Remarks on the Preceding Paper by D. Gilbarg

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In the preceding paper D. Gilbarg communicated an important generalization of the Phragmén-Lindelöf theorem to linear elliptic differential equations of second order and in two independent variables. It is our object to prove this theorem for n independent variables. Suppose, as Gilbarg does, that the coefficients of the differential operator

$$(1) \quad L(u) = a_{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} + b_i(x) \frac{\partial u}{\partial x_i}$$

are continuous in the half space $x_n > 0$, $x = (x_1, \dots, x_n)$, and that L is uniformly elliptic in $x_n > 0$:

$$(2) \quad \frac{\text{Max}_{i,k} |a_{ik}(x)|}{\lambda(x)} < A$$

is bounded in $x_n > 0$ if $\lambda(x) > 0$ denotes the smallest proper value of the quadratic form $a_{ik}(x)\xi_i\xi_k$. Uniform ellipticity means, equivalently, that

$$\frac{\Lambda(x)}{\lambda(x)}$$

is bounded in $x_n > 0$ if $\Lambda(x)$ is the greatest proper value.

Theorem. Suppose that the coefficients $a_{ik}(x)$ and $b_i(x)$ are continuous and that L is uniformly elliptic in $x_n > 0$. Suppose moreover that there is a decreasing function $p(r) > 0$ such that

$$(3) \quad \int_0^\infty p(r) dr < \infty, \quad \sum_i \frac{|b_i(x)|}{\lambda(x)} < p(r), \quad r^2 = x_i x_i.$$

Then any solution $u(x)$ of $L(u) = 0$ in $x_n > 0$ which is continuous and ≥ 0 on $x_n = 0$ and which satisfies

$$(4) \quad \lim_{R \rightarrow \infty} \frac{\mu_R}{R} = 0, \quad \mu_R = \min_{r=R} u(x),$$

is ≥ 0 everywhere in $x_n > 0$.

The theorem remains valid if the limit relation (4) is satisfied merely for an infinite sequence $R = R_\nu \rightarrow \infty$. It remains valid also if the condition (3) is imposed merely upon b_n while $x_n b_i(x)/\lambda(x)$ is bounded for $i < n$. Under more relaxed conditions on the a_{ik} and b_i it remains valid if at the same time the growth condition (4) on the solution u is suitably sharpened.

Proof of the theorem. Gilbarg's proof in two dimensions is a remarkable application of the method of differential inequalities which we used twenty-five years ago to prove the sharp theorem on the extrema of the solutions and which is probably not yet fully exploited. The idea is to find a function $v(x) = v_R(x)$ in the hemisphere $r \leq R$, $x_n \geq 0$ with the following properties. v is of class C'' in the open hemisphere and continuous on the boundary except at the edge $r = R$, $x_n = 0$, where it satisfies

$$(5) \quad \begin{aligned} v &= 0, & r < R, & & x_n &= 0, \\ v &= 1, & r &= R, & & x_n > 0; \end{aligned}$$

further, in the open hemisphere

$$(6) \quad 0 < v < 1, \quad L(v) < 0;$$

and, lastly, for each fixed point x

$$(7) \quad Rv_R(x) \text{ is bounded as } R \rightarrow \infty.$$

Let

$$(8) \quad \sigma_R = \min(0, \mu_R) \leq 0.$$

By virtue of the continuity of u on the entire boundary of the hemisphere $r < R$, $x_n > 0$ and its assumed non-negativeness on $x_n = 0$ and by (5) and the first part of (6),

$$u_R(x) = u(x) - \sigma_R v_R(x)$$

is easily seen to have only non-negative limit values on the entire boundary $r = R$ or $x_n = 0$ of the hemisphere. By (6) and (8),

$$L(u_R) = 0 - \sigma_R L(v_R) \leq 0$$

holds for $r < R$, $x_n > 0$. Therefore $u_R(x)$ cannot attain its minimum in the interior of the hemisphere. Hence, $u_R \geq 0$ holds everywhere in it, and for all

large R . In other words, for each fixed x ,

$$u(x) \geq \frac{\sigma_R}{R} Rv_R(x)$$

holds for all large R . But by (4), (8), and (7) this implies that $u \geq 0$ holds everywhere in $x_n > 0$.

The key to the proof is thus the construction of a function $v_R(x)$. Gilbarg's success in choosing it in the form of a function of the harmonic measure in two dimensions seems surprising. Let us see why this works. We confine ourselves at first to the case $R = 1$, i.e. to the semicircular area $r \leq 1$, $x_2 \geq 0$ in the x_1 - x_2 plane, and we consider the arcs in this area of the circles which pass through both corners $r = 1$, $x_2 = 0$. The harmonic measure in this region is the function

$$h(x) = h(x_1, x_2) = \frac{2}{\pi} [\pi - \phi^+(x_1, x_2) + \phi^-(x_1, x_2)]$$

where ϕ^+ , ϕ^- are the polar angles of the point (x_1, x_2) with $(1, 0)$ and $(-1, 0)$ as poles respectively. The simple properties needed are these:

- a) h is constant along each of those arcs,
- b) $h = 0$ on $x_2 = 0$, $h = 1$ on $r = 1$,
- c) $dh/dx_2 > 0$ along the axis of x_2 , $0 \leq x_2 \leq 1$,
- d) along each arc the gradient of h is smallest on the axis of x_2 ,

$$h = \text{const. and } |h'(x)| = \sqrt{\frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_i}} \geq \frac{\partial h}{\partial x_2} \text{ on axis of } x_2.$$

d) is obvious since the level lines are farthest apart on the axis of x_2 . The first and second derivatives of a polar angle are of the order δ^{-1} and δ^{-2} , respectively, where δ is the distance from the pole. Hence and from d), c) and the continuity properties of h' , h'' it follows that

$$(9) \quad \sum_{i,k} \frac{|h''_{ik}(x)|}{|h'(x)|^2} < C, \quad r < 1, \quad x_2 > 0$$

holds with a finite constant C in the semicircular area.

In n dimensions but still in the case $R = 1$, i.e. in the case of the hemispherical region $r \leq 1$, $x_n \geq 0$, we replace the circular arcs by the caps of the hyperspheres which pass through the edge $r = 1$, $x_n = 0$ and which have for extreme members the hemisphere $r = 1$, $x_n \geq 0$ and the plane part $r \leq 1$, $x_n = 0$. We use the function

$$h(x) = h(\rho, x_n), \quad \rho = \sqrt{x_1^2 + \cdots + x_{n-1}^2} = \sqrt{r^2 - x_n^2}$$

where $h(x_1, x_2)$ is the function chosen above for the two-dimensional case. Those hyperspheres are now the level surfaces of h . h is analytic in the closed hemispherical region except on the edge. The appearance of the square root

does not invalidate this fact, since $h(x_1, x_2)$ is analytic and even in x_1 . The n -dimensional h has the properties a)–d) if therein arcs are replaced by hyperspheres and x_2 by x_n . It is easily seen that (9) too holds in the hemispherical region.

We assert that there exists a function

$$f(h) = f_R(h)$$

of a single variable h , $0 \leq h \leq 1$, and of class C'' such that

$$(10) \quad v_R(x) = f_R\left(h\left(\frac{x}{R}\right)\right), \quad r \leq R, \quad x_n \geq 0,$$

has all the required properties. If

$$(11) \quad f_R(0) = 0, \quad f_R(1) = 1$$

then (5) is satisfied. If

$$(12) \quad 0 < f_R < 1, \quad 0 < h < 1,$$

then the first half of (6) is satisfied. If, furthermore,

$$(13) \quad f'_R(h) \text{ is bounded}$$

for all h, R then (7) is satisfied because then

$$Rv_R(x) = \frac{f_R\left(h\left(\frac{x}{R}\right)\right) - f_R(0)}{h\left(\frac{x}{R}\right) - 0} \cdot \frac{h\left(\frac{x}{R}\right)}{\frac{1}{R}}$$

remains bounded as $R \rightarrow \infty$.

We choose for $f(h)$ a solution of the ordinary differential equation

$$(14) \quad \frac{f''(h)}{f'(h)} = -AC - \frac{p(x_n)}{\frac{dh}{dx_n}}, \quad 0 \leq h \leq 1,$$

where A, C are the constants of (2) and (9), where the function p is taken from the hypothesis of the theorem, and where the variables x_n and h are interrelated by the relation

$$(14') \quad h = h\left(\frac{x}{R}\right) \text{ along axis of } x_n, \quad 0 \leq x_n \leq R.$$

This is a one-to-one relation by virtue of c). The solution with the boundary values (11) is easily found. We obtain that

$$(15) \quad f'_R(h) = K \exp(-ACH - P(x_n)), \quad f_R(0) = 0$$

where

$$(16) \quad K^{-1} = \int_0^1 \exp(-ACh - P(x_n)) dh, \quad P(x_n) = \int_0^{x_n} p(t) dt,$$

and where (14') is used. (12) is obviously satisfied. (3) implies that the function P lies between the numerical bounds 0 and $\int_0^\infty p(r) dr$. Hence (13) too is satisfied since

$$f'_R(h) \leq K, \quad K^{-1} > \int_0^1 \exp(-ACh - P(\infty)) dh.$$

What remains to be proved is the decisive differential inequality (6). We have

$$(17) \quad L(v_R) = f''(h)Q(h) + f'(h)L(h), \quad f = f_R,$$

where Q, L are differential operators acting on $h = h(x/R)$ and where

$$Q(h) = a_{ik}(x)h'_i(\xi)h'_k(\xi)R^{-2} > 0, \quad \xi = \frac{x}{R}.$$

Using the least proper value $\lambda(x)$ we obtain

$$(18) \quad Q(h) \geq \lambda(x) |h'(\xi)|^2 R^{-2}.$$

On the other hand,

$$L(h) = a_{ik}(x)h''_{ik}(\xi)R^{-2} + b_i(x)h'_i(\xi)R^{-1}.$$

By (2) and (9), the first sum on the right is in absolute value less than

$$AC\lambda(x) |h'(\xi)|^2 R^{-2}.$$

The second sum is by (3) and by virtue of $|h'_i| \leq |h'|$ absolutely less than

$$\lambda(x) |h'(\xi)| p(r)R^{-1}.$$

Hence and from (17), (18) we infer that

$$(19) \quad \frac{L(v_R)}{Q(h)f'_R(h)} = \frac{f''(h)}{f'(h)} + \frac{L(h)}{Q(h)} < \frac{f''(h)}{f'(h)} + AC + \frac{Rp(r)}{|h'(\xi)|}, \quad \xi = \frac{x}{R}.$$

The third term is estimated from above in keeping h fixed, i.e. on a level surface $h = h(\xi) = h(x/R) = \text{constant}$. r is smallest and, therefore, $p(r)$ largest (p is, by hypothesis, decreasing) on the axis of x_n . $|h'(\xi)|$ is by d) smallest on the same axis,

$$\frac{1}{R} |h'(\xi)| \geq \frac{dh}{dx_n},$$

where $h = h(x/R)$ is taken on that axis. The third term on the right of (19) is therefore less than

$$\frac{p(x_n)}{\frac{dh}{dx_n}}.$$

It is now obvious from (14), (14') that $L(v_R) < 0$ in the region $r < R$, $x_n > 0$. The theorem is thereby completely proved.