The Phragmén-Lindelöf Theorem for Elliptic Partial Differential Equations

DAVID GILBARG

Graduate Institute for Applied Mathematics, Indiana University, Bloomington, Indiana

We consider linear elliptic partial differential equations of the form

$$(1) \quad L(u) \equiv au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y = 0, \quad (a > 0).$$

It is known that these equations obey the maximum principle, namely that a non-constant solution cannot take on a maximum or minimum in the interior of its domain of definition. For harmonic functions a strengthening of the usual maximum principle is provided by the classical Phragmén-Lindelöf theorem. We state the analogue for equation (1) in the following rough form: if a solution of (1) defined in a region lying in a half plane is non-negative on the boundary and is $o(r)$ as $r \rightarrow \infty$, it must be non-negative in the entire region. In the present note this result is proved for solutions of (1) under some mild assumptions concerning the coefficients. The theorem remains true if $L(u)$ is replaced by $L(u) + \lambda u$, where $\lambda(x, y) \leq 0$. The method is patterned after a well known proof of the Phragmén-Lindelöf theorem based on the harmonic measure as majorant.¹ The essential idea of our proof is to exploit the fact that the majorant need not be chosen among solutions of (1) but rather among the wider class of functions obeying only the inequality $L(u) \leq 0$. This is an attack made possible by Hopf's form of the maximum principle,² which asserts that a non-constant function satisfying the inequality $L(u) \leq 0$ cannot take on an interior minimum. Among the functions available as majorant it suffices to find one with certain growth properties possessed in particular by the harmonic majorant of the classical

¹ See R. NEVANLINNA, *Eindeutige Analytische Funktionen*, Springer, Berlin 1936, p. 43.

² E. HOPF, *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Sitzungsberichte Akad. Berlin 1927, 147-152; see also BATEMAN, *Partial Differential Equations*, Cambridge 1932, Dover Publ. 1944, p. 135-137.

Phragmén-Lindelöf theorem, and we even find it possible to select as majorant a simple function of the classical one. From this it is clear that the harmonic character of the latter is not strictly relevant to the proof of either the older or the present theorem. The method of this note applies as well to the case of n -independent variables, as shown in the following paper by E. Hopf.

Having the Phragmén-Lindelöf theorem for (1), we are then able to achieve analogous results for certain classes of non-linear equations. Among the possible applications is a comparison theorem generalizing to compressible fluids a result of Lavrentieff which has proved useful in hydrodynamics.

Conditions on $L(u)$. In $L(u)$ let the domain of definition of u and the coefficients be a region D contained in a half plane. The coefficients a, b, c, d, e are assumed bounded and continuous in every finite portion of D . As $r \rightarrow \infty$ ($r^2 = x^2 + y^2$) we require that a, b, c remain bounded, although they need not approach a limit, while d and e tend to zero as $O(r^{-1-\epsilon})$ for some $\epsilon > 0$ ³. The condition of ellipticity guarantees that at each point $z = (x, y)$ the inequality

$$a\lambda^2 + 2b\lambda\mu + c\mu^2 \geq \alpha(\lambda^2 + \mu^2)$$

is satisfied by some $\alpha > 0$ for all real λ, μ . We make the further *uniform ellipticity* requirement that for some fixed $\alpha > 0$ this inequality holds at every point in the domain D . In view of the boundedness of a, b, c this condition is equivalent to $ac - b^2$ being bounded away from zero in D . Under these hypotheses concerning $L(u)$ we prove the following theorem.

Theorem: Let $u(z)$ satisfy in D the inequality $L(u) \leq 0$, and let the lower limit of $u(z)$ be non-negative as z approaches any point on the boundary of D . Then if

$$\lim_{r \rightarrow \infty} \frac{m(r)}{r} = 0, \quad \text{where} \quad m(r) = \min_{|z|=r} u(z),$$

it follows that $u(z) \geq 0$ throughout D .

Proof: The principal tool in the proof is Hopf's form of the maximum principle mentioned in the opening paragraph; namely, a non-constant function $u(z)$ satisfying the inequality $L(u) \leq 0$ ($L(u) \geq 0$) cannot take on a minimum (maximum) in the interior of its domain of definition. This theorem depends only on the continuity of the coefficients and the local ellipticity of $L(u)$.

Without loss of generality the domain D may be assumed to lie in the half plane $y \geq y_0 > 0$. We seek a function $V_r(z)$ which is defined for each $r > y_0$ in the intersection, G_r , of D with the circle $|z| < r$, and which satisfies the following conditions in this region:

³ It is clear from order of magnitude considerations that for a general growth theorem of the type sought here the coefficients d, e must go to zero at least as rapidly as $1/r$. However, weaker conditions than $O(r^{-1-\epsilon})$ suffice for proof of the theorem.

- (i) $L[V_r(z)] \leq 0$ for $z \in G_r$;
 - (ii) $V_r(z) \geq 0$ on the common boundary of G_r and D , and $V_r(z) = 1$ for $|z| = r$;
 - (iii) for fixed z , $V_r(z) = O(1/r)$ as $r \rightarrow \infty$.
- To see that such a function suffices for the theorem, we consider

$$U_r(z) \equiv u(z) - m(r)V_r(z), \quad z \in G_r,$$

and let us assume contrary to the theorem that $m(r) < 0$ for some r , and hence for all larger r . The hypotheses on $u(z)$ and conditions (i) and (ii) on $V_r(z)$ give us that $L[U_r(z)] \leq 0$ in the interior of G_r , and that $U_r(z) \geq 0$ on the boundary. Thus $U_r(z) \geq 0$ throughout G_r . Now consider $U_r(z)$ for fixed z as $r \rightarrow \infty$. We have

$$u(z) \geq m(r)V_r(z) = o(r)O(1/r) \rightarrow 0 \text{ as } r \rightarrow \infty;$$

hence

$$u(z) \geq 0,$$

thereby contradicting the assumed negativeness of $m(r)$. The proof of the theorem is thus reduced to the determination of a function $V_r(z)$ with properties (i)–(iii).

Let

$$v_r(z) \equiv 2 \left(1 - \frac{\theta(z)}{\pi} \right), \quad |z| \leq r,$$

where, for each circle of radius r about the origin as center, $\theta(z)$ is the angle subtended from the point z by the diameter between $(-r, 0)$ and $(r, 0)$. This is the harmonic majorant which is used in the proof of the classical Phragmén-Lindelöf theorem. The function $v_r(z)$ is positive in G_r and is equal to 1 for $|z| = r$. Since

$$(2) \quad \begin{aligned} v_r(z) &= \frac{2}{\pi} \left(\arccot \cot \frac{y}{r+x} + \arccot \cot \frac{y}{r-x} \right) \\ &\sim \frac{4ry}{\pi(r^2 - x^2)} \quad \text{as } r \rightarrow \infty, \end{aligned}$$

we see that $v_r = O(1/r)$ for fixed z as $r \rightarrow \infty$. Thus v_r satisfies conditions (ii) and (iii) above.

We attempt now to find $V_r(z)$ in the form $V_r(z) = f(v_r)$.⁴ In other words we seek a V_r satisfying (i)–(iii) among those functions that are constant on the circular arcs through $(-r, 0)$ and $(r, 0)$. Because of the harmonic property of v_r the particular form of the function, namely $f(v_r)$, will prove to simplify the

⁴ I am grateful to W. GUSTIN for the suggestion which led to this form of the majorant.

formal details, but is otherwise inessential. Since the function f is to satisfy the inequality

$$f''(av_x^2 + 2bv_xv_y + cv_y^2) + f'L(v) \leq 0$$

in G_r , let it be chosen to obey the stricter inequality,

$$(3) \quad \frac{f''}{f'} \leq -\frac{|L(v)|}{\alpha(v_x^2 + v_y^2)} \leq -\frac{L(v)}{av_x^2 + 2bv_xv_y + cv_y^2} \quad (f' > 0).$$

(The subscript r has been dropped with the understanding that r is fixed in the discussion.) We are thus led to obtain bounds on the right member of the inequality,

$$(4) \quad |L(v)| \leq (a^2 + 2b^2 + c^2)^{\frac{1}{2}}(v_{xx}^2 + 2v_{xy}^2 + v_{yy}^2)^{\frac{1}{2}} + (d^2 + e^2)^{\frac{1}{2}}(v_x^2 + v_y^2)^{\frac{1}{2}}.$$

Since

$$\theta(z) = \operatorname{Im} \log \frac{z-r}{z+r},$$

we have

$$(5) \quad (v_{xx}^2 + 2v_{xy}^2 + v_{yy}^2)^{\frac{1}{2}} = \frac{2\sqrt{2}}{\pi} \left| \frac{d^2}{dz^2} \left(\log \frac{z-r}{z+r} \right) \right| = \frac{2\sqrt{2}}{\pi} \left| \frac{4rz}{(z^2 - r^2)^2} \right|$$

while

$$(6) \quad v_x^2 + v_y^2 = \left(\frac{2}{\pi} \right)^2 \left| \frac{d}{dz} \left(\log \frac{z-r}{z+r} \right) \right|^2 = \left(\frac{2}{\pi} \right)^2 \left| \frac{4r^2}{(z^2 - r^2)^2} \right| \geq \left(\frac{2}{\pi} \right)^2 \frac{1}{r^2}.$$

Also, at each point z in G_r lying on the arc $\theta = \text{const.}$, we have

$$|z| \geq r \cot \frac{\theta}{2} = r \tan \frac{\pi v}{4} \geq \frac{rv}{2}.$$

Consequently, for suitable positive constants K, L , independent of z and r ,

$$(7) \quad (d^2 + e^2)^{\frac{1}{2}} \leq \frac{K}{|z|^{1+\epsilon}} \leq \frac{L}{(rv)^{1+\epsilon}}, \quad (z \in G_r, v = v_r(z)).$$

From (4), (5), (6), (7), and the boundedness of the coefficients a, b, c we obtain at all points in G_r ,

$$(8) \quad \frac{|L(v)|}{\alpha(v_x^2 + v_y^2)} \leq A + \frac{Bv^{-1-\epsilon}}{r^\epsilon}.$$

where A and B are positive constants, independent of z and r .

Let $f(v)$ now be chosen to satisfy the equation,

$$\frac{f''}{f'} = -A - \frac{Bv^{-1-\epsilon}}{r^\epsilon},$$

namely,

$$(9) \quad f(v) = \frac{\int_{v_0}^v \exp [-A\xi + C(r\xi)^{-\epsilon}] d\xi}{\int_{v_0}^1 \exp [-A\xi + C(r\xi)^{-\epsilon}] d\xi}, \quad C = B/\epsilon.$$

The lower limit $v_0 = v_0(r)$ is the minimum value (>0) of v_r on the line $y = y_0$, so that $v_r(z) > v_0$ throughout G_r . On the portion of the boundary where $|z| = r$, we have $f(v_r) = 1$, and on the portion common to G_r and D , $f(v_r) \geq 0$; thus condition (ii) is fulfilled by $V_r = f(v_r)$. From the fact that $f(v)$ satisfies (8) and therefore (3), and since $f'(v) > 0$, it follows that $L[f(v)] \leq 0$ in G_r ; hence condition (i) is also satisfied. Now keeping z fixed, it remains only to establish that $f(v_r) = O(1/r)$ as $r \rightarrow \infty$. It is apparent from (2) that

$$\lim_{r \rightarrow \infty} rv_0(r) = \frac{4y_0}{\pi},$$

therefore in (9)

$$\int_{v_0}^{v_r} \exp [-A\xi + C(r\xi)^{-\epsilon}] d\xi = O\{\exp [C(rv_0)^{-\epsilon}]v_r\} = O\left(\frac{1}{r}\right)$$

as $r \rightarrow \infty$ and $v_r \rightarrow 0$. Thus the numerator in the expression for $f(v_r)$ is $O(1/r)$ while the denominator is bounded away from zero. This gives finally that conditions (i)–(iii) are satisfied by $V_r(z) \equiv f(v_r)$, and from this the theorem follows.

From the preceding proof we infer readily that *the same theorem holds if u satisfies the more general condition,*

$$L(u) + \lambda u \leq 0, \quad \lambda(x, y) \leq 0 \text{ in } D.$$

For, let $V_r(z)$ be the majorant determined previously, and define

$$\begin{aligned} \mu(r) &= \min (0, m(r)) \leq 0 \\ &= o(r); \end{aligned}$$

then we have on the boundary of G_r ,

$$u(z) - \mu(r)V_r(z) \geq 0$$

while in the interior,

$$L[u - \mu(r)V_r] \leq -\lambda u \leq -\lambda[u - \mu(r)V_r].$$

From this it follows that $u(z) - \mu(r)V_r(z)$ cannot be negative in the interior of G_r , and therefore by the same argument as before, $u(z) \geq 0$.

We remark also that the theorem remains true if the hypothesis requires only the existence of some sequence $r_i \rightarrow \infty$ for which

$$\lim_{r_i \rightarrow \infty} \frac{m(r_i)}{r_i} = 0.$$

The analogous theorem can be stated for any equation of the form (1) whether linear or not provided of course that the coefficients have the required behavior as functions of x and y for the solutions under consideration.

An example of particular interest for its applications to gas dynamics arises from quasi-linear equations of the form

$$(10) \quad au_{xx} + 2bu_{xy} + cu_{yy} = 0$$

where a, b, c may be functions of x, y, u_x, u_y , but not of u . The equations for the stream and potential function in the irrotational plane flow of a compressible fluid are of this type. Let two solutions u and v be given in a domain D contained in a half-plane, and let these functions and the coefficients a, b, c satisfy the following conditions:

(i) The coefficients a, b, c corresponding to u are bounded and obey the uniform ellipticity condition,

$$ac - b^2 \geq \alpha > 0,$$

where α is a constant independent of x, y . The coefficients are continuously differentiable in their arguments with uniformly bounded derivatives for all values of the arguments in D .

(ii) The difference $v - u$ is $o(r)$ as $r \rightarrow \infty$, and the second derivatives of v are $O(r^{-1-\epsilon})$ for some $\epsilon > 0$ as $r \rightarrow \infty$.

(iii) $v \geq u$ on the boundary of D .

Under these assumptions we have that $v \geq u$ throughout (D) . If the inequality is reversed in (iii) then it is reversed also in the conclusion. (This is not as immediately evident as in the linear case.)

We remark that the character of the differential equation (10) with respect to the second solution v plays no part in the theorem, and that the theorem may in principle compare solutions, one of which is elliptic and the other say mixed elliptic-hyperbolic.

The proof is based on a well known artifice which displays the difference $\omega \equiv v - u$ as a solution of a linear equation of the form (1). With the notation, $a(u) \equiv a(x, y, u_x, u_y)$, etc., we have

$$\begin{aligned} 0 &= a(u + \omega)(u_{xx} + \omega_{xx}) + \cdots - \{a(u)u_{xx} + \cdots\} \\ &= a(u)\omega_{xx} + 2b(u)\omega_{xy} + c(u)\omega_{yy} + [a(u + \omega) - a(u)]v_{xx} + \cdots. \end{aligned}$$

If we apply the mean value theorem to the bracketed expressions, while recalling the regularity conditions on a , b , c and the assumed behavior of v_{xx} , v_{xy} , v_{yy} as $r \rightarrow \infty$, we see that ω is the solution of a linear equation of type (1) whose coefficients satisfy the conditions of the main theorem. Since ω itself also meets the hypotheses of the theorem the required result follows.

As an application of the above result we have the following comparison theorem, for which the details are omitted here.

Consider two irrotational flows of a compressible medium, both of the flows being purely subsonic, and both flows having the same uniform velocity at infinity with the same limit values for all flow quantities. Let D and $\bar{D}$ be the two flow regions, lying in a half-plane, such that D is contained in $\bar{D}$ and each is bounded by a single streamline extending to $x = \pm \infty$; let these two streamlines have a regular point P in common. Then, designating by ρ and $\bar{\rho}$ the respective densities, and by q and $\bar{q}$ the flow speeds, we have at P the inequality

$$(\rho q)_P \leq (\bar{\rho} \bar{q})_P ;$$

the equality holds if and only if the two flows are identical.

This is the analogue for compressible fluids of a comparison theorem of Lavrentieff⁵ which has proved exceedingly valuable in the free boundary problem of hydrodynamics. A study of the above theorem and of its applications to gas dynamics will be the subject of a later work.

⁵ M. LAVRENTIEFF, *On certain properties of univalent functions and their application to wake theory* (in Russian), Mat. Sbornik, **46**, 391–458 (1938); also J. SERRIN, *Existence theorems for some hydrodynamical free boundary problems*, Journal of Rational Mechanics and Analysis, **1**, 1–48 (1952), and D. GILBARG, *Uniqueness of axially symmetric flows with free boundaries*, Journal of Rational Mechanics and Analysis, **1**, 309–320 (1952).