can be found by seeking rotational flow fields in which the actual velocity vector $\mathbf{V}$, instead of the reduced velocity vector $\mathbf{W}$, is a function of θ alone. To settle this question we employ the formulation of the continuity equation (53) in terms of $\mathbf{V}$ and a :

$$\operatorname{div} [(a^2 - V^2)^{1/(\gamma-1)} \mathbf{V}] = 0, \quad \mathbf{V} \cdot \operatorname{grad} a = 0 \quad (134), (135)$$

and set

$$\mathbf{V} = \mathbf{V}(\theta) = \mathbf{V}_1 \mathfrak{U}(\theta) + \theta_1 \mathfrak{V}(\theta). \quad (136)$$

Expansion of (134) and use of (135) yields

$$(a^2 - V^2) \operatorname{div} \mathbf{V} - \frac{1}{\gamma - 1} \mathbf{V} \cdot \operatorname{grad} V^2 = 0. \quad (137)$$

Substitution of (136) in (137) leads to

$$a^2 = \frac{1}{\gamma - 1} \frac{\mathfrak{V}(\mathfrak{U}^2 + \mathfrak{V}^2)'}{\mathfrak{U} + \mathfrak{V}'} + \mathfrak{U}^2 + \mathfrak{V}^2. \quad (138)$$

If $(\mathfrak{U}^2 + \mathfrak{V}^2)'$ and $(\mathfrak{U} + \mathfrak{V}')$ are not simultaneously zero, (138) implies $a^2 = a^2(\theta)$, which with (136) in turn implies $\mathbf{W} = \mathbf{W}(\theta)$, the case already investigated. If $(\mathfrak{U}^2 + \mathfrak{V}^2)' = (\mathfrak{U} + \mathfrak{V}') = 0$, then $\mathfrak{V} = A \cos(\theta - \theta_0)$ and $\mathfrak{U} = A \sin(\theta - \theta_0)$, so that we have only the degenerate case of flow in parallel straight lines at uniform velocity magnitude A (a can be assigned, of course, an arbitrary value on each streamline).

Thus the generalization (102) contains all significant cases of the generalization (136).

Section F. Flow Fields for which the Reduced Velocity Field is of the Form $\mathbf{W} = iu(y) + jv_1(x)v_2(y)$.* We shall now study that class of flow fields of a perfect gas which has a reduced velocity field of the form

$$\mathbf{W}(x, y) = iu(y) + jv_1(x)v_2(y), \quad (139)$$

where i and j are the unit vectors in the x and y directions, respectively, in the ordinary rectangular cartesian coordinate system.

Substitution of (139) into the continuity equation (53) yields the condition:

$$u \frac{\partial}{\partial x} \ln(1 - u^2 - v_1^2 v_2^2) + v_1 v_2 \frac{\partial}{\partial y} \ln(1 - u^2 - v_1^2 v_2^2) + (\gamma - 1)v_1 v_2' = 0. \quad (140)$$

(Here and in the sequel primes indicate differentiation with respect to the argument of the function involved *e.g.*, $u' \equiv du/dy$ while $v_1' \equiv dv_1/dx$.) The compatibility equation (35) leads to the further requirement:

$$u \frac{\partial}{\partial x} \ln \left[\frac{v_1' v_2 - u'}{1 - u^2 - v_1^2 v_2^2} \right] + v_1 v_2 \frac{\partial}{\partial y} \ln \left[\frac{v_1' v_2 - u'}{1 - u^2 - v_1^2 v_2^2} \right] + v_1 v_2' = 0. \quad (141)$$

* The family of solutions (167)–(173) is here discussed in detail for the first time. It was first discovered, however, as a special case of a larger class of solutions in an investigation [19] to be treated in Chapter V.

This equation simplifies by virtue of (140) to

$$u \frac{\partial}{\partial x} \ln (v_1' v_2 - u') + v_1 v_2 \frac{\partial}{\partial y} \ln (v_1' v_2 - u') + \gamma v_1 v_2' = 0. \quad (142)$$

Upon simplification, (140) and (142) become the following two equations limiting the choice of the three functions $u(y)$, $v_1(x)$, and $v_2(y)$:

$$2u v_2'^2 v_1' + 2u u' v_2 + (\gamma + 1) v_2'^2 v_2' v_1'^2 - (\gamma - 1)(1 - u^2) v_2' = 0, \quad (143)$$

$$u v_2 v_1'' + (\gamma + 1) v_2 v_2' v_1 v_1' - u'' v_2 v_1 - \gamma u' v_2' v_1 = 0. \quad (144)$$

We shall assume throughout the analysis that $u \neq 0$, $v_1 \neq 0$, and $v_2 \neq 0$, because the vanishing of either component of $\mathbf{W}$ leads to the trivial case of flow in parallel straight lines.

Dividing (143) by $2v_2'^2$ and differentiating with respect to x , we obtain as a necessary condition:

$$u v_1'' + (\gamma + 1) v_2' v_1 v_1' = 0. \quad (145)$$

By virtue of (145), (144) simplifies to

$$(u' v_2')' = 0. \quad (146)$$

We now must consider two cases depending on whether or not $v_1' = 0$.

Case I. $v_1' \neq 0$. For this case (145) implies

$$\frac{v_1''}{v_1 v_1'} = - \frac{(\gamma + 1) v_2'}{u} = A \text{ (a constant)}, \quad (147)$$

whence

$$v_1' = \frac{1}{2} A v_1^2 + B, \quad A u = - (\gamma + 1) v_2'. \quad (148), (149)$$

Sub-Case IA. $v_1' \neq 0$, $v_2' \neq 0$. A is not zero and (149) permits integrations of (146), yielding

$$v_2'' = D v_2^{-\gamma}, \quad (v_2')^2 = E - \frac{2D}{\gamma - 1} v_2^{-(\gamma-1)}. \quad (150), (151)$$

Substitution of (148) and (149) into (143) leads to

$$v_2' \left[\frac{2(\gamma + 1)^2}{A^2} v_2 v_2'' + \frac{(\gamma - 1)(\gamma + 1)^2}{A^2} (v_2')^2 - \frac{2(\gamma + 1)B}{A} v_2^2 - (\gamma - 1) \right] = 0. \quad (152)$$

Elimination of v_2' and v_2'' from the second factor of (152) by the use of (150) and (151) gives

$$v_2' \left\{ \frac{2(\gamma+1)B}{A} v_2^2 + (\gamma-1) \left[\frac{(\gamma+1)^2 E}{A^2} - 1 \right] \right\} = 0. \quad (153)$$

Since $v_2 \neq 0$ by hypothesis for this sub-case, (153) requires

$$B = 0, \quad E = \frac{A^2}{(\gamma+1)^2};$$

(148), (149) and (151), and (151) readily yield:

$$v \equiv v_1 v_2 = -\frac{2}{A} \frac{v_2}{x - x_0}, \quad (154)$$

$$u = \mp \frac{\gamma+1}{A} \left[\frac{A^2}{(\gamma+1)^2} - \frac{2D}{\gamma-1} v_2^{-(\gamma-1)} \right]^{\frac{1}{2}}, \quad (155)$$

$$y - y_0 = \pm \int \left[\frac{A^2}{(\gamma+1)^2} - \frac{2D}{\gamma-1} v_2^{-(\gamma-1)} \right]^{-\frac{1}{2}} dv_2, \quad (156)$$

where x_0 and y_0 are integration constants. Introducing the streamline x -coordinate through the relation $v dx = u dy$, we obtain by use of (154)–(156)

$$x - x_0 = F v_2^{(\gamma+1)/2}, \quad (157)$$

where F is a parameter singling out a particular streamline. The set of equations (154)–(157) is a complete representation of the solution of this sub-case in terms of the parametric variable v_2 . The number of constants appearing in these equations gives a false impression of the variety of flow types represented. All of these solutions can be by a translation of the origin together with a uniform change of length scale be brought into coincidence with the single flow pattern given by:

$$v = \frac{1}{\eta} m^{-(\gamma-1)/(\gamma+1)}, \quad u = [1 - m^{-2(\gamma-1)/(\gamma+1)}]^{\frac{1}{2}}, \quad (158), (159)$$

$$y = \int [m^{2(\gamma-1)/(\gamma+1)} - 1]^{-\frac{1}{2}} dm, \quad x = \eta m, \quad (160), (161)$$

where η is a parameter (≥ 1) singling out a particular streamline.

By eliminating m between (158) and (159) we obtain the hodograph equation

$$u^2 + \eta^2 v^2 = 1. \quad (162)$$

The hodograph plane mapping of each streamline $\eta = \text{const.}$ is therefore an ellipse through the points $u = \pm 1$ and $v = \pm 1/\eta$. On the limiting streamline

$\eta = 1$, $u^2 + v^2 = W^2 = 1$; that is, the velocity is ultimate at every point of the streamline.

For certain rational values of γ the quadrature (160) can be evaluated explicitly to give y directly as a function of m . In particular, for $\gamma = 1.4 = \frac{7}{5}$ we obtain

$$y = 6(m^{\frac{1}{3}} - 1)^{\frac{1}{3}}[1 + \frac{2}{3}(m^{\frac{1}{3}} - 1) + \frac{1}{5}(m^{\frac{1}{3}} - 1)^2]. \quad (163)$$

The streamline pattern given by (161) and (163) is shown in Fig. 24. The solution exists only in that portion of the plane within the curved streamline

FIGURE 24

$\eta = 1$. Inasmuch as the streamline curvature along this line implies an outward pressure gradient, it is evident that the condition $W^2 = 1$ is here associated not with zero pressure, but with infinite stagnation pressure.

Sub-Case IB. $v_1' \neq 0$, $v_2' = 0$. For this sub-case we let $v_2 = 1$ and $v = v_1(x)$. Equation (143) then becomes $v'(x) + u'(y) = 0$, whence

$$v = J \cdot (x - x_0), \quad u = -J \cdot (y - y_0). \quad (164)$$

From (164), the streamline slope is $-(x - x_0)/(y - y_0)$; hence the flow takes place in a concentric circular pattern—a familiar degenerate case.

Case II. $v_1' = 0$. For this case we let $v_1 = 1$ and $v = v_2(y)$ and we have the case

$\mathbf{W} = \mathbf{W}(y)$, in one sense a limiting case of the flows considered in Section E. Equations (140) and (146) now require

$$[(1 - u^2 - v^2)v^{\gamma-1}]' = 0, \quad (u'v^\gamma)' = 0, \quad (165), (166)$$

from which we obtain

$$u = [1 - v^2 - Lv^{-(\gamma-1)}]^\frac{1}{2}, \quad u' = Mv^{-\gamma}, \quad (167), (168)$$

where L and M are integration constants. Substituting (167) into (168) we obtain

$$v' = \frac{M[1 - v^2 - Lv^{-(\gamma-1)}]^\frac{1}{2}}{\frac{\gamma-1}{2}L - v^{\gamma+1}} \quad (169)$$

from which we get

$$y = \frac{1}{M} \int_{v_0}^v \frac{\left(\frac{\gamma-1}{2}L - \bar{v}^{\gamma+1}\right) d\bar{v}}{[1 - \bar{v}^2 - L\bar{v}^{-(\gamma-1)}]^\frac{1}{2}}. \quad (170)$$

Introducing the streamline x -coordinate through $u dy = v dx$, we obtain from (167) and (169):

$$\begin{aligned} x - x_0 &= \frac{1}{M} \int_{v_0}^v \left(\frac{\gamma-1}{2} L \bar{v}^{-1} - \bar{v}^\gamma \right) d\bar{v}, \\ &= \frac{1}{M} \left[\frac{\gamma-1}{2} L \ln \frac{v}{v_0} - \frac{(v^{\gamma+1} - v_0^{\gamma+1})}{\gamma+1} \right]. \end{aligned} \quad (171)$$

We observe from (170) and (171) that M is only a scale factor which may be taken as unity. We can also let $x_0 = 0$ without real loss of generality and for convenience take v_0 as the least positive value of v for which u (from (167)) is zero. (171) then becomes

$$x = \frac{\gamma-1}{2} L \ln \frac{v}{v_0} - \frac{(v^{\gamma+1} - v_0^{\gamma+1})}{\gamma+1}, \quad (172)$$

and (170), after an integration by parts,

$$y = v^\gamma [1 - v^2 - Lv^{-(\gamma-1)}]^\frac{1}{2} - \gamma \int_{v_0}^v [1 - \bar{v}^2 - L\bar{v}^{-(\gamma-1)}]^\frac{1}{2} \bar{v}^{\gamma-1} d\bar{v}. \quad (173)$$

Equations (167), (172), and (173) represent the complete parametric solution for a one-parameter (L) family of flows.

In Fig. 25 the solution trajectories in the hodograph plane are shown for several values of L (for $\gamma = 1.4$). The locus of points of zero slope of the tra-

jectories is the ellipse $u^2 + [(\gamma + 1)/(\gamma - 1)]v^2 = 1$. For $L \rightarrow 0$ the trajectories approach the circle $u^2 + v^2 = W^2 = 1$ and for

$$L \rightarrow 2(\gamma - 1)^{-1}[(\gamma - 1)/(\gamma + 1)]^{\frac{1}{2}(\gamma+1)}$$

they approach the point $v = \sqrt{(\gamma - 1)/(\gamma + 1)}$ (uniform parallel flow at sonic speed). Only values of L in this range are significant. The formal similarity between these solutions and those of Fig. 2 should be noted. In Fig. 26 is shown a representative streamline of the family under discussion. The pattern is periodic

FIGURE 25

with respect to y . Limiting lines resembling those of Section E occur when the y -component of flow velocity is sonic ($u^2 + [(\gamma + 1)/(\gamma - 1)]v^2 = 1$). Figures 27 and 28 exhibit the effect of the choice of L on the streamline shape.

The interesting singular solution (158)–(163) was first discovered by I. A. Kiebel [6] using a formulation of the equations of plane gas flow based on taking as independent variables the coordinate x and the streamfunction ψ defined by

$$\rho u = \left(\frac{\partial \psi}{\partial y} \right)_x; \quad \rho v = \left(\frac{\partial \psi}{\partial x} \right)_y \quad (174)$$

(u and v here are components of the *actual* velocity $\mathbf{V}$). With these independent

variables it can be easily shown that the following relations hold:

$$\frac{\partial y}{\partial x} = \frac{v}{u}; \quad \frac{\partial y}{\partial \psi} = \frac{1}{\rho u}; \quad \frac{\partial p}{\partial \psi} = -\frac{\partial v}{\partial x}; \quad (175)$$

and the Bernoulli equation assumes the form

$$u^2 + v^2 + \frac{2\gamma}{\gamma-1} \theta(\psi) p^{(\gamma-1)/\gamma} = 2h_0(\psi), \quad (176)$$

FIGURE 26

where

$$\theta(\psi) \equiv \frac{p^{1/\gamma}}{\rho} \quad (177)$$

is essentially an entropy function. The last relation of (175) is satisfied by the introduction of the function χ through

$$p = \frac{\partial \chi}{\partial x}; \quad v = -\frac{\partial \chi}{\partial \psi}. \quad (178)$$

(178) can now be substituted into (176) and (177) and the resulting expressions used to eliminate u and ρ from the first two relations of (175) to obtain the equations

$$\frac{\partial y}{\partial x} = -\frac{\partial \chi}{\partial \psi} \left[2h_0(\psi) - \frac{2\gamma}{\gamma-1} \theta(\psi) \left(\frac{\partial \chi}{\partial x} \right)^{(\gamma-1)/\gamma} - \left(\frac{\partial \chi}{\partial \psi} \right)^2 \right]^{-\frac{1}{2}}, \quad (179)$$

$$\frac{\partial y}{\partial \psi} = \theta(\psi) \left(\frac{\partial \chi}{\partial x} \right)^{-1/\gamma} \left[2h_0(\psi) - \frac{2\gamma}{\gamma-1} \theta(\psi) \left(\frac{\partial \chi}{\partial x} \right)^{(\gamma-1)/\gamma} - \left(\frac{\partial \chi}{\partial \psi} \right)^2 \right]^{-\frac{1}{2}}, \quad (180)$$

FIGURE 27

between which y can be eliminated by cross-differentiation. From these equations Kiebel found the flow (158)–(163) by seeking a product solution $\chi = H(\psi)X(x)$. He restricted himself to the case $h_0 = \text{const.}$, but this restriction is removable by use of the Substitution Principle.

Section G. Crocco's Streamfunction Equation for Plane Flows. L. Crocco [2] was the first to deduce a single differential equation governing the streamfunction of plane, rotational flow fields of a perfect gas. This derivation was limited to flow fields of uniform stagnation enthalpy. That this restriction is not essential follows [15] from the Substitution Principle of Chapter II. The following derivation does not involve this assumption.

The continuity equation (53) permits the introduction of a streamfunction ψ through

$$(1 - W^2)^{1/(\gamma-1)} u = \frac{\partial \psi}{\partial y}, \quad (1 - W^2)^{1/(\gamma-1)} v = -\frac{\partial \psi}{\partial x}, \quad (181), (182)$$

FIGURE 28

where $W = iu + jv$. From (181) and (182) one has

$$F(W^2) \equiv (1 - W^2)^{2/(\gamma-1)} W^2 = (\text{grad } \psi)^2, \quad (183)$$

which determines W^2 implicitly as a function of $(\text{grad } \psi)^2$:

$$W^2 = F^{-1}[(\text{grad } \psi)^2]. \quad (184)$$

From (183) we also have

$$\frac{\partial W^2}{\partial x} = \frac{\frac{\partial}{\partial x} (\text{grad } \psi)^2}{F'(W^2)}, \quad \frac{\partial W^2}{\partial y} = \frac{\frac{\partial}{\partial y} (\text{grad } \psi)^2}{F'(W^2)}, \quad (185), (186)$$

where

$$F'(W^2) \equiv \frac{dF}{dW^2} = (1 - W^2)^{(3-\gamma)/(\gamma-1)} \left(1 - \frac{\gamma+1}{\gamma-1} W^2 \right). \quad (187)$$

Also, from (181) and (182),

$$\text{curl} [(1 - W^2)^{1/(\gamma-1)} \mathbf{W}] = -\mathbf{k} \text{div grad } \psi. \quad (188)$$

We shall require the vector identity

$$\begin{aligned} \text{curl } \mathbf{W} = (1 - W^2)^{-1/(\gamma-1)} \{ \text{curl} [(1 - W^2)^{1/(\gamma-1)} \mathbf{W}] \\ + \mathbf{W} \times \text{grad}(1 - W^2)^{1/(\gamma-1)} \}. \end{aligned} \quad (189)$$

Making use of (181)–(188), we can write (189) in the form:

$$\begin{aligned} \text{curl } \mathbf{W} = -\mathbf{k}(1 - W^2)^{1/(\gamma-1)} \left[\text{div grad } \psi + \frac{1}{\gamma-1} (1 - W^2)^{-2/(\gamma-1)} \right. \\ \left. \cdot \left(1 - \frac{\gamma+1}{\gamma-1} W^2 \right)^{-1} \text{grad } \psi \cdot \text{grad} (\text{grad } \psi)^2 \right]. \end{aligned} \quad (190)$$

However, by (54) and (66) we have

$$\text{curl } \mathbf{W} = \mathbf{k}f(\psi)(1 - W^2)^{\gamma/(\gamma-1)} \quad (191)$$

Equating (190) and (191) and using (184) we finally have the desired equation:

$$\begin{aligned} \text{div grad } \psi + \frac{1}{\gamma-1} [1 - F^{-1}((\text{grad } \psi)^2)]^{-2/(\gamma-1)} \\ \cdot \left[1 - \frac{\gamma+1}{\gamma-1} F^{-1}((\text{grad } \psi)^2) \right]^{-1} \text{grad } \psi \cdot \text{grad} (\text{grad } \psi)^2 \\ + f(\psi)[1 - F^{-1}((\text{grad } \psi)^2)]^{(\gamma+1)/(\gamma-1)} = 0. \end{aligned} \quad (192)$$

Unfortunately, the usefulness of this equation is considerably impaired by the fact that the inverse function F^{-1} cannot in general be written in finite form.

Chapter IV. AXIALLY-SYMMETRIC FLOW FIELDS

Section A. Generalized Form of Crocco's Pressure Theorem for Axially-Symmetric Flow Fields. The analysis of Section A of Chapter III can be easily modified to apply to axially-symmetric flow fields. For this case a streamfunction can be defined by

$$\frac{d\psi}{dl} = 0, \quad \frac{d\psi}{dn} = rP(p)W, \quad (193)$$

where r denotes the radius from the axis of symmetry and d/dl and d/dn denote differentiation along and normal to the vector $\mathbf{W}$ in a meridian plane.

The counterpart of the equation (65) relating the vorticity of the reduced velocity field and the pressure in a shock-free region of flow of a gas having a product state equation is then

$$\Omega = f(\psi)rH(p)P(p) \quad (194)$$

which specializes for the case of a perfect gas to

$$\Omega = \frac{\gamma - 1}{2\gamma} p_0^{-(\gamma-1)/\gamma} \frac{d \ln p_0}{d\psi} r p. \quad (195)$$

The latter relation is a direct extension [15] of Crocco's theorem [3] concerning vorticity and pressure in iso-energetic axially-symmetric flow fields.

Section B. Axially-Symmetric Flows Having a Constant Velocity Magnitude Along Each Streamline [18]. In this section we shall investigate the possibility of steady axially-symmetric flows with constant velocity magnitude along each streamline for inviscid fluids subject to no extraneous force field. The analysis will apply to all substances for which $\rho = \rho(\psi, V)$ and $p = p(\psi, V)$. This includes homogenous and inhomogenous incompressible fluids and thermally non-conducting compressible fluids.

The continuity equation (1) is satisfied by the introduction of a streamfunction ψ through

$$\frac{d\psi}{dn} = r\rho V, \quad (196)$$

where r denotes radius from the axis of symmetry and dn denotes the differential outward normal to the streamline. For the class of fluids under consideration,

the condition $V = V(\psi)$ implies also $\rho = \rho(\psi)$ and $p = p(\psi)$. The dynamic equation (2) then simplifies to

$$\frac{dp}{dn} = \kappa \rho V^2 \quad (197)$$

where $\kappa = \kappa(x, \psi)$, x being the coordinate along the axis, denotes the streamline curvature, taken as positive when the streamlines are concave with respect to the axis.

Together (196) and (197) imply

$$\kappa(x, \psi) = A(\psi)r(x, \psi), \quad A(\psi) \equiv \frac{1}{V(\psi)} \frac{dp(\psi)}{d\psi}. \quad (198)$$

For convenience letting q denote the local streamline slope,

$$q(x, \psi) \equiv \left(\frac{\partial r}{\partial x} \right)_{\psi}, \quad (199)$$

we have from elementary calculus the formula

$$\kappa(x, \psi) = -(1 + q^2)^{-\frac{1}{2}} \left(\frac{\partial q}{\partial x} \right)_{\psi} \quad (200)$$

Substitution of (198) into (200) and integration of the resulting differential equation yields

$$q^2 = \left[\frac{2}{A(\psi)r^2 + B(\psi)} \right]^2 - 1, \quad (201)$$

where $B(\psi)$ is an arbitrary suitably differentiable function of ψ .

We now observe that the streamfunction equation (196) can be written in the form

$$d\psi = \frac{\rho V r}{\sqrt{q^2 + 1}} (dr - q dx),$$

whence

$$dr(x, \psi) = q dx + \frac{\sqrt{q^2 + 1}}{\rho V r} d\psi. \quad (202)$$

The condition of integrability for the function $r(x, \psi)$ now leads to the equation

$$\rho V \left(\frac{\partial q}{\partial \psi} \right)_x - \frac{q^2}{r \sqrt{q^2 + 1}} \left(\frac{\partial q}{\partial r} \right)_{\psi} + \frac{q \sqrt{q^2 + 1}}{r^2} = 0,$$

which by virtue of the relations

$$\left(\frac{\partial q}{\partial \psi}\right)_x = \left(\frac{\partial q}{\partial \psi}\right)_r + \left(\frac{\partial r}{\partial \psi}\right)_x \left(\frac{\partial q}{\partial r}\right)_\psi, \quad \left(\frac{\partial r}{\partial \psi}\right)_x = \frac{\sqrt{q^2 + 1}}{\rho V r}$$

can be written

$$\frac{1}{2}\rho V \left(\frac{\partial q^2}{\partial \psi}\right)_r + \frac{1}{\sqrt{q^2 + 1}} \left(\frac{\partial q^2}{\partial r^2}\right)_\psi + \frac{q^2 \sqrt{q^2 + 1}}{r^2} = 0. \quad (203)$$

Substituting (201) into (203) we obtain the condition

$$\left[2\rho(\psi)V(\psi) \frac{dA(\psi)}{d\psi} + 3A^2(\psi) \right] r^4 + \left[2\rho(\psi)V(\psi) \frac{dB(\psi)}{d\psi} + 4A(\psi)B(\psi) \right] r^2 + [B^2(\psi) - 4] = 0. \quad (204)$$

Two cases arise.

Case I. $B^2 = 4$ and $A = 0$. In this case (204) is identically zero and from (201) $q = (\partial r / \partial x)_\psi = 0$, implying $r = r(\psi)$.

Case II. $B^2 \neq 4$ or $A \neq 0$. In this case (204) can be solved for r^2 in terms of the three coefficients. But these coefficients are functions of ψ alone, so that again $r = r(\psi)$.

It is thus established that for an inviscid fluid subject to no extraneous force the only steady axially-symmetric flows in which the velocity magnitude, pressure, and density are constant along each individual streamline are purely axial flows.

Section C. Axially-Symmetric Flows Having a Reduced Velocity Field which is a Function of Radius Alone. We now investigate for a perfect gas the possibility of axially-symmetric flow fields in which the reduced velocity field depends only on the radius. That is, we seek flows whose reduced velocity field referred to an ordinary cylindrical coordinate system is of the form

$$\mathbf{W} = \mathbf{W}(r) = r_1 u(r) + z_1 q(r), \quad (205)$$

where r_1 and z_1 denote, respectively, unit vectors in the radial and axial directions.

Substitution of (205) into the continuity equation (53) leads to the condition

$$[ru(1 - u^2 - q^2)^{1/(\gamma-1)}]' = 0, \quad (206)$$

and the compatibility equation (35) yields the additional requirement

$$\left[\frac{uq'}{1 - u^2 - q^2} \right]' = 0. \quad (207)$$

From (206) and (207) we obtain

$$q = \pm[1 - u^2 - A(ru)^{-(\gamma-1)}]^\frac{1}{2}, \quad q' = \frac{B}{u}(1 - u^2 - q^2), \quad (208), (209)$$

where A and B are constants of integration. Substituting (208) into (209) and solving for u' we obtain

$$\frac{du}{dr} = \frac{\pm \frac{2}{\gamma-1} B[1 - u^2 - A(ru)^{-(\gamma-1)}]^\frac{1}{2} - \frac{u}{r}}{1 - \frac{2}{\gamma-1} \frac{1}{A} r^{\gamma-1} u^{\gamma+1}}. \quad (210)$$

Inspection of (210) reveals that the constant A only plays the role of a uniform scale factor for length, and hence can be set equal to one without real loss of generality. Furthermore, since the constant B can be either positive or negative, there is no objection to dropping the $\pm$ signs simultaneously in (208) and (210). Thus the problem is finally formulated in terms of the solution of a single first-order ordinary differential equation involving an arbitrary parameter:

$$\frac{du}{dr} = \frac{\frac{2}{\gamma-1} Br^{(3-\gamma)/2} [r^{\gamma-1}(u^{\gamma-1} - u^{\gamma+1}) - 1]^\frac{1}{2} - u^{(\gamma+1)/2}}{ru^{(\gamma-1)/2} \left(1 - \frac{2}{\gamma-1} r^{\gamma-1} u^{\gamma+1} \right)}, \quad (211)$$

$$q = \frac{[r^{\gamma-1}(u^{\gamma-1} - u^{\gamma+1}) - 1]^\frac{1}{2}}{(ru)^{(\gamma-1)/2}}. \quad (212)$$

This solution will be generalized in Chapter V, Section D.

Equation (211) requires, in general, numerical or graphical methods of solution and no study of its properties has yet been made.

For the special case $B = 0$, it follows from (209) that the axial component of reduced velocity is constant and from (206) we have

$$r = Ru^{-1}(1 - Q^2 - u^2)^{-1/(\gamma-1)}$$

where R and Q are constants. This simple flow field is a superposition of a source flow (from a line source of uniform strength) and a uniform flow parallel to the source line. It is a special case of the class of spatial flows obtainable from plane flow field solutions by the application of Poritsky's Superposition Principle which will be discussed in Chapter V. It should be noted that this particular flow field is irrotational in $\mathbf{W}$.

Section D. Crocco's Streamfunction Equation for Axially-Symmetric Flows. In addition to the streamfunction equation for plane gas flows discussed in

Section G of Chapter III, Crocco [2] developed an equation governing the streamfunction for axially-symmetric flows of a perfect gas. As in the plane case, Crocco unnecessarily restricted his attention to isoenergetic flows. We present a derivation free from this assumption and paralleling that given for the plane case.

For $\mathbf{W} = r_1 u + z_1 q$ the continuity equation (53) implies the existence of a streamfunction ψ such that

$$r(1 - W^2)^{1/(\gamma-1)} q = \frac{\partial \psi}{\partial r}, \quad r(1 - W^2)^{1/(\gamma-1)} u = -\frac{\partial \psi}{\partial z}. \quad (213), (214)$$

From (213) and (214) we have

$$F(W^2) \equiv (1 - W^2)^{2/(\gamma-1)} W^2 = \frac{1}{r^2} (\text{grad } \psi)^2, \quad (215)$$

or, implicitly,

$$W^2 = F^{-1} \left[\frac{1}{r^2} (\text{grad } \psi)^2 \right]. \quad (216)$$

Also,

$$\frac{\partial W^2}{\partial r} = \frac{\frac{\partial F}{\partial r}}{F'[W^2]}, \quad \frac{\partial W^2}{\partial z} = \frac{\frac{\partial F}{\partial z}}{F'[W^2]}, \quad (217), (218)$$

where

$$F'[W^2] \equiv \frac{dF}{dW^2} = (1 - W^2)^{(3-\gamma)/(\gamma-1)} \left(1 - \frac{\gamma+1}{\gamma-1} W^2 \right). \quad (219)$$

Making use of the relations (213)–(219) in the vector identity

$$\begin{aligned} \text{curl } \mathbf{W} = r^{-1}(1 - W^2)^{-1/(\gamma-1)} \{ \text{curl } [(1 - W^2)^{1/(\gamma-1)} r \mathbf{W}] \\ + \mathbf{W} \times \text{grad } [(1 - W^2)^{1/(\gamma-1)} r] \}, \end{aligned}$$

we obtain

$$\begin{aligned} \text{curl } \mathbf{W} = -\theta_1 r^{-1}(1 - W^2)^{-1/(\gamma-1)} \left\{ \text{div grad } \psi + \frac{1}{\gamma-1} (1 - W^2)^{-2/(\gamma-1)} \right. \\ \left. \left(1 - \frac{\gamma+1}{\gamma-1} W^2 \right)^{-1} \text{grad } \psi \cdot \text{grad } \left[\frac{1}{r^2} (\text{grad } \psi)^2 \right] + \text{grad } \ln r \cdot \text{grad } \psi \right\}. \end{aligned} \quad (220)$$

But from the pressure theorem of Section A it follows that

$$\text{curl } \mathbf{W} = \theta_1 f(\psi) (1 - W^2)^{\gamma/(\gamma-1)} r. \quad (221)$$

Equating (220) and (221) and using (216) we obtain the desired equation:

$$\begin{aligned} \operatorname{div} \operatorname{grad} \psi + \frac{1}{\gamma - 1} \left[1 - F^{-1} \left(\left(\frac{\operatorname{grad} \psi}{r} \right)^2 \right) \right]^{-2/(\gamma-1)} \\ \left[1 - \frac{\gamma + 1}{\gamma - 1} F^{-1} \left(\left(\frac{\operatorname{grad} \psi}{r} \right)^2 \right) \right]^{-1} \operatorname{grad} \psi \cdot \operatorname{grad} \left(\frac{\operatorname{grad} \psi}{r} \right)^2 \quad (222) \\ + \operatorname{grad} \ln r \cdot \operatorname{grad} \psi + r^2 f(\psi) \left[1 - F^{-1} \left(\left(\frac{\operatorname{grad} \psi}{r} \right)^2 \right) \right]^{(\gamma+1)/(\gamma-1)} = 0. \end{aligned}$$

As in the plane case, the implicit nature of (216) is a major difficulty in application of (222).

Chapter V. SPATIAL FLOW FIELDS

Section A. Generalization of Poritsky's Superposition Principle. H. Poritsky [13] has discussed the construction of steady spatial gas flow solutions from steady plane flow solutions by the superposition of a uniform velocity field normal to the given plane flow field. In particular, he pointed out that if

$$\mathbf{V}_p = iu(x, y) + jv(x, y) \quad (223)$$

is the (plane) velocity field of a solution to the equations ((1)–(4)) of gas dynamics, then the spatial velocity field

$$\mathbf{V} = \mathbf{V}_p + \mathbf{k}V_n \quad (224)$$

where V_n is a constant will also satisfy the equations.

The validity of this Superposition Principle follows at once from the Newtonian Relativity Principle that the laws of mechanics are the same for all observers moving at uniform velocities with respect to an inertial system. With respect to an observer moving with a velocity $-\mathbf{k}V_n$ with respect to the given flow field (223), that flow field has the velocity vector given by (224). It is essential that the given field not only be of the form $\mathbf{V}_p = iu + jv$, but that u and v be independent of z . Otherwise the flow field will be unsteady in time with respect to the moving observer.

By application of the Substitution Principle of Chapter II flows can be derived from (224) in which the normal velocity component varies from place to place in the field. However, the Substitution Principle can also be applied before the superposition. This suggests the advisability of formulating a Superposition Principle in terms of the reduced velocity fields.

We note that

$$a^2 = V^2 + 2h = V_p^2 + V_n^2 + 2h = a_p^2 + V_n^2$$

and hence

$$\frac{a_p}{a} = (1 - W_n^2)^{\frac{1}{2}}, \quad W_n \equiv \frac{V_n}{a} = \frac{V_n}{(a_p^2 + V_n^2)^{\frac{1}{2}}}. \quad (225), (226)$$

From (224) we then have

$$\frac{\mathbf{V}}{a} = \frac{\mathbf{V}_p}{a_p} \frac{a_p}{a} + \mathbf{k} \frac{V_n}{a},$$

or

$$\mathbf{W} = (1 - W_n^2)^{\frac{1}{2}} \mathbf{W}_p + \mathbf{k} W_n, \quad (227)$$

where, from (226) and the Substitution Principle, W_n need only satisfy the conditions

$$\frac{\partial W_n}{\partial z} = \mathbf{W}_p \cdot \text{grad } W_n = 0. \quad (228)$$

(That is, $W_n = W_n[\psi_p(x, y)]$ where ψ_p is a streamfunction of the given plane flow.)

Equations (227) and (228) permit the generation of a great variety of spatial reduced velocity fields $\mathbf{W}$ from the reduced velocity field $\mathbf{W}_p$ of any known plane flow solution. The freedom of choice of the function $W_n[\psi_p]$ is a consequence of the possibility of applying the Substitution Principle to the given plane flow field without changing its reduced velocity field $\mathbf{W}_p$. After the function W_n has been chosen, the Substitution Principle yields from (227) a great variety of actual velocity fields.

Section B. Doubly-Laminar Flow Fields Having a Constant Velocity Magnitude Along Each Streamline. We now turn our attention to the problem of determining what spatial flow fields have the property that along each streamline the velocity magnitude is a constant. For inhomogeneous incompressible fluids and for compressible fluids having a product equation of state (15), this problem can be reduced to the study of flow fields of a homogeneous incompressible fluid having the property under consideration. This reduction is accomplished by application of the Substitution Principle as follows.

The constancy of velocity magnitude along each streamline implies the constancy of density *along* each individual streamline. The Substitution Principle then can be applied to obtain a flow in which this density constant is the same from one streamline to the other.

Our problem then is reduced to the study of velocity fields satisfying the conditions

$$\text{div } \mathbf{V} = 0, \quad \text{curl } (\mathbf{V} \times \text{curl } \mathbf{V}) = 0, \quad \mathbf{V} \cdot \text{grad } V = 0. \quad (229), (230), (231)$$

For the special cases of plane and of axially-symmetric flow fields this problem has been solved in Chapter III, Section B, and Chapter IV, Section B. For the case of irrotational spatial flow the problem was solved by Hamel [4], who proved that the only such flows possible were those irrotational helical flows obtainable by normal superposition of a uniform parallel flow and a potential vortex.

The answer to the general rotational problem is not yet known. We shall consider here the case of doubly-laminar flow fields, i.e., flow fields of the type $\mathbf{V} = \lambda \text{ grad } \phi$. This is not the most general possible case, but is sufficiently general to include the above mentioned results for plane, axially-symmetric, and irrotational fields as special cases.

We first introduce the unit normal vector field for the family of surfaces $\phi = \text{const.}$ through

$$\mathbf{n} \equiv \frac{\text{grad } \phi}{|\text{grad } \phi|}. \quad (232)$$

We can now write

$$\mathbf{V} = \mathbf{n}V. \quad (233)$$

Substituting (233) into (229) we obtain

$$V \text{div } \mathbf{n} + \mathbf{n} \cdot \text{grad } V = 0,$$

whence by virtue of (231)

$$\text{div } \mathbf{n} = 0. \quad (234)$$

This is a well known equation for the unit normal of a one-parameter field of minimal surfaces. Hence, a necessary and sufficient condition for a velocity field satisfying (231)–(233) to comply with the continuity equation is that the surfaces $\phi = \text{const.}$ be minimal surfaces. This result was known to Hamel [4].

We now substitute the form (233) into the compatibility condition (230) and obtain the expansion

$$\begin{aligned} V^2(\mathbf{n} \cdot \text{grad } \ln |\text{grad } \phi|)(\mathbf{n} \times \text{grad } \ln |\text{grad } \phi|) \\ + V^2 \text{grad } (\mathbf{n} \cdot \text{grad } \ln |\text{grad } \phi|) \times \mathbf{n} \\ + (\text{grad } V^2 \cdot \mathbf{n} \times \text{grad } \ln |\text{grad } \phi|)\mathbf{n} = 0. \end{aligned} \quad (235)$$

Two successive vector pre-multiplications of (235) with $\mathbf{n}$ yield the *necessary* condition

$$\mathbf{n} \times \text{grad } \ln \left(\frac{\mathbf{n} \cdot \text{grad } \ln |\text{grad } \phi|}{|\text{grad } \phi|} \right) = 0,$$

which implies

$$\frac{\mathbf{n} \cdot \text{grad } \ln |\text{grad } \phi|}{|\text{grad } \phi|} = f(\phi). \quad (236)$$

By suitable parametrization of the surfaces $\phi = \text{const.}$, the function appearing on the right side of (236) can be made zero. (Replace ϕ by

$$\phi(\phi) \equiv \int \exp \left[- \int f(\phi) d\phi \right] d\phi.)$$

Our necessary condition then becomes

$$\mathbf{n} \cdot \text{grad } |\text{grad } \phi| = 0. \quad (237)$$

Geometrically, this condition means that the surfaces $\phi = \text{const.}$ can be so parametrized as to cut off equal arcs along each individual streamline.

Substituting (237) back into (235) we obtain the additional condition

$$\text{grad } V \cdot \mathbf{n} \times \text{grad } |\text{grad } \phi| = 0,$$

which by virtue of (231) and (237) implies that either

$$V = V[|\text{grad } \phi|] \quad \text{or} \quad \text{grad } |\text{grad } \phi| = 0. \quad (238), (239)$$

The latter case is associated with the obvious degenerate case of flow in parallel straight lines and will be dismissed without further attention.

We now observe that substitution of (232) into (234) yields the equation for a family of minimal surfaces in the form

$$\text{div grad } \phi - \text{grad } \phi \cdot \text{grad } \ln |\text{grad } \phi| = 0. \quad (240)$$

The simultaneous validity of (237) and (240) implies

$$\text{div grad } \phi = 0: \quad (241)$$

that is, the surfaces $\phi = \text{const.}$ are a family of potential surfaces. Any two of the following properties of the family of surfaces $\phi = \text{const.}$ are seen to imply the third:

- (a) the surfaces are a family of minimal surfaces,
- (b) the surfaces cut off equal arc lengths along each orthogonal trajectory,
- (c) the surfaces are a family of potential surfaces.

This geometrical result, known to Hamel, brings us to a point where we can make use of the results he obtained for the case of irrotational flow fields. (Whereas in our case property (c) appears as a consequence of (a) and (b), in Hamel's case property (b) appeared as a consequence of (a) and (c).)

Our analysis shows that the geometric variety of doubly-laminar flow fields having a constant velocity magnitude along each streamline is exactly the same as that of potential flow fields having the same property. Specifically, the only possible streamline patterns for such flows are those associated with the family of potential flow fields

$$\mathbf{V} = \text{grad } (Az + B\theta). \quad (242)$$

referred to ordinary cylindrical coordinates r, θ, z . Insofar as the streamline geometry is concerned, this is a one-parameter family of helical flow fields. However, the kinematic variety of the possible doubly-laminar flow fields is much greater than that of the potential flows of (242). By (238), for the doubly-laminar case any velocity field of the form

$$\mathbf{V} = f(r) \text{grad } (Az + B\theta) \quad (243)$$

is possible. (In the degenerate case $B = 0$, $\mathbf{V} = g(r, \theta) \text{grad } Az$.)

For flows of compressible fluids or inhomogeneous incompressible fluids, the velocity magnitude can be assigned arbitrarily on each streamline. However, because of the invariance of the product ρV^2 under the group of substitution flows, the density must satisfy a relation of the type $\rho = h(r)/V^2$.

Section C. Generalized Beltrami Flow Fields. Beltrami fields, Beltrami flows, and generalized Beltrami flows were defined in Section C of Chapter II, where we have already discussed the relations between the vectors $\mathbf{V} \times \text{curl } \mathbf{V}$ and $\mathbf{W} \times \text{curl } \mathbf{W}$ and the basic thermodynamic quantities, and between Beltrami fields and generalized Beltrami fields. In the present Section we shall establish some general properties of Beltrami vector fields, present a method of deriving generalized Beltrami flows from plane flow fields, investigate the analogs for gas flow of certain classes of incompressible Beltrami flows, and examine some specific examples of interesting generalized Beltrami flow fields.

If $\mathbf{f}$ is a Beltrami vector field and $\sigma \mathbf{f}$ is a solenoidal vector field, then $\text{curl } \mathbf{f}$ is proportional to $\sigma \mathbf{f}$ along each vector line of $\mathbf{f}$. Beltrami's proof [1] is essentially as follows. The condition $\mathbf{f} \times \text{curl } \mathbf{f} = 0$ implies that $\mathbf{f} = \lambda \text{curl } \mathbf{f}$. Substitution of this formula into the condition $\text{div } \sigma \mathbf{f} = 0$ gives

$$0 = \text{div } (\sigma \lambda \text{curl } \mathbf{f}) = \text{grad } \sigma \lambda \cdot \text{curl } \mathbf{f}.$$

Hence the quantity $\sigma \lambda$ is constant along each vector line of $\mathbf{f}$.

Applying this theorem to Beltrami gas flow fields we find that for such fields $\text{curl } \mathbf{V} \propto \rho \mathbf{V}$ along each streamline.

Applying the theorem to generalized Beltrami flows of a gas with a product equation of state (15) we find that $\text{curl } \mathbf{W} \propto P(p) \mathbf{W}$ along each streamline. For a perfect gas, it follows from (53) that $\text{curl } \mathbf{W} \propto (1 - W^2)^{1/(\gamma-1)} \mathbf{W}$ along each streamline.

We shall now establish the following theorem concerning Beltrami vector fields in which the vector lines are straight lines:

If a rotational vector field $\mathbf{f}$ possesses any two of the following three properties,

- (a) *it is a Beltrami field,*
- (b) *it is a constant-magnitude field,*
- (c) *it is a rectilinear field,*

it possesses also the third. These properties are formalized as follows:

- (a) $\mathbf{f} \times \text{curl } \mathbf{f} = 0$;
- (b) $\text{grad } f^2 = 0$;
- (c) $\mathbf{f} \times (\mathbf{f} \cdot \text{grad } \mathbf{f}) = \mathbf{f} \times \text{grad } f^2/2 - \mathbf{f} \times (\mathbf{f} \times \text{curl } \mathbf{f}) = 0$.

That (a) and (b) imply (c) is immediate. That (b) and (c) imply (a) also follows at once from the fact that $\mathbf{f}$ is perpendicular to $\mathbf{f} \times \text{curl } \mathbf{f}$. To show that (a) and (c) imply (b) we observe that the former two conditions imply

$$\mathbf{f} \times \text{grad } f^2 = \text{curl } \mathbf{f} \times \text{grad } f^2 = 0,$$

whence $\mathbf{f} = \eta \text{grad } f^2$, and $\text{curl } \mathbf{f} = \text{grad } \eta \times \text{grad } f^2$; by scalar multiplication

by $\text{grad } f^2$:

$$\text{curl } \mathbf{f} \cdot \text{grad } f^2 = |\text{curl } \mathbf{f}| |\text{grad } f^2| = 0,$$

from which it follows that $\text{curl } \mathbf{f} \neq 0$ implies $\text{grad } f^2 = 0$.

If the field $\mathbf{f}$ of this theorem be identified with the velocity field of a Beltrami flow or with the reduced velocity field of a generalized Beltrami flow, it must satisfy the additional condition $\text{div } \mathbf{f} = 0$. It is geometrically evident that this condition together with (b) and (c) limits $\mathbf{f}$ to fields of the type

$$\mathbf{f} = ib(z) + jc(z),$$

where (b) further requires

$$b^2(z) + c^2(z) = C,$$

giving as the only possible rectilinear, solenoidal Beltrami vector field

$$\mathbf{f} = ib(z) + j(C - b^2(z))^{\frac{1}{2}}. \quad (244)$$

Substitution of (244) into (a) leads to no restriction of $b(z)$.

We shall now show that *the only Beltrami vector fields $\mathbf{f}$ such that $\text{curl } \mathbf{f}$ is also a Beltrami vector field are those in which $\text{curl } \mathbf{f}$ is proportional to $\mathbf{f}$* . From $\mathbf{f} \times \text{curl } \mathbf{f} = 0$ we have $\mathbf{f} = \alpha \text{curl } \mathbf{f}$, whence

$$\text{curl } \mathbf{f} = \alpha \text{curl curl } \mathbf{f} + \text{grad } \alpha \times \text{curl } \mathbf{f}.$$

Vector multiplication of this expression by $\text{curl } \mathbf{f}$ leads because of the condition $\text{curl } \mathbf{f} \times \text{curl curl } \mathbf{f} = 0$ to

$$\text{curl } \mathbf{f} \times (\text{grad } \alpha \times \text{curl } \mathbf{f}) = 0,$$

which for $\text{curl } \mathbf{f} \neq 0$ implies $\text{grad } \alpha \times \text{curl } \mathbf{f} = 0$, and hence

$$\mathbf{f} \times \text{grad } \alpha = 0. \quad (245)$$

Now, if $\text{grad } \alpha$ were not zero, this equation would require $\mathbf{f} = \lambda \text{grad } \alpha$. But this would mean that $\mathbf{f}$ would be simultaneously a doubly-laminar vector field and a Beltrami vector field, an obvious impossibility if $\text{curl } \mathbf{f} \neq 0$. Therefore (245) implies $\text{grad } \alpha = 0$, and the above statement is proved. It is further evident that for $\alpha = \text{const.}$ all successive curls of $\mathbf{f}$ are Beltrami fields.

We now focus our attention on generalized Beltrami flows of ideal gases having a product equation of state (15). The dominant result in the present theory of such flows is a method of generation based on (50) and the general Superposition Principle of Section A.

First it must be noted that the spatial flow fields $\mathbf{W}$ derived from a plane flow field $\mathbf{W}_p$ by (227) have the same pressure field as $\mathbf{W}_p$. This follows from the fact that the pressure is invariant under the Substitution Principle and under

the normal superposition (224) of a uniform actual velocity field. Therefore we have from (23)

$$H(p) = H(p_0) (1 - W^2) = H(p_{0p}) (1 - W_p^2), \quad (246)$$

where subscript p 's always designate quantities associated with the given plane flow field. Now it is easily shown from (227) that

$$1 - W^2 = (1 - W_p^2) (1 - W_n^2), \quad (247)$$

so that (246) yields the following relation between the stagnation pressures of the derived flow field and of the given plane flow field and the normal reduced velocity component W_n :

$$W_n = \left[1 - \frac{H(p_{0p})}{H(p_0)} \right]^{\frac{1}{2}}. \quad (248)$$

Using this relation, we can write (227) in the form

$$\mathbf{W} = \left[\frac{H(p_{0p})}{H(p_0)} \right]^{\frac{1}{2}} \mathbf{W}_p + \mathbf{k} \left[1 - \frac{H(p_{0p})}{H(p_0)} \right]^{\frac{1}{2}}. \quad (249)$$

In this form the arbitrariness of the function $W_n[\psi_p]$ manifests itself as the arbitrary assignability of the function $p_0[\psi_p]$ so long as $H(p_0) \geq H(p_{0p})$.

Now from (50) it is known that a necessary and sufficient condition for a flow field of the type under consideration to have $\mathbf{W} \times \text{curl } \mathbf{W} = 0$ is that it have a uniform stagnation pressure. From (248) it is clear that the function W_n can always be so chosen as to make p_0 a constant, P_0 . Therefore each rotational plane flow field for which $H(p_{0p})$ is finite throughout generates through (249) a one-parameter family of generalized Beltrami flow fields as we set $p_0 = P_0$, where P_0 is any positive constant such that $H(P_0) \geq H(p_{0p})$ throughout. If the given plane flow field is irrotational in $\mathbf{W}_p$, p_{0p} is a constant, so by (248) $p_0 = \text{const.}$ implies $W_n = \text{const.}$ and only irrotational spatial flows are obtained.

For the special case of a perfect gas the relations (248) and (249) take on the forms

$$W_n = \left[1 - \left(\frac{p_{0p}}{p_0} \right)^{(\gamma-1)/\gamma} \right]^{\frac{1}{2}},$$

$$\mathbf{W} = \left(\frac{p_{0p}}{p_0} \right)^{(\gamma-1)/(2\gamma)} \mathbf{W}_p + \mathbf{k} \left[1 - \left(\frac{p_{0p}}{p_0} \right)^{(\gamma-1)/\gamma} \right]^{\frac{1}{2}} \quad (250), (251)$$

while the limitation on the assignment of the constant P_0 is that $P_0 \geq p_{0p}$ throughout.

No generalized Beltrami flow solutions are yet known that are not given by the method just described.

The solutions (244) are generated by the family of plane flow fields with parallel straight streamlines (for which the pressure is uniform and either the stag-

nation pressure or the reduced velocity magnitude can be assigned arbitrarily for each streamline).

The other extremely simple family of plane flow solutions, the general vortex flows having concentric circular streamlines, generate a large variety of generalized Beltrami flow fields having concentric helical streamlines and constant velocity magnitude on each surface of constant helical radius. Referring these generalized Beltrami solutions to an ordinary cylindrical coordinate system (r, θ, z) , it is easy to show that the tangential reduced velocity component $W_\theta(r)$ and the axial component $W_z(r)$ are restricted only by the differential condition

$$(W_\theta^2 + W_z^2)' + \frac{2}{r} W_\theta^2 = 0, \quad (252)$$

or

$$W_z^2(r) = A - W_\theta^2(r) - 2 \int \frac{1}{r} W_\theta^2(r) dr, \quad (253)$$

where A is an integration constant. It is a straight-forward and simple matter [11] using (252) so to determine the arbitrary function $W_\theta(r)$ that the resulting solution has some preassigned geometric or kinematic property such as, for example, streamlines of uniform slope or $\text{curl } \mathbf{W} \propto \mathbf{W}$.

We shall now use the generating method to derive a family of generalized Beltrami flows from the special corner flow solution (118), (119) of Chapter III. From (124) and (125) we find that, except for a change of scale in r ,

$$p_{0p} = \left(r \cos^\lambda \frac{\theta}{\lambda v_0} \right)^{(\lambda+1)(v_0^2 - (1/\lambda))/(1-v_0^2)} \quad (254)$$

Using (118), (119) and (254) in (251) we obtain the two-parameter (P_0 and v_0) family of generalized Beltrami flow fields

$$\begin{aligned} \mathbf{W} = & \mathbf{r}_1 P_0^{-1/(\lambda+1)} \sin \frac{\theta}{\lambda v_0} \left(r \cos^\lambda \frac{\theta}{\lambda v_0} \right)^{(v_0^2 - 1/\lambda)/(1-v_0^2)} \\ & + \boldsymbol{\theta}_1 P_0^{-1/(\lambda+1)} v_0 \cos \frac{\theta}{\lambda v_0} \left(r \cos^\lambda \frac{\theta}{\lambda v_0} \right)^{(v_0^2 - 1/\lambda)/(1-v_0^2)} \\ & + \mathbf{z}_1 \left[1 - P_0^{-2/(\lambda+1)} \left(r \cos^\lambda \frac{\theta}{\lambda v_0} \right)^{2(v_0^2 - 1/\lambda)/(1-v_0^2)} \right]^{1/2}, \end{aligned} \quad (255)$$

where $\mathbf{r}_1$, $\boldsymbol{\theta}_1$, and $\mathbf{z}_1$ are unit vectors in an ordinary cylindrical coordinate system (r, θ, z) . In contrast with the solutions of (244) and (252), this family of explicit solutions does not have a constant velocity magnitude along each streamline.

The solutions (244) and (252) (with $\mathbf{f}$ and $\mathbf{W}$ replaced by $\mathbf{V}$) are included in the families of Beltrami flow fields of homogeneous, incompressible fluids discovered by Beltrami [1]. As a matter of fact, for any flow field having a constant velocity magnitude along each streamline (and only for such flows) the conditions on $\mathbf{W}$ for generalized Beltrami gas flow fields are exactly the same as the conditions on $\mathbf{V}$ for Beltrami flow fields of homogeneous, incompressible fluids: $\mathbf{W} \times \text{curl } \mathbf{W} = 0$, $\text{div } \mathbf{W} = 0$. Hence solutions of these two problems are identical except for interpretation.

Beltrami's investigation of specific Beltrami flow fields of homogeneous, incompressible fluids centered on two types of *pseudo-plane* flow fields. We recall that a velocity field $\mathbf{V} = \mathbf{i}V_x + \mathbf{j}V_y + \mathbf{k}V_z$ is plane if it satisfies the conditions

$$V_z = 0, \quad \frac{\partial V_x}{\partial z} = \frac{\partial V_y}{\partial z} = 0. \quad (256), (257)$$

A flow is *pseudo-plane* if one but not both of (256), (257) holds.

One type of pseudo-plane flow fields considered by Beltrami is obtained by retaining the second of the above two conditions and weakening the first to $\partial V_z / \partial z = 0$. We investigate the obvious analog of these fields for the case of generalized Beltrami flows by setting

$$\mathbf{W} = \mathbf{i}W_x(x, y) + \mathbf{j}W_y(x, y) + \mathbf{k}W_z(x, y). \quad (258)$$

It is then easily verified that the vanishing of the z -component of $\mathbf{W} \times \text{curl } \mathbf{W}$ implies

$$W_x \frac{\partial W_z}{\partial x} + W_y \frac{\partial W_z}{\partial y} = 0; \quad (259)$$

i.e.,

$$\mathbf{W} \cdot \text{grad } W_z = 0, \quad (259)$$

and W_z must be constant along each streamline. But it is clear that any reduced velocity field of the type (258) for which (259) also holds is derivable from a plane flow field by the general Superposition Principle of Section A. (If the ultimate velocity is properly assigned the z -component of velocity can be made a constant throughout the field.) Therefore, all generalized Beltrami flows of the type (258) are generated by the process described in connection with (249).

In a second type of pseudo-plane flow fields investigated by Beltrami for the case of homogeneous, incompressible fluids, the condition $V_z = 0$ is retained while V_x and V_y are allowed to depend on all three coordinates. To investigate the analogous case for generalized Beltrami fields of a perfect gas we set

$$\mathbf{W} = iu(x, y, z) + jv(x, y, z). \quad (260)$$

The Beltrami condition then yields

$$v_x - u_y = 0, \quad (u^2 + v^2)_z = (W^2)_z = 0. \quad (261), (262)$$

We satisfy (261) by introducing a function $\phi(x, y, z)$, *not* the familiar potential function, such that $u = \phi_x$, $v = \phi_y$. (262) now becomes

$$(\phi_x^2 + \phi_y^2)_z = 0. \quad (263)$$

The continuity equation (53) takes on the form

$$[(1 - \phi_x^2 - \phi_y^2)^{1/(\gamma-1)} \phi_x]_x + [(1 - \phi_x^2 - \phi_y^2)^{1/(\gamma-1)} \phi_y]_y = 0. \quad (264)$$

Except for the fact that ϕ is here a function of z as well as of x and y this is the familiar equation for the potential function for plane irrotational flow fields of a perfect gas. Thus the generalized Beltrami flow fields we seek can be regarded as a stack of plane flow fields each one of which is irrotational. The additional condition (263) requires that the reduced velocity magnitude in the stack not vary from layer to layer for given values of x and y . The simplest solution $u = u(z)$ and $v = v(z)$ gives the solutions (244). No other continuous rotational solutions are known. In the incompressible case treated by Beltrami the analysis parallels that given above except that instead of (264) one has $\phi_{xx} + \phi_{yy} = 0$. Because of the superposition of solutions permitted by the linearity of this equation and of the fact that conjugate harmonic functions have the same moduli, a great variety of solutions exist for this case, in contrast to the apparent paucity of solutions to the gas flow problem.

Finally, we shall derive an interesting formula for the angle σ between the vortex lines (of the actual velocity field) and the streamlines of any generalized Beltrami flow field [12]. From the definition of the vector product we can write

$$\csc \sigma = \frac{V |\operatorname{curl} \mathbf{V}|}{|\mathbf{V} \times \operatorname{curl} \mathbf{V}|}.$$

But

$$\operatorname{curl} \mathbf{V} = a (\operatorname{curl} \mathbf{W} - \mathbf{W} \times \operatorname{grad} \ln a), \quad \mathbf{V} \times \operatorname{curl} \mathbf{V} = W^2 a^2 \operatorname{grad} \ln a,$$

whence, taking into account the fact that $\mathbf{W} \cdot \operatorname{grad} a = 0$,

$$\begin{aligned} \csc \sigma &= \frac{Wa(a\sqrt{|\operatorname{curl} \mathbf{W}|^2 + W^2} |\operatorname{grad} \ln a|^2)}{W^2 a^2 |\operatorname{grad} \ln a|}, \\ &= \left(\frac{|\operatorname{curl} \mathbf{W}|^2}{W^2 |\operatorname{grad} \ln a|^2} + 1 \right)^{\frac{1}{2}}, \end{aligned}$$

or

$$\cot \sigma = \frac{\frac{|\operatorname{curl} \mathbf{W}|}{W}}{\frac{|\operatorname{grad} a|}{a}}. \quad (265)$$

Section D. Flow Fields for Which the Three Reduced Velocity Components in a Cylindrical Isometric Coordinate System Depend Only on One Isometric Coordinate [19]. We shall now investigate a broad class of flow fields of a perfect gas containing as special cases many of the solutions discussed in detail in earlier sections. We shall consider a general cylindrical coordinate system (ξ, η, z) based on a plane isometric net (ξ, η) . The squared element of arc length in this system is given by

$$(ds)^2 = g^2(\xi, \eta)[(d\xi)^2 + (d\eta)^2] + (dz)^2 \quad (266)$$

where the condition that the (ξ, η) net lie in a plane requires that

$$\frac{\partial^2 \ln g}{\partial \xi^2} + \frac{\partial^2 \ln g}{\partial \eta^2} = 0. \quad (267)$$

We will study those flow fields which, referred to these coordinates, have a reduced velocity field of the form

$$\mathbf{W} = \xi_1 u(\xi) + \mathbf{n}_1 v(\xi) + \mathbf{z}_1 q(\xi) \quad (268)$$

where ξ_1 , $\mathbf{n}_1$, and $\mathbf{z}_1$ are unit vectors in the ξ , η , and z -directions, respectively.

Substitution of (268) into the continuity equation (53) and the compatibility equation (35) yields four conditions limiting the choice of the functions $g(\xi, \eta)$, $u(\xi)$, $v(\xi)$, and $q(\xi)$:

$$v(\ln g)_\eta + u\{\ln [gu(1 - W^2)^{1/(\gamma-1)}]\}_\xi = 0, \quad (269)$$

$$\left[\frac{uq'}{g(1 - W^2)} \right]_\xi = 0, \quad \left[\frac{uq'}{g(1 - W^2)} \right]_\eta = 0, \quad (270), (271)$$

$$v(\ln \Omega)_\eta + u \left(\ln \frac{u\Omega}{1 - W^2} \right)_\xi = 0, \quad (272)$$

where

$$\Omega \equiv v' + v(\ln g)_\xi - u(\ln g)_\eta. \quad (273)$$

The two equations (270) and (271) together require that one or more of the conditions $u = 0$, $q' = 0$, or $g_\eta = 0$ be satisfied. When the various possible combinations of these conditions are examined in the light of the restrictions imposed by (269)–(272), the following four distinct cases emerge.

Case I. $u = 0$, $g_\eta \neq 0$. For this case (270) requires also that $v = 0$ and we are led only to the familiar trivial case of flow in parallel straight lines.

Case II. $u = 0$, $g_\eta = 0$. In this case (269) and (272) impose no further conditions, but (267) limits the permissible isometric nets to those for which $\ln g = A\xi + C$ (A and C constants). For $A \neq 0$, the lines $\xi = \text{const.}$ form a set of concentric circles; for $A = 0$, a set of parallel straight lines. We are thus led to the helical and pseudo-plane flow fields obtainable by the general Superposition Principle from plane vortex flows and plane flows in parallel straight lines.

Case III. $u \neq 0$, $q' = 0$. For this case, the streamlines no longer lie in the surfaces $\xi = \text{const.}$ However, the z -component of $\mathbf{W}$ is a constant. By virtue of the Superposition Principle we can limit our attention to the case $q = 0$.

Introducing, for convenience,

$$\mathcal{L}(\xi, \eta) \equiv (\ln g)_\eta \quad (274)$$

we obtain the following necessary conditions from (267), (269), and (272) by differentiation with respect to η :

$$(\ln \mathcal{L}_\xi)_{\xi\eta} = (\ln \mathcal{L}_\xi)_{\eta\eta} = 0, \quad (\ln \mathcal{L}_\eta)_{\xi\eta} = (\ln \mathcal{L}_\eta)_{\eta\eta} = 0, \quad (275), (276)$$

$$\mathcal{L}_{\xi\xi} + \mathcal{L}_{\eta\eta} = 0. \quad (277)$$

This set of necessary conditions, together with (267), leaves the following possibilities for the function $\ln g$:

$$\ln g = \frac{E}{F^2} e^{F\eta} \cos F(\xi - D) + A\xi + B\eta + C, \quad \text{where } F \neq 0; \quad (278)$$

$$\ln g = \frac{1}{2}G(\eta^2 - \xi^2) + H\xi\eta + A\xi + B\eta + C. \quad (279)$$

Substitution of these necessary conditions into the stronger, undifferentiated forms of (269) and (272) finally restricts the choice of $\ln g$ to

$$\ln g = A\xi + B\eta + C. \quad (280)$$

By use of the methods of Section B of Chapter III it is easily shown that (280) implies that *the lines $\xi = \text{const.}$ of the basic coordinate net form a set of logarithmic spirals or their limiting cases* ($A = 0$, radial straight lines; $B = 0$, concentric circles; $A = B = 0$, parallel straight lines).

Substituting (280) and the condition $q = 0$ into (269) and (272), we obtain a pair of ordinary differential equations restricting $u(\xi)$ and $v(\xi)$:

$$Bv + Au + u[\ln u(1 - u^2 - v^2)^{1/(\gamma-1)}]' = 0, \quad (280)$$

$$v' + Av - Bu - \frac{D}{u}(1 - u^2 - v^2) = 0, \quad (281)$$

These equations determine a family of plane gas flow solutions involving three

essential parameters, a point in the (u, v) plane, the constant D , and one of the constants A and B . (One of the constants A and B serves only as a scale factor for ξ .) The special choice $A = 0$ leads to the Generalized Prandtl-Meyer Corner Flows studied in Section E of Chapter III. The choice $D = 0$ leads to the irrotational, plane, spiral flows studied by Tollmien [20].

Case IV. $u \neq 0$, $g_\eta = 0$. Here, as in Case II, the lines $\xi = \text{const.}$ of the basic coordinate net are, respectively, concentric circles or straight lines as $A \neq 0$ or $A = 0$. However in the present case the streamlines do not lie in the surfaces $\xi = \text{const.}$ and the z -component of reduced velocity is in general not constant.

For $A = 0$, the surfaces $\xi = \text{const.}$ are planes parallel to the z -axis. Without loss of generality we may refer our flows to an ordinary Cartesian coordinate system by setting $\xi = x$ and $\eta = y$. The conditions (269), (270), and (272) then yield the following three restrictions on $u(x)$, $v(x)$, and $q(x)$:

$$q' = E(1 - W^2)u^{-1}, \quad v' = F(1 - W^2)u^{-1}, \quad Gu^{-(\gamma-1)} = 1 - W^2. \quad (282), (283), (284)$$

Substituting (284) into (282) and (283) we obtain

$$q' = GEu^{-\gamma}, \quad v' = GFu^{-\gamma}. \quad (285), (286)$$

For $F \neq 0$, we then have

$$q' = \frac{E}{F} v',$$

or

$$q = \frac{E}{F} v + H. \quad (287)$$

Substitution of (287) into (284) yields

$$\left(1 + \frac{E^2}{F^2}\right)v^2 + \left(\frac{2EH}{F}\right)v + (u^2 + Gu^{-(\gamma-1)} + H^2 - 1) = 0. \quad (288)$$

This quadratic equation in v can be solved to give v as a function of u , denoted for convenience by

$$v = f(u); \quad (289)$$

(287) and (289) then give q also as a function of u :

$$q = \frac{E}{F} f(u) + H. \quad (290)$$

Finally, from (286) and (289) we have

$$x = \int_{u_0}^u \frac{1}{GF} \tilde{u}^\gamma f'(\tilde{u}) d\tilde{u}. \quad (291)$$

(289)–(291) constitute the complete solution to this family of flows in terms of u as a parameter. This family of flows is a spatial generalization of the plane flows studied in Case II of Section F of Chapter III.

For $F = 0$, we have $v = \text{const.}$ and by the Superposition Principle the flow can be reduced to a flow in the (x, z) plane with W depending on x alone. This is equivalent to the solution of Case II, Section F, Chapter III.

For $A \neq 0$, the surfaces $\xi = \text{const.}$ are a family of concentric circular cylinders and it is convenient to introduce as independent variable the cylinder radius $r \equiv e^c A^{-1} e^{A\xi}$. The conditions (269), (270), and (272) then yield the following three relations restricting $u(r)$, $v(r)$, and $q(r)$:

$$q' = I(1 - W^2)u^{-1}, \quad (rv)' = J(1 - W^2)u^{-1}, \quad 1 - W^2 = Kr^{-(\gamma-1)}u^{-(\gamma-1)}. \quad (292), (293), (294)$$

Substituting (294) into (292) and (293) we obtain

$$q' = KIr^{-(\gamma-1)}u^{-\gamma}, \quad (rv)' = KJr^{-(\gamma-1)}u^{-\gamma}. \quad (295), (296)$$

For $J \neq 0$, (295) and (296) yield

$$q = \frac{I}{J}rv + L, \quad (297)$$

which substituted into (294) gives

$$\left(1 + \frac{I^2}{J^2}r^2\right)v^2 + \left(\frac{2IL}{J}r\right)v + (u^2 + Kr^{-(\gamma-1)}u^{-(\gamma-1)} + L^2 - 1) = 0. \quad (298)$$

This quadratic equation in v can be solved to obtain v as a function of u and r , for convenience denoted by

$$rv = j(u, r). \quad (299)$$

(297) and (299) now give q as a function of u and r :

$$q = \frac{I}{J}j(u, r) + L, \quad (300)$$

while (296) and (299) yield

$$d(rv) = j_u(u, r) du + j_r(u, r) dr,$$

whence

$$\frac{du}{dr} = \frac{KJr^{-(\gamma-1)}u^{-\gamma} - j_r(u, r)}{j_u(u, r)}. \quad (301)$$

The solutions $u = u(r)$ of the ordinary first order differential equation (301) give with (299) and (300) the desired solutions, spatial generalizations of the plane source flow.

For $J = 0$, we have

$$rv = M, \quad (302)$$

which substituted into (294) gives

$$q^2 + u^2 + Kr^{-(\gamma-1)}u^{-(\gamma-1)} + M^2r^{-2} - 1 = 0. \quad (303)$$

This relation gives q as a function of u and r :

$$q = k(u, r). \quad (304)$$

From (295) and (304) we then have

$$\begin{aligned} dq &= k_u(u, r) du + k_r(u, r) dr, \\ &= KI r^{-(\gamma-1)} u^{-\gamma} dr \end{aligned}$$

whence

$$\frac{du}{dr} = \frac{KI r^{-(\gamma-1)} u^{-\gamma} - k_r(u, r)}{k_u(u, r)}. \quad (305)$$

The integrals $u = u(r)$ of (305), together with (302) and (304), constitute the desired solutions.

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