Extremal Methods in Cavitation Flow

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1. Introduction. In recent years a great interest has been revived in the classical free surface flow problems of hydrodynamics. The advent of missiles moving through water at high velocities has brought the need for a more extensive theory of cavitation flow. While the classical Helmholtz theory of constant pressure free surfaces was not found to agree with experimental evidence on water wakes, the new data on flows past gas or water vapor cavities has been in good agreement with the mathematical theory. The study of free surfaces in a gravity field has also received a new stimulus from the wide variety of new problems on surface waves. Last, but not least, the entry into water of objects projected from the air yields a theoretical free surface problem of practical importance.

We attempt in this paper to make available for the investigation of these hydrodynamical problems the modern theory of variational techniques in conformal mapping. We derive a number of extremal properties of the flow past constant pressure free surfaces, and we use these properties to study the flow. To a large extent it is found that the minimum problems which we associate with free stream flow are of a type which has already appeared in the more recent work on variational methods in conformal mapping [1, 4].

A decisive step towards the application of minimum problems to the determination of free surfaces was made by Friedrichs [3], who based uniqueness of the free surface on such a method. Schiffer was the first to point out that an extremal problem might be used to obtain the existence of cavitation flow, and it is to discussions with him that we owe the original idea of working in this direction. The principle of minimum virtual mass for free surfaces, of which we make extensive use here, is a direct result of Schiffer's theory. We use this principle in a form which combines Friedrichs' approach through a wide class of competing functions and the Dirichlet principle with Schiffer's approach through the narrower class of harmonic functions.¹

The first two sections of the paper are devoted to a formal presentation of the basic principles of minimum virtual mass and of maximum least velocity for free surfaces, and they are intended for the reader with wider interests. The next section takes up a few results on the behavior of virtual mass under symmetrization which are of importance for the variational approach to hydrodynamics. There follow three sections in which the existence of plane cavitation flows and free surface flows in a gravity field is obtained by the principle of minimum energy. We remark here that after we obtained these existence proofs, Schiffer indicated to us an independent method by which he arrived at similar results for cavitation flow. His method appears to be more general for plane flow, whereas

¹ Since this paper was submitted for publication, the authors' attention has been called to a paper by D. RIABOUCHINSKY, "Sur un problème de variation," *C. R.*, Vol. 185 (1927), pp. 840-841, in which the principle of minimum virtual mass appears to have been stated for the first time.

ours, which is based on symmetrization, would seem to be better adapted for a generalization to the study of axially symmetric free surface flow. In our treatment of the free flow in a gravity field, we make decisive use of an argument due to Lewy [7] which proves the analyticity of the free streamline.

In sections 8 and 9, we develop the existence of plane cavital flows by a method based on the principle of maximum least velocity. The proofs are quite short, but are given in a form which fails to indicate the real scope of the procedure. The work here uses a lemma of Lavrentieff [5] which has proved to be of the utmost help throughout our researches. We mention in this connection the work of Serrin [9], in which Lavrentieff's lemma is exploited in a very simple form.

The question of existence and uniqueness can then be handled by our method of minimum virtual mass and Friedrichs' paper [3] using the principle of minimum energy. An alternate treatment of both existence and uniqueness is available from our principle of maximum least velocity and Lavrentieff's lemma for comparison of velocities.

Section 10 is devoted to a discussion of generalizations to axially symmetric free surface flow of the methods developed earlier in detail for plane flow. Some progress can be reported in this direction, and especially we discuss computational procedures which can be based on the minimum principles. A final section takes up a few remarks on the problem of water entry.

We note at this point that the number of physical models to which we have applied our extremal principles is quite limited. Far wider classes of problems can be treated using the same methods, but in a first report on the subject it seemed most appropriate to illustrate the main ideas with a few examples where they appear in the simplest context. Thus no pretense is made at a complete discussion of all results that can be obtained, and no pretense is made at giving theorems in a form with the weakest or most general hypotheses and assumptions.

2. The principle of minimum virtual mass. Let us consider the flow of an incompressible, irrotational fluid past an object B in three-dimensional space. The flow is completely described in terms of its velocity potential $\phi = \phi(x, y, z)$, which is a harmonic function whose gradient is the velocity of the fluid. We shall denote by C the boundary of B and by D the region exterior to B , or region of flow. If we suppose that the fluid moves at infinite distance with uniform velocity 1 in the direction of the positive x -axis, then ϕ can be defined in precise mathematical terms to be that function which is harmonic in D , has near infinity an expansion

$$(2.1) \quad \phi = x + \frac{ax}{r^3} + \cdots, \quad r^2 = x^2 + y^2 + z^2,$$

and has on the boundary C a vanishing outer normal derivative,

$$\frac{\partial \phi}{\partial n} = 0.$$

We shall be concerned here with the study of flows about an object B which consists in part of cavities, or wakes. This is a classical problem of Helmholtz which derives a new significance from the agreement of its mathematical solution with experimental data obtained for high velocity flow past objects followed by cavities of air or water vapor. Indeed, cavitation flow of water means a flow with such high velocity that the pressure falls to vapor pressure and a cavity of water vapor appears.

Let us denote by W that portion of B which consists of the cavity or cavities, and let L denote the part of C bounding W . Then in W , the pressure p will be constant, and hence on L we obtain from Bernoulli's law

$$(2.2) \quad \frac{1}{2}(\nabla \phi)^2 + p = \text{const.}$$

the condition that the velocity $|\nabla \phi|$ is constant. This additional boundary condition must be satisfied by suitable choice of the unknown free surface L , and it is to the determination of L that we devote our attention in this paper.

The principal result of this section is based on an alternate formulation of the above free surface problem in terms of minimum energy. We suppose that there is originally in space a uniform flow in the direction of the x -axis with velocity potential x . The flow past B with cavities W is obtained when we disturb the original flow by superposition of the flow with velocity potential $\phi - x$. The kinetic energy

$$(2.3) \quad M = \iiint_D (\nabla \phi - \nabla x)^2 dx dy dz$$

required to set the new flow in motion is termed the virtual mass of B in the direction of the x -axis. We shall show here that the free surface L is characterized by the property that the virtual mass M is a minimum for fixed volume

$$A = \iiint_W dx dy dz$$

of the cavity. Thus to find L we must choose amongst all shapes for W with a fixed volume that which yields a flow past the complete object B with smallest virtual mass M .

In order to show the equivalence of the new formulation of the problem in terms of virtual mass with the classical condition of constant velocity on L , we need to develop a number of identities and variational formulas (cf. [12]).

Let us transform (2.3) by Green's theorem. We obtain from the condition

that the normal derivative of ϕ vanishes on C

$$\begin{aligned} M &= \iint_C (\phi - x) \frac{\partial(\phi - x)}{\partial n} d\sigma \\ &= \iint_C \left\{ \phi \frac{\partial\phi}{\partial n} - x \frac{\partial\phi}{\partial n} - \phi \frac{\partial x}{\partial n} + x \frac{\partial x}{\partial n} \right\} d\sigma \\ &= \iint_C \left\{ x \frac{\partial\phi}{\partial n} - \phi \frac{\partial x}{\partial n} \right\} d\sigma + \iint_C x \frac{\partial x}{\partial n} d\sigma, \end{aligned}$$

where $d\sigma$ denotes the surface element of C . Since the first integral is independent of the surface of integration, it can be evaluated over a large sphere at infinite distance to yield

$$(2.4) \quad \iint_C \left\{ x \frac{\partial\phi}{\partial n} - \phi \frac{\partial x}{\partial n} \right\} d\sigma = 4\pi a,$$

where a is the coefficient appearing in the expansion (2.1). The second integral merely represents the negative of the volume V of B ,

$$\iint_C x \frac{\partial x}{\partial n} d\sigma = - \iiint_B dx dy dz = -V.$$

Thus we obtain the classical formula

$$(2.5) \quad a = \frac{1}{4\pi} \{M + V\}$$

expressing a in terms of the sum of the mass and the virtual mass of B .

We now proceed to discuss variational formulas for a and V under the assumption that C is an analytic surface and that ϕ is harmonic in a region including C . We suppose given a function ρ on C which is analytic as a real function of the real variables x, y, z . For small values of the real parameter ϵ , we construct the surface C^* obtained by shifting each point of C a distance $\delta n = \epsilon\rho$ along the outer normal n of C with respect to D . The surface C^* bounds new regions D^* and B^* which are infinitesimal variations of D and B . We define ϕ^* to be the velocity potential in D^* of the flow past B^* with uniform velocity 1 near infinity,

$$\phi^* = x + \frac{a^*x}{r^3} + \cdots,$$

and we define V^* to be the volume of B^* .

It is quite evident that the variation of volume V in the above shift has the form

$$(2.6) \quad \delta V = V^* - V = - \iint_C \delta n d\sigma + o(\epsilon),$$

where $o(\epsilon)$ denotes higher order terms such that

$$\lim_{\epsilon \rightarrow 0} \frac{o(\epsilon)}{\epsilon} = 0.$$

The determination of a similar formula for $\delta a = a^* - a$ is a more difficult question to which we must proceed in some detail.

Let R denote the exterior of a large sphere S contained in D and D^* . Then on the one hand we have

$$\begin{aligned} \iiint_{D^*-R} \nabla \phi \nabla \phi^* dx dy dz - \iiint_{D-R} \nabla \phi \nabla \phi^* dx dy dz \\ = \iint_C \nabla \phi \nabla \phi^* \delta n d\sigma + o(\epsilon) = \iint_C (\nabla \phi)^2 \delta n d\sigma + o(\epsilon), \end{aligned}$$

since $\nabla \phi$ and $\nabla \phi^*$ differ very little on C . On the other hand, by Green's theorem

$$\begin{aligned} \iiint_{D^*-R} \nabla \phi \nabla \phi^* dx dy dz - \iiint_{D-R} \nabla \phi \nabla \phi^* dx dy dz \\ = \iint_{C^*} \phi \frac{\partial \phi^*}{\partial n} d\sigma - \iint_C \phi^* \frac{\partial \phi}{\partial n} d\sigma + \iint_S \left\{ \phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right\} d\sigma \\ = \iint_S \left\{ \phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right\} d\sigma, \end{aligned}$$

since

$$\frac{\partial \phi}{\partial n} = 0 \text{ on } C, \quad \frac{\partial \phi^*}{\partial n} = 0 \text{ on } C^*.$$

Using the fact that the integral over S is independent of the surface of integration, we find

$$\iint_S \left\{ \phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right\} d\sigma = 4\pi a - 4\pi a^*.$$

Combining these formulas, we obtain the final result

$$(2.7) \quad \delta a = a^* - a = -\frac{1}{4\pi} \iint_C (\nabla \phi)^2 \delta n d\sigma + o(\epsilon).$$

From (2.6) and (2.7) we derive immediately the formula

$$(2.8) \quad \delta M = \iint_C \{1 - (\nabla \phi)^2\} \delta n d\sigma + o(\epsilon)$$

for the variation of the virtual mass, in view of the identity (2.5).

We are now in a position to attack our original extremal problem for the free surface L . Thus suppose that L has such a shape that for fixed volume A of the cavities W , the virtual mass M of the complete object B is a minimum. Of course, the solid portion of the object, $B - W$, is fixed throughout the discussion, and therefore we may only perform our variations on the variable free surface L . In this case,

$$\delta A = \delta V.$$

Now the method of Lagrange multipliers for isoperimetric problems in the calculus of variations states that there exists a proportionality constant μ with

$$\delta M = \mu \delta A$$

for every variation of L . Thus by (2.6) and (2.8) we find that

$$\iint_L \{1 - (\nabla\phi)^2\} \delta n \, d\sigma + \mu \iint_L \delta n \, d\sigma = 0$$

for every function δn on L which vanishes on those portions of C bounding the solid object $B - W$. We conclude from the arbitrary nature of δn on L that the identity

$$(2.9) \quad (\nabla\phi)^2 - 1 - \mu = 0$$

must hold there. But this is precisely the condition of constant velocity on the free surface L arrived at earlier using Bernoulli's equation. Thus we have demonstrated the equivalence of the minimum problem and the classical Helmholtz free boundary problem.

Before proceeding to further formulations of the extremum problem, let us discuss a special example in which it is heuristically clear that a minimum configuration exists. We take for the solid portion of B two circular discs of equal size placed perpendicular to the x -axis with their centers on the axis. The surface L is supposed to extend from the circumference of the leading disc to that of the tail disc, and the whole interior of the object is supposed to consist of a water vapor cavity. We restrict L further to lie between the two planes containing the discs. Thus L can extend to large distances only in directions perpendicular to that of the flow, in which case the virtual mass increases indefinitely, and indeed at a rate exceeding that at which the volume of the cavity can grow. Thus we are assured of a finite cavity for the minimum configuration, and the extremal free surface flow will be axially symmetric about the x -axis. An analogous two-dimensional model has been introduced by Riabouchinsky and gives rise to calculations which are in good agreement with experimental data for cavitation flow.

The principle of minimum virtual mass as formulated thus far gives for each volume A of the cavity a free surface flow. A more natural parameter than volume of the cavity from the physical point of view would be the square of the

velocity of the fluid along the free surface. Indeed, since the velocity at infinity is unity, this parameter is a linear function of the cavitation constant. To obtain a given value λ for the square of the velocity on the free stream L , it suffices to choose L in such a way that

$$(2.10) \quad 4\pi a - \lambda A = \text{minimum.}$$

This extremal problem has a sense when the cavity is restrained to lie between a fixed nose and a fixed tail, since a is of higher order of magnitude than A for a variable cavity which stretches to infinity in a direction perpendicular to the flow. For the extremal surface L we obtain from the variational formulas (2.6) and (2.7)

$$\delta(4\pi a - \lambda A) = - \iint_c \{(\nabla\phi)^2 - \lambda\} \delta n \, d\sigma + o(\epsilon) = 0.$$

Since δn is arbitrary on L , we obtain, upon letting ϵ tend to zero, the relation

$$(2.11) \quad (\nabla\phi)^2 = \lambda, \quad \text{on } L,$$

as was desired.

For a given free surface L , the parameters λ and μ in (2.9) and (2.11) are connected by the relation

$$\lambda = 1 + \mu.$$

Thus the extremal problem (2.10) can also be stated in the form

$$(2.12) \quad \frac{1}{2}M - \frac{1}{2}\mu A = \text{minimum.}$$

If we normalize the pressure p so that $p = \mu/2$ in the cavity W , then the second term in (2.12) can be interpreted as a potential energy of the cavity, since it represents pressure times volume. Therefore (2.12) means simply that the difference between kinetic energy and potential energy is a minimum. Our work with the variational formulas, on the other hand, shows that the boundary condition (2.2) with constant pressure on L is equivalent to the statement that this difference of energies is stationary.

It is natural at this stage to attempt generalizing the minimum principle to include flows in a gravity field. Such flows with a free surface occur, for example, in the theory of surface waves. If g denotes the gravitational constant and gravity acts in the direction of the negative y -axis, Bernoulli's equation now becomes

$$(2.13) \quad \frac{1}{2}(\nabla\phi)^2 + gy + p = \text{const.}$$

Thus on a free surface L of constant pressure in a gravity field, one obtains the non-linear boundary condition

$$(2.14) \quad \frac{1}{2}(\nabla\phi)^2 + gy = \text{const.}$$

The principle of minimum energy contained in formula (2.12) is generalized to this case by introduction of the term

$$E = \iiint_w gy \, dx \, dy \, dz,$$

which represents the increase of potential energy of the fluid caused by the expansion of vapor in the cavity. We assume here that the fluid displaced by the cavity has been raised, and that the region of flow lies above the free surface L with respect to the direction of increasing y .

The variational formula for E is obviously

$$(2.15) \quad \delta E = - \iint_L gy \, \delta n \, d\sigma + o(\epsilon).$$

We choose the free surface L so that

$$(2.16) \quad \frac{1}{2}M + E - \frac{1}{2}\mu A = \text{minimum},$$

and we prove that on such an extremal surface (2.14) is satisfied. In fact, the variational formulas (2.6), (2.8) and (2.15) can be combined to yield

$$\delta\{\frac{1}{2}M + E - \frac{1}{2}\mu A\} = \iint_L \left\{ \frac{1+\mu}{2} - \frac{1}{2}(\nabla\phi)^2 - gy \right\} \delta n \, d\sigma + o(\epsilon),$$

and since for the extremal surface L this expression is non-negative with an arbitrary displacement δn , we obtain the relation

$$(2.17) \quad \frac{1}{2}(\nabla\phi)^2 + gy = \frac{1+\mu}{2}.$$

This equation is equivalent to (2.14), and thus it becomes clear that (2.16) provides an extremal characterization for gravity flows with a free stream surface L .

The general remarks which we have developed up to this point have immediate application to the important special cases of two-dimensional plane flow past an object and axially symmetric flow past an object. By a limiting process, they can further be extended to the classical Helmholtz flow with an infinite cavity. However, in this section we shall confine ourselves to formal discussion of only one further type of flow, namely, the plane flow with fixed circulation about a body due to a vortex at infinity.

The stream function ψ of such a flow is found to be the Green's function G of the region of flow D , with logarithmic pole at infinity. We recall that the Green's function G is harmonic in D , vanishes on the boundary C of D , and has near infinity an expansion

$$(2.18) \quad G(z) = \log |z| - \gamma + \cdots, \quad z = x + iy,$$

where the dots indicate regular terms which vanish at infinity. The constant γ is usually referred to as the logarithmic capacity of the object B enclosed by the curve C .

This constant γ can also be viewed as a modified virtual mass, since it is closely related to the kinetic energy

$$\iint_D (\nabla G - \nabla \log |z|)^2 dx dy$$

required in order to shift the uniformly rotating flow with stream function $\log |z|$ into the circulating flow about B described by the stream function G . Since G vanishes on C , Green's theorem yields here the identity

$$\begin{aligned} \iint_D (\nabla G - \nabla \log |z|)^2 dx dy &= \int_C (G - \log |z|) \frac{\partial(G - \log |z|)}{\partial n} ds \\ &= \int_C \left\{ G \frac{\partial G}{\partial n} - \log |z| \frac{\partial G}{\partial n} - G \frac{\partial \log |z|}{\partial n} + \log |z| \frac{\partial \log |z|}{\partial n} \right\} ds \\ &= \int_C \left\{ G \frac{\partial \log |z|}{\partial n} - \log |z| \frac{\partial G}{\partial n} \right\} ds + \int_C \log |z| \frac{\partial \log |z|}{\partial n} ds \\ &= -2\pi\gamma + \int_C \log |z| \frac{\partial \log |z|}{\partial n} ds, \quad ds^2 = dx^2 + dy^2, \end{aligned}$$

provided the origin lies in B . Thus $2\pi\gamma$ differs from the virtual mass only by the logarithmic area of B .

Let us now suppose that B consists of a fixed object B' plus certain variable cavities W . For fixed area A of W , we choose C in such a way that the capacity γ of $B = B' + W$ is a minimum, and we prove that as a result the arcs L of C bounding W yield a free streamline for the flow about B in the sense that

$$(2.19) \quad (\nabla G)^2 = \text{const.}, \quad \text{on } L.$$

Thus the curve C which encloses B' together with an additional set W of fixed area A and has minimum capacity provides the outermost streamline of a circulating flow about B in which W appears as a cavity bounded by arcs at constant pressure.

To prove this statement, we note first that a shift of L along the outer normal by $\delta n = \epsilon\rho$ units can be performed for small values of ϵ . This yields a varied curve C^* and a new region of flow D^* with stream function G^* . The variation of area A caused by this shift can be written in the form

$$(2.20) \quad \delta A = - \int_L \delta n ds + o(\epsilon).$$

To find the variation of γ , we cut the exterior R of a large circle S from D and D^* and obtain, as in the derivation of (2.7),

$$\begin{aligned} \iint_{D^*-R} \nabla G \nabla G^* \, dx \, dy - \iint_{D-R} \nabla G \nabla G^* \, dx \, dy \\ = \int_C \nabla G \nabla G^* \delta n \, ds + o(\epsilon) = \int_C (\nabla G)^2 \delta n \, ds + o(\epsilon). \end{aligned}$$

On the other hand, by Green's theorem

$$\begin{aligned} \iint_{D^*-R} \nabla G \nabla G^* \, dx \, dy - \iint_{D-R} \nabla G \nabla G^* \, dx \, dy \\ = \int_{C^*} G^* \frac{\partial G}{\partial n} \, ds - \int_C G \frac{\partial G^*}{\partial n} \, ds + \int_s \left\{ G^* \frac{\partial G}{\partial n} - G \frac{\partial G^*}{\partial n} \right\} ds \\ = \int_s \left\{ G^* \frac{\partial G}{\partial n} - G \frac{\partial G^*}{\partial n} \right\} ds = 2\pi\gamma - 2\pi\gamma^*, \end{aligned}$$

since $G = 0$ on C and $G^* = 0$ on C^* . Thus the final result becomes

$$(2.21) \quad \delta\gamma = \gamma^* - \gamma = -\frac{1}{2\pi} \int_L (\nabla G)^2 \delta n \, ds + o(\epsilon).$$

For the extremal arc L there is a Lagrange multiplier λ such that

$$2\pi\delta\gamma = \lambda\delta A$$

for any variation. Hence

$$(2.22) \quad \int_C \{(\nabla G)^2 - \lambda\} \delta n \, ds + o(\epsilon) = 0$$

for any shift δn of L , and we obtain by letting ϵ tend to zero the relation

$$(\nabla G)^2 = \lambda, \quad \text{on } L,$$

equivalent to (2.19). This yields the desired characterization of the free surface L as an arc of minimum capacity.

It is evident that the variational method described in this section for determining free stream surfaces applies to many special cases which we do not treat here, such as flow from a nozzle or flow past an object in a channel. Our chief aim in this paper will be to apply and illustrate the method with a few examples where it is most easily understood, and it is to be hoped that the reader will be able to carry through for himself the corresponding theory for the many particular problems which arise in practice.

3. The principle of maximum least velocity. Cavitational flow occurs in hydrodynamics when the pressure falls to the vapor pressure of water, so that the water boils and a vapor cavity appears. Bernoulli's law shows that this phenome-

non will occur for high velocity flows, and that the high velocities will appear on the free surface bounding the cavity. Thus it seems natural to attempt a characterization of cavitation flow in terms of an extremal problem which states that velocity on the free streamline is as large as possible.

In our work in this direction we shall consider only plane flows and axially symmetric flows, in which case there is in addition to a velocity potential ϕ also a stream function ψ . For the plane flows, the complex potential $\zeta = \phi + i\psi$ is an analytic function of the complex variable z in the physical plane. For flows with axial symmetry in the x -axis, ϕ and ψ are connected by the generalized Cauchy-Riemann equations

$$\frac{\partial \phi}{\partial x} = \frac{1}{y} \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{1}{y} \frac{\partial \psi}{\partial x},$$

where y represents the radial distance in cylindrical coordinates. In this case, ψ therefore satisfies the second order elliptic partial differential equation

$$\frac{\partial^2 \psi}{\partial y^2} - \frac{1}{y} \frac{\partial \psi}{\partial y} + \frac{\partial^2 \psi}{\partial x^2} = 0,$$

and it is well known that any solution of this equation in a region D assumes its maximum and minimum values on the boundary C of D . We shall refer to this result as the maximum principle.

Let us suppose that we are given in the (x, y) -plane a pair of arcs C_1 and C_2 which are symmetric in the x -axis and are reflected images of one another in the y -axis. Let L_1 and L_2 denote two arcs joining the end-points of C_1 and C_2 which are reflected images of one another in the x -axis, and denote by C the simple closed curve made up of all four arcs. We shall consider here the plane flow past C with uniform velocity 1 at infinity in the horizontal direction, and we shall consider also the axially symmetric flow past the surface generated by rotation of C about the x -axis and having uniform velocity 1 at infinity. Both these flows have stream functions ψ which vanish on C and on the x -axis. For the plane flow, ψ has near infinity the form

$$\psi = y - \frac{ay}{r^2} + \cdots, \quad r^2 = x^2 + y^2,$$

while in the axially symmetric case

$$\psi = \frac{y^2}{2} - \frac{by^2}{r^4} + \cdots$$

in the neighborhood of infinity. The square of the velocity of the plane flow is

$$v^2 = (\nabla \psi)^2,$$

while for the axially symmetric flow

$$v^2 = \frac{1}{y^2} (\nabla\psi)^2.$$

We suppose that C_1 and C_2 are fixed, while $L = L_1 + L_2$ is allowed to vary. Let us denote by α the minimum of v^2 for values of (x, y) lying on L . Then let us choose L in such a way that α is a maximum. We propose to show that if this is done, the extremal arcs L which we obtain are free streamlines for the flow past C in the sense that v^2 , and hence the pressure p , is constant on L . This will be proved in both the case of plane flow and the case of axial symmetry. Thus we can construct cavitation flows between the plates C_1 and C_2 with a cavity consisting of the entire interior of C if we determine the free streamlines L so that

$$(3.1) \quad \min_L v^2 = \text{maximum}.$$

The main idea of the proof of this extremal property of free streamlines is the following. Suppose that α is the minimum of v^2 for the extremal curves L , and suppose that there are arcs Γ of L on which $v^2 > \alpha$. We shift Γ by a small amount into the interior of C with symmetry in the x -axis, thus obtaining new arcs Γ^* and a new pair of curves L^* . For a small enough shift, the velocity v^* of the flow past C_1 , C_2 and L^* will satisfy on Γ^* the inequality $v^{*2} > \alpha$. On the other hand, if we denote by ψ^* the stream function of the new flow, then by the maximum principle $\psi^* \geq 0$ on Γ in the upper half-plane $y > 0$, with the strong inequality holding in at least one point. Hence $\psi^* - \psi > 0$ in the portion of the original flow region which lies above the x -axis, since $\psi^* - \psi \geq 0$ on the boundary of this region and since strong inequality holds on part of that boundary.

Now $\psi = \psi^* = 0$ on $L - \Gamma$, and thus we obtain there

$$(\nabla\psi)^2 < (\nabla\psi^*)^2,$$

by virtue of the inequality $\psi^* - \psi > 0$ just proved. It follows immediately that $v^{*2} > \alpha$ on $L - \Gamma$, or, what is the same thing, on $L^* - \Gamma^*$. Since this inequality is already known to hold on Γ^* , we find that

$$\min_{L^*} v^{*2} > \alpha = \min_L v^2.$$

But this contradicts the extremal characterization of L , and we conclude that $v^2 = \alpha$ on L . This shows, finally, that L is a free streamline in the sense that pressure and velocity are constant on L .

The above extremal characterization can be developed for many other types of free streamline flows. Among these are the classical Helmholtz flow, the flow from a nozzle, and the flow about an object with circulation due to a vortex at infinity. Rather than outline all these applications, however, we illustrate a

modification of the basic idea with an example which represents, not true cavitation flow, but flow with a wake of gas at higher pressure.

Let C denote a fixed simple closed curve which is symmetric in the x -axis, and let L denote a pair of variable arcs joining two fixed points P_1 and P_2 on C to a variable point Q on the x -axis. For points P_1 and P_2 sufficiently close to the x -axis, we can choose L so that the velocity v of the flow past the body bounded by C and L has on L values less than one. Here the flow past L is taken to have uniform velocity 1 in the horizontal direction at infinity, and it can be either axially symmetric or plane flow.

We denote by β the maximum of v^2 on the variable arcs L , and we choose L so that β is a minimum. We maintain that for this choice of L , v^2 is constant on L , so that we obtain in effect a pair of free streamlines bounding a wake behind C . In other words, the extremal problem

$$(3.2) \quad \max_L v^2 = \text{minimum}$$

characterizes L as the free stream boundary of a finite wake behind C ending in a cusp on the x -axis.

To prove the above statement, we suppose that on the extremal curves L there are arcs Γ where $v^2 < \beta$. We make an infinitesimal shift of these arcs to a new position Γ^* inside the region of flow in such a way that symmetry in the x -axis is preserved. This yields a new set of curves L^* and a corresponding new flow with stream function ψ^* . For a small enough shift, the velocity v^* of the new flow still satisfies on Γ^* the inequality $v^{*2} < \beta$.

By the maximum principle, the stream function ψ of the original flow is positive on Γ^* in the upper half-plane. Hence, again by the maximum principle, $\psi > \psi^*$ in that portion of the new flow region which lies above the x -axis, since $\psi \geq \psi^*$ on the boundary of this region, with strong inequality holding along Γ^* . Since $\psi = \psi^* = 0$ on $L^* - \Gamma^* = L - \Gamma$, we have there

$$(\nabla\psi)^2 > (\nabla\psi^*)^2.$$

It follows that

$$\max_{L^*} v^{*2} < \beta = \max_L v^2,$$

and thus we obtain a contradiction of the extremal characterization of L . The conclusion is that $v^2 \equiv \beta$ on L , which is therefore a free streamline in the sense that pressure is constant along L , as stated.

The present discussion is intended to give a heuristic idea of the extremal nature of a free surface. We shall return later to a more rigorous treatment of the mathematical theory which can be based on these observations.

4. Symmetrization and virtual mass. Steiner symmetrization of an object B with respect to a plane T is defined as follows. Each line Q perpendicular to T

intersects B in a set q of linear measure b . This set is replaced by a line segment m of Q which is bisected by T and is b units long. The set which consists of all the segments m is defined to be the symmetrization B^* of the original object B .

The symmetrized object B^* is by its very definition symmetric in T ; it has the further property that each line perpendicular to T intersects it in a line segment. It is obvious that the volume of B^* is the same as the volume of B . Steiner's fundamental theorem states that the area of the surface C^* of B^* is not greater than the area of the surface C of B . Thus if $z = f(x, y)$ is the equation of C and $z = f^*(x, y)$ is the equation of C^* , then

$$(4.1) \quad \iint_C \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy \geq \iint_{C^*} \sqrt{1 + f_x^{*2} + f_y^{*2}} \, dx \, dy.$$

Similar statements hold in 2-dimensional space and in 4-dimensional space or in higher spaces. Thus in the plane, symmetrization preserves the area of a figure and decreases the length of its perimeter.

The behavior of length, area and volume under symmetrization is a classical study which is well known from its application to the isoperimetric inequality. Recently it has been shown that many physical quantities change monotonically with symmetrization [8]. In particular, the logarithmic capacity of a plane object B decreases when B is symmetrized with respect to any line.

It will be our object here to investigate the change of the virtual mass M of the body B under symmetrization. The results which we shall obtain in this direction will be of decisive importance for our later study of cavitation flow by means of the principle of minimum virtual mass discussed in section 2. The investigation is necessarily limited to the cases of plane flow and axially symmetric flow, since proofs are based on properties of the stream function ψ .

We consider first the plane flow in a channel past an object B . Let the channel consist of those points for which $-h < y < h$, and suppose that the object B is already symmetric in the x -axis. We consider the flow past B in this channel with a stream function ψ such that $\psi - y$ vanishes at infinity and such that $\psi = 0$ on the portion of the x -axis which lies in the region of flow D . Clearly $\psi = h$ on the upper edge of the channel, $\psi = -h$ on the lower edge of the channel, and $\psi = 0$ on the curve C bounding B . We propose to show that the virtual mass

$$M = \iint_D (\nabla\psi - \nabla y)^2 \, dx \, dy$$

of this flow decreases when B is symmetrized with respect to the x -axis. Note that if the boundary curve C cannot be expressed in the form $y = \pm f(x)$, then B transforms under symmetrization into a body B^* whose boundary C^* does have such a representation.

For large values of k , we introduce the function Ψ which is harmonic in the region R common to D and the rectangle $-k < x < k$, $0 < y < h$ and which

satisfies the boundary conditions

$$(4.2) \quad \begin{cases} \Psi(x, h) &= h, \\ \Psi(-k, y) &= \Psi(k, y) = y, \\ \Psi(x, 0) &= 0, \\ \Psi(x, y) &= 0, (x, y) \text{ on } C. \end{cases}$$

We have by Green's theorem

$$(4.3) \quad \begin{aligned} \iint_R (\nabla \Psi - \nabla y)^2 dx dy &= \iint_R (\nabla \Psi)^2 dx dy + \iint_R dx dy - 2 \int_S \Psi \frac{\partial y}{\partial n} ds \\ &= \iint_R (\nabla \Psi)^2 dx dy + \iint_R dx dy - 4hk, \end{aligned}$$

where S is the boundary of R .

We define Ψ in the reflection of R in the x -axis by the rule

$$\Psi(x, y) = \Psi(x, -y),$$

and we define Ψ to be zero inside B . Thus the derivatives of Ψ jump as we cross the x -axis. Let us consider the region A of space bounded by the surface

$$z = \epsilon \Psi(x, y)$$

and the planes $z = \epsilon h$, $x = k$, $x = -k$, where ϵ is a positive number. We symmetrize the configuration A in the (x, z) -plane, thus in particular producing a symmetrization B^* of the object B and a symmetrization $z = \epsilon \Psi^*$ of the harmonic surface $z = \epsilon \Psi$. The area of the plane surfaces of A is evidently unchanged in the symmetrization, since Ψ is already symmetrized on the lines $x = \pm k$, $y = \pm h$. Hence from (4.1) we derive the inequality

$$(4.4) \quad \iint_R \sqrt{1 + \epsilon^2 \Psi_x^2 + \epsilon^2 \Psi_y^2} dx dy \geq \iint_{R^*} \sqrt{1 + \epsilon^2 \Psi_x^{*2} + \epsilon^2 \Psi_y^{*2}} dx dy.$$

By letting ϵ tend to zero and comparing the terms of order ϵ^2 in (4.4) we deduce from this the result

$$\iint_R (\nabla \Psi)^2 dx dy \geq \iint_{R^*} (\nabla \Psi^*)^2 dx dy.$$

If we replace Ψ^* on the right by the harmonic function Ψ^{**} in R^* with the same boundary values as Ψ^* , we improve the inequality by virtue of Dirichlet's principle. Taking (4.3) into account, we now obtain

$$(4.5) \quad \iint_R (\nabla \Psi - \nabla y)^2 dx dy \geq \iint_{R^*} (\nabla \Psi^{**} - \nabla y)^2 dx dy.$$

There is a flow ψ^* in the channel $-h < y < h$ past the symmetrized object B^* . Clearly for $y > 0$ the harmonic function Ψ converges to ψ in D and the harmonic function Ψ^{**} converges to ψ^* in D^* as the parameter k tends to infinity. Thus in the limit, (4.5) yields the basic inequality

$$(4.6) \quad \iint_D (\nabla\psi - \nabla y)^2 dx dy \geq \iint_{D^*} (\nabla\psi^* - \nabla y)^2 dx dy,$$

or

$$M \geq M^*.$$

Thus the virtual mass decreases with symmetrization perpendicular to the direction of flow.

If we let $h \rightarrow \infty$ in (4.6), we obtain the same inequality for the plane flow past B due to a dipole at infinity. The virtual mass M of this flow, which has a stream function

$$\psi = y - \frac{ay}{r^2} + \cdots, \quad r^2 = x^2 + y^2,$$

is related to the coefficient a and the area V of B by the identity

$$(4.7) \quad a = \frac{1}{2\pi} \{M + V\},$$

analogous to (2.5). It follows that the coefficient a for an object B symmetric in the x -axis decreases under symmetrization in that axis.

We turn next to the study of the behavior of the virtual mass under symmetrization in the y -axis. For this case no great change is required in the discussion just given. It suffices to notice that the harmonic function Ψ satisfies in R the inequality

$$\Psi < y,$$

by the maximum principle. Thus the boundary of the surface $z = e\Psi$, which projects onto the lines $x = \pm k$, $y = \pm h$, is already symmetrized in the (y, z) -plane. Therefore (4.4) follows as before, and so also does (4.5), where now B^* denotes the symmetrization of B in the y -axis. Passage to the limit $k \rightarrow \infty$ yields (4.6), and passage to the limit $h \rightarrow \infty$ again shows that the virtual mass of the dipole flow past B decreases under symmetrization in the y -axis. Finally, by (4.7), the coefficient a for a body B symmetric in the x -axis is seen to decrease under symmetrization in the y -axis.

It remains to generalize these results to the case of axially symmetric flow. Here the region $0 \leq y < h$ is to be interpreted as a pipe in the 3-dimensional space with cylindrical coordinates x, y, θ , and we consider the flow with a stream function ψ in this pipe past the axially symmetric object B . The virtual mass of

the flow is

$$M = \iiint_D \frac{1}{y} \left(\nabla \psi - \frac{1}{2} \nabla y^2 \right)^2 dx dy d\theta,$$

where D denotes the region of flow. We show that M decreases under symmetrization of B with respect to either the x -axis or the (y, θ) -plane.

We introduce the intersection R of D with the finite cylinder $-k < x < k$, $0 \leq y < h$, and we define Ψ to be the solution in R of the stream function equation

$$\frac{\partial^2 \Psi}{\partial y^2} - \frac{1}{y} \frac{\partial \Psi}{\partial y} + \frac{\partial^2 \Psi}{\partial x^2} = 0$$

which satisfies the boundary conditions

$$(4.8) \quad \begin{cases} \Psi(x, h) = \frac{h^2}{2}, \\ \Psi(-k, y) = \Psi(k, y) = \frac{y^2}{2}, \\ \Psi(x, 0) = 0, \\ \Psi(x, y) = 0, \quad (x, y) \text{ on } C. \end{cases}$$

The function Ψ converges to ψ as $k \rightarrow \infty$, whence it will suffice to show that

$$(4.9) \quad \begin{aligned} & \iiint_R \frac{1}{y} (\nabla \Psi - \tfrac{1}{2} \nabla y^2)^2 dx dy d\theta \\ &= \iiint_R \frac{1}{y} (\nabla \Psi)^2 dx dy d\theta + \iiint_R y dx dy d\theta - 2 \iint_s \frac{1}{y} \Psi \frac{\partial y}{\partial n} d\sigma \\ &= \iiint_R \frac{1}{y} (\nabla \Psi)^2 dx dy d\theta + \iiint_R y dx dy d\theta - 4\pi h^2 k \end{aligned}$$

decreases under symmetrization.

We set $\Psi = yu$, and from (4.9) we derive

$$(4.10) \quad \begin{aligned} & \iiint_R \frac{1}{y} (\nabla \Psi - \tfrac{1}{2} \nabla y^2)^2 dx dy d\theta = \iiint_R \frac{1}{y^2} (y \nabla u + u \nabla y)^2 dx dy d\theta \\ & \quad + \iiint_R y dx dy d\theta - 4\pi h^2 k \\ &= \iiint_R (\nabla u)^2 y dx dy d\theta + 2 \iiint_R u \frac{\partial u}{\partial y} dx dy d\theta \\ & \quad + \iiint_R \frac{u^2}{y} dx dy d\theta + \iiint_R y dx dy d\theta - 4\pi h^2 k \\ &= \iiint_R (\nabla u)^2 y dx dy d\theta + \iiint_R \frac{u^2}{y} dx dy d\theta + \iiint_R y dx dy d\theta - 3\pi h^2 k. \end{aligned}$$

We introduce the 3-dimensional surface $z = \epsilon u$ defined over R in the 4-dimensional space with coordinates x, y, θ and z . This surface bounds together with the hyperplanes $x = \pm k, z = 0$ and $z = \epsilon h/2$ a certain 4-dimensional region A . Symmetrization of this region in the (y, θ, z) -hyperplane induces a symmetrization u^* of u and a symmetrization B^* of B . Since u is already symmetrized for $x = \pm k$ and for $y = h$, and since the 3-dimensional surface area of A decreases under symmetrization, we obtain the inequality

$$(4.11) \quad \iiint_R \sqrt{1 + \epsilon^2 (\nabla u)^2} y \, dx \, dy \, d\theta \geq \iiint_{R^*} \sqrt{1 + \epsilon^2 (\nabla u^*)^2} y \, dx \, dy \, d\theta,$$

where R^* is the symmetrization of R . The second integral on the right in (4.10) is invariant under symmetrization in the (y, θ) -plane, and therefore comparison of terms of order ϵ^2 in (4.11) yields

$$(4.12) \quad \iiint_R \left\{ (\nabla u)^2 + \frac{u^2}{y^2} \right\} y \, dx \, dy \, d\theta \geq \iiint_{R^*} \left\{ (\nabla u^*)^2 + \frac{u^{*2}}{y^2} \right\} y \, dx \, dy \, d\theta.$$

Symmetrization in the x -axis is somewhat more involved in this case. It consists in replacing each intersection of A with a plane Q parallel to the (y, θ) -plane by a circular disc in Q of the same area with its center on the (x, z) -plane. Such a symmetrization is essentially equivalent to repeated symmetrization of A in hyperplanes through the (x, z) -plane, and therefore Steiner's fundamental theorem on the decrease of surface area still holds. We deduce that the symmetrization u^* of u in the x -axis again satisfies (4.11). On the other hand, the second integral on the right in (4.10) also decreases in the symmetrization, since for each fixed x , the area under the curve $z = u^2$ in the (y, z) -plane is moved away from the origin. Thus (4.12) holds as before.

Notice that the third integral on the right in (4.10), which represents a volume, is invariant under both symmetrization in the (y, θ) -plane and symmetrization in the x -axis, as described above. Also note that the symmetrized object B^* has the property that each vertical or horizontal line intersects it in at most one line segment.

In order to show that the virtual mass M of B in the pipe $0 \leq y < h$ decreases under symmetrization, we remark that by Dirichlet's principle, the inequality (4.12) is improved if we replace u^* by the solution u^{**} of

$$\Delta u^{**} = \frac{1}{y^2} u^{**}$$

in R^* with the same boundary values as u^* . The function $\Psi^* = y u^{**}$ is readily verified to satisfy

$$\frac{\partial^2 \Psi^*}{\partial y^2} - \frac{1}{y} \frac{\partial \Psi^*}{\partial y} + \frac{\partial^2 \Psi^*}{\partial x^2} = 0,$$

and thus if we define Ψ^* to be the symmetrization of Ψ , we see that the integral (4.9) decreases with symmetrization of B in the (y, θ) -plane or the x -axis,

$$(4.13) \quad \iiint_R \frac{1}{y} (\nabla \Psi - \tfrac{1}{2} \nabla y^2)^2 dx dy d\theta \geq \iiint_{R^*} \frac{1}{y} (\nabla \Psi^* - \tfrac{1}{2} \nabla y^2)^2 dx dy d\theta.$$

Passage to the limit $k \rightarrow \infty$ shows by (4.13) that the virtual mass M of the flow in a pipe $0 \leq y < h$ decreases under vertical or horizontal symmetrization of the object B , and passage to the limit $h \rightarrow \infty$ proves the same result for the axially symmetric flow past B in the whole plane with uniform velocity at infinity. These are results, of interest in their own right, which will prove to be of special importance for our later investigations of free stream flows.

5. Restricted analytic variations. We shall have need in later sections for some variational techniques in conformal mapping which we develop here (cf. [4]).

Let R be a plane region bounded by an analytic curve Γ , and let w and t be two fixed points in R . We define

$$\Phi(z; w, t) = \frac{1}{z - w} - \frac{1}{t - w} + (z - t)(\text{analytic terms})$$

to be a conformal mapping of R onto the entire plane slit along a circular arc about the origin, and we define

$$\Psi(z; w, t) = \frac{1}{z - w} - \frac{1}{t - w} + (z - t)(\text{analytic terms})$$

to be a conformal mapping of R° onto the entire plane slit along a radial segment pointed toward the origin. Setting

$$P(z; w, t) = \tfrac{1}{2} \log \Phi(z; w, t) - \tfrac{1}{2} \log \Psi(z; w, t),$$

$$Q(z; w, t) = \tfrac{1}{2} \log \Phi(z; w, t) + \tfrac{1}{2} \log \Psi(z; w, t),$$

we find that for z on Γ

$$(5.1) \quad P(z; w, t) = -\overline{Q(z; w, t)} + \text{const.}$$

Differentiation with respect to arc length along Γ yields

$$(5.2) \quad P'(z; w, t)z' = -\overline{Q'(z; w, t)z'},$$

where $z' = z'(s)$ is the unit tangent on Γ .

One sees easily that P is regular in R and $P(w; w, t) = 0$. The function Q has logarithmic singularities at w and t , and Q' has there simple poles with residues

-1 and $+1$, respectively. By Cauchy's theorem and (5.1), (5.2) we have

$$\begin{aligned} 2\pi i P(z; w, t) &= \int_{\Gamma} P(\zeta; w, t) dQ(\zeta; w, z) \\ &= \overline{\int_{\Gamma} \overline{P(\zeta; w, t)} d\overline{Q(\zeta; w, z)}} = \overline{\int_{\Gamma} Q(\zeta; w, t) dP(\zeta; w, z)} \\ &= -\int_{\Gamma} \overline{P(\zeta; w, z) dQ(\zeta; w, t)} = 2\pi i \overline{P(t; w, z)}. \end{aligned}$$

This proves the anti-symmetry rule

$$(5.3) \quad P(z; w, t) = \overline{P(t; w, z)}$$

and shows that $P(z; w, t)$ is an analytic function of $\bar{t}$. If we set

$$Q(z; w, t) = \log \frac{z - t}{z - w} + q(z; w, t),$$

then

$$\begin{aligned} 0 &= \overline{\int_{\Gamma} P(\zeta; w, z) dP(\zeta; w, t)} = \int_{\Gamma} Q(\zeta; w, z) dQ(\zeta; w, t) \\ &= \int_{\Gamma} q(\zeta; w, z) dQ(\zeta; w, t) + \int_{\Gamma} \log \frac{\zeta - z}{\zeta - w} dq(\zeta; w, t) \\ &\quad + \int_{\Gamma} \log \frac{\zeta - z}{\zeta - w} d \log \frac{\zeta - t}{\zeta - w}. \end{aligned}$$

The last integral vanishes, since the integrand is regular in the exterior of Γ . Since $q(w; w, t) = 0$, this yields

$$\begin{aligned} 2\pi i q(t; w, z) &= -\int_{\Gamma} \log \frac{\zeta - z}{\zeta - w} dq(\zeta; w, t) \\ &= \int_{\Gamma} q(\zeta; w, t) d \log \frac{\zeta - z}{\zeta - w} = 2\pi i q(z; w, t), \end{aligned}$$

and we obtain the symmetry rule

$$(5.4) \quad q(z; w, t) = q(t; w, z).$$

These identities remain true if we take for R the region exterior to Γ and if we take for w the point at infinity.

We denote by

$$p(z) = \log \frac{1}{z - w} + \text{analytic terms}$$

the analytic function whose real part is the Green's function of R . Thus

$$\operatorname{Re} \{p(z)\} = 0, \quad z \text{ on } \Gamma,$$

and also on Γ

$$(5.5) \quad p'(z)^2 z'^2 = \text{real}$$

does not vanish.

Consider the expression

$$(5.6) \quad z^{**} = z + \frac{\bar{\epsilon}P'(z; w, t) - \epsilon Q'(z; w, t)}{p'(z)^2}$$

for small values of the complex parameter ϵ . This function performs a schlicht transformation of Γ into a nearby curve Γ^* , and it maps R conformally onto the region R^* bounded by Γ^* , except in the neighborhood of $z = t$. On Γ we have

$$\begin{aligned} z^{**} - z &= \frac{\bar{\epsilon}P'(z; w, t)z' - \epsilon Q'(z; w, t)z'}{p'(z)^2 z'^2} z' \\ &= \frac{\bar{\epsilon}P'(z; w, t)z' + \epsilon \overline{P'(z; w, t)z'}}{p'(z)^2 z'^2} z' \\ &= (\text{real}) z', \end{aligned}$$

by (5.2) and (5.5). Thus the shift from z on Γ to z^{**} on Γ^* is tangential and of order of magnitude ϵ . The actual normal shift from Γ to Γ^* therefore has only the order of magnitude ϵ^2 .

It is easily shown by standard methods that there is a conformal mapping $z^* = f(z^{**})$ of R^* onto R which has the form

$$z^* = f(z^{**}) = z^{**} + O(\epsilon^2),$$

where $O(\epsilon^2)$ denotes terms of order ϵ^2 , or, more precisely, terms such that

$$\epsilon^{-2}O(\epsilon^2)$$

is bounded. This conclusion is based on the fact that Γ and Γ^* have a maximum normal distance of order ϵ^2 . Applying the transformation f to (5.6), we find that

$$(5.7) \quad z^* = z + \frac{\bar{\epsilon}P'(z; w, t) - \epsilon Q'(z; w, t)}{p'(z)^2} + O(\epsilon^2)$$

is an analytic function which maps the curve Γ onto itself. This mapping of Γ onto itself, with a pole at $z = t$ inside Γ , will prove quite useful for studying cavitation flow by variational methods. It is easily extended to the case where Γ is only piece-wise analytic.

We consider here the variation in the area A of an open set W under the trans-

formation (5.7). We treat, in particular, the case where the pole $z = t$ lies in W , and we define the variation W^* of the set W to be that region which is enclosed by the varied boundary of W . Let γ be a circle about $z = t$, and let Ω be the interior of this circle. Then the area A^* of W^* is given by the formula

$$A^* = \iint_{W-\Omega} |z^{*\prime}(z)|^2 dx dy + \frac{1}{2i} \int_{\gamma} \overline{z^*(z)} dz^*(z).$$

Hence

$$\begin{aligned} \delta A &= A^* - A = \iint_{W-\Omega} \{|z^{*\prime}|^2 - 1\} dx dy + \frac{1}{2i} \int_{\gamma} \{\overline{z^*} dz^* - \bar{z} dz\} \\ &= 2 \operatorname{Re} \iint_{W-\Omega} \frac{d}{dz} \frac{\bar{\epsilon} P'(z; w, t) - \epsilon Q'(z; w, t)}{p'(z)^2} dx dy \\ &\quad + \operatorname{Im} \int_{\gamma} \frac{\overline{\epsilon P'(z; w, t)} - \bar{\epsilon} Q'(z; w, t)}{p'(z)^2} dz + O(\epsilon^2). \end{aligned}$$

Expansion of these integrands in Laurent series about $z = t$ shows that the integral over γ tends to zero as γ shrinks toward t , and the area integral exists over W in the sense of the Cauchy principal value. Therefore

$$(5.8) \quad \delta A = 2 \operatorname{Re} \left\{ \epsilon \iint_W \left[\left(\frac{d}{dz} \frac{P'(z; w, t)}{p'(z)^2} \right) - \frac{d}{dz} \frac{Q'(z; w, t)}{p'(z)^2} \right] dx dy \right\} + O(\epsilon^2).$$

From (5.3) we see that

$$\left(\frac{d}{dz} \frac{P'(z; w, t)}{p'(z)^2} \right)$$

is an analytic function of t in R , and from (5.4) we see that

$$\frac{d}{dz} \frac{Q'(z; w, t)}{p'(z)^2} = -\frac{1}{p'(t)^2} \frac{1}{(z-t)^2} + \dots$$

is analytic as a function of t in R .

Therefore we can write

$$\left(\frac{d}{dz} \frac{P'(z; w, t)}{p'(z)^2} \right) - \frac{d}{dz} \frac{Q'(z; w, t)}{p'(z)^2} = \frac{1}{p'(t)^2} \frac{1}{(z-t)^2} + l(t; w, z),$$

where $l(t; w, z)$ is a regular function of t throughout R . We obtain from (5.8) the final variational formula

$$(5.9) \quad \delta A = 2 \operatorname{Re} \left\{ \frac{\epsilon}{p'(t)^2} \iint_W \frac{dx dy}{(z-t)^2} + \epsilon \iint_W l(t; w, z) dx dy \right\} + O(\epsilon^2),$$

in which the second integral on the right is a regular function of t in R .

6. Existence of cavitation flow by minimizing virtual mass. In this section we shall prove by means of the principle of minimum virtual mass the existence of a plane flow past two symmetric curves with a finite cavity between the curves. We choose such a model as an example of the use of the principle of minimum virtual mass because of the agreement of mathematical and experimental data computed for such flows with finite cavity.

Let C be a fixed simple closed analytic curve in the z -plane bounding a region which is symmetrized in the x -axis and in the y -axis. Let C_1 and C_2 be the two symmetric arcs of C which lie to the left and to the right, respectively, of the lines $x = -k$, $x = +k$. We denote by L a variable pair of curves which join the end-points of C_1 to the end-points of C_2 and which lie outside C and inside the strip $-k \leq x \leq k$. We denote by B the body enclosed by C_1 , C_2 and L , and we denote by W the cavity lying between L and C .

Denote by A the area of W , by

$$\phi = x + \frac{ax}{r^2} + \cdots, \quad r^2 = x^2 + y^2,$$

the velocity potential of the flow past B , and by M the virtual mass of this flow. Then in analogy with (2.5), we have

$$(6.1) \quad a = \frac{1}{2\pi} (M + V),$$

where V is the area of B .

Let us attempt to choose the variable curves L in such a way that for a given positive constant λ the expression

$$(6.2) \quad 2\pi a - \lambda A = \text{minimum},$$

in analogy with (2.10). It is first of all clear that we can restrict the competing shapes for the body B to be symmetrized in the x -axis and in the y -axis. For symmetrization in these axes leaves the region enclosed by C invariant, by assumption, and it leaves V and A unchanged, while by the results of section 4 it decreases the virtual mass M and the coefficient a . For these symmetrized shapes, the curves L are monotonic in each quadrant, since each vertical or horizontal line intersects B in at most one line segment.

Let h denote the height of L above the x -axis. Then B contains the segment $x = 0$, $-h \leq y \leq h$, and the coefficient a for B is larger than the corresponding coefficient a_0 for the segment. This can either be seen by direct computation with the Dirichlet integral or by noticing that in view of (2.7) for plane flows the coefficient a decreases monotonically when B diminishes in size. Hence

$$(6.3) \quad a \geq a_0 = \frac{1}{2}h^2.$$

On the other hand,

$$A \leq 4kh,$$

so the competing shapes B cannot tend to infinity, and must, indeed, be uniformly bounded.

If we now apply the theory of normal families of schlicht analytic functions to the conformal transformation

$$\zeta(z) = \phi + i\psi$$

by the complex potential, we see that a pair of minimum curves L for (6.2) must exist. Indeed, for any minimum sequence of such curves one has a corresponding sequence of schlicht mappings by the complex potential, and from these one can select a convergent subsequence and thus in the limit find the extremal region of flow. The minimum curves must be monotonic in each quadrant, in view of our previous application of symmetrization.

If the extremal curves L do not degenerate to the curve C itself, then the extremal cavity W consists of an open set bounded by rectifiable curves. If L has no arcs in common with the lines $x = \pm k$, we denote by Γ simply the curve C . If L has four segments in common with $x = \pm k$, which is the only alternate possibility, we denote by Γ the simple closed curve consisting of these segments, of adjacent horizontal segments, and of arcs of C . The region R exterior to Γ includes in all cases the region of flow D and the portions of the cavity W bordering on those arcs of L which are distinct from the lines $x = \pm k$.

Let us apply the variational formula (5.7) to the region R , with $w = \infty$ and with t inside the cavity W . In general, this conformal transformation takes the extremal flow for (6.2) into another flow which is admissible for the minimum problem (6.2). However, when Γ does not have corners on the lines $x = \pm k$, then W may be mapped in part outside the strip $-k \leq x \leq k$, and we must verify that such an occurrence does not affect the admissibility of the varied flow in an essential way. In order to keep W within the strip $-k \leq x \leq k$, we can impose restraints on the variation so that the four points on Γ with $x = \pm k$ remain fixed, and these restraints introduce terms with Lagrange multipliers in later formulas. But it is found that because the error in computing the change of virtual mass and area without a correction by such restraints is of higher order than the first in ϵ , the Lagrange multipliers vanish. Thus it is permissible to neglect such an error in the first place, although a direct proof that the effect of a shift of W outside the strip $-k \leq x \leq k$ is of higher order gives more serious difficulties.

Near infinity we have

$$z^* = z - \epsilon \left[z + \alpha_0(t) + \frac{\alpha_1(t)}{z} + \cdots \right] + \bar{\epsilon} \left[\overline{\beta_0(t)} + \frac{\overline{\beta_1(t)}}{z} + \cdots \right] + O(\epsilon^2),$$

where $\alpha_0, \alpha_1, \cdots, \beta_0, \beta_1, \cdots$, are analytic functions of t in R . Hence the coefficient a^* for the new flow is

$$a^* = a + \operatorname{Re} \{ \epsilon \alpha_1(t) - \bar{\epsilon} \overline{\beta_1(t)} - 2\epsilon a \} + O(\epsilon^2),$$

whence

$$(6.4) \quad \delta a = \operatorname{Re} \{ \epsilon \alpha_1(t) - \epsilon \beta_1(t) - 2\epsilon a \} + O(\epsilon^2).$$

Since the curves L are extremal for (6.2), we must have

$$2\pi\delta a - \lambda\delta A = \delta(2\pi a - \lambda A) = 0,$$

or, by (5.9) and (6.4),

$$\begin{aligned} \operatorname{Re} \left\{ 2\pi\epsilon\alpha_1(t) - 2\pi\epsilon\beta_1(t) - 4\pi\epsilon a - \frac{2\lambda\epsilon}{p'(t)^2} \iint_w \frac{dx dy}{(z-t)^2} \right. \\ \left. - 2\lambda\epsilon \iint_w l(t; w, z) dx dy \right\} + O(\epsilon^2) = 0. \end{aligned}$$

Since ϵ is an arbitrarily small complex number, this yields the variational identity

$$(6.5) \quad \iint_w \frac{dx dy}{(z-t)^2} = F'(t),$$

where $F'(t)$ is the analytic function in R given by

$$F'(t) = -p'(t)^2 \iint_w l(t; w, z) dx dy + \frac{\pi\alpha_1(t)}{\lambda} p'(t)^2 - \frac{\pi\beta_1(t)}{\lambda} p'(t)^2 - \frac{2\pi a}{\lambda} p'(t)^2.$$

Set

$$I = \iint_w \frac{dx dy}{z-t} - F(t).$$

We introduce the operators

$$\frac{\partial}{\partial t} = \frac{1}{2} \left\{ \frac{\partial}{\partial \tau} - i \frac{\partial}{\partial \sigma} \right\}, \quad \frac{\partial}{\partial \bar{t}} = \frac{1}{2} \left\{ \frac{\partial}{\partial \tau} + i \frac{\partial}{\partial \sigma} \right\}, \quad t = \tau + i\sigma,$$

and we note that

$$4 \frac{\partial^2}{\partial t \partial \bar{t}} = \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \sigma^2}.$$

Hence, by Poisson's equation for logarithmic potentials,

$$(6.6) \quad \frac{\partial I}{\partial \bar{t}} = 2 \frac{\partial^2}{\partial t \partial \bar{t}} \iint_w \log \frac{1}{|z-t|} dx dy - \frac{\partial F}{\partial \bar{t}} = -\pi,$$

since F is an analytic function of t . But (6.5) can also be written

$$\frac{\partial I}{\partial t} = 0.$$

Integration of these two partial differential equations for I gives in W

$$(6.7) \quad I = -\pi\bar{t} + \text{const.}$$

In the region of flow D , the integral I is clearly an analytic function of t . Also I is clearly continuous across those arcs L' of the rectifiable curves L in R which bound W . By (6.7) we have on such arcs

$$I + \pi\bar{t} = \text{const.}$$

Thus if we define analytic functions g and f in D by the formulas

$$f(t) = I(t) + \pi t,$$

$$g(t) = I(t) - \pi t,$$

we find that on the arcs L' of L separating W from D , and distinct from $x = \pm k$,

$$\text{Re } f(t) = \text{const.},$$

$$\text{Im } g(t) = \text{const.}$$

If we consider f and g as functions of $\zeta = \phi + i\psi$, we see by the Schwarz principle of reflection that they are analytic on the segments of the ϕ -axis corresponding to L' . Hence

$$t = \frac{1}{2\pi} \{f - g\}$$

is analytic there, and it follows that L' consists of analytic arcs.

Having proved that the arcs L' are analytic, one is justified in applying the formal variational technique of section 2 in order to deduce that

$$(6.8) \quad (\nabla\phi)^2 = \lambda, \quad \text{on } L'.$$

This can likewise be proved directly from (6.7) if one studies more carefully the exact nature of the analytic integral $I(t)$. In fact, it is not hard to see that

$$\lambda I'(t) + \pi \zeta'(t)^2 = 0$$

in D .

Our conclusion is that the extremal curves L consist in part of segments with $x = \pm k$, in part of arcs of C , and in part of constant pressure free streamlines L' . Thus the minimum problem (6.2) leads to the existence of a free stream flow for each value of the cavitation constant λ . We proceed to discuss the dependence of the flows on λ in order to infer that segments on $x = \pm k$ do not occur on the boundary of all such flows.

Consider the intersections $\bar{D}$ and $\bar{D}^*$ of any two extremal flow regions D and D^* with the upper half-plane $y > 0$. We translate $\bar{D}^*$ in the vertical direction

until it includes $\bar{D}$, with at least one point of tangency P_1 of the boundaries. By the maximum principle, the stream functions ψ and ψ^* satisfy in D the inequality $\psi^* > \psi$, and therefore at P_1

$$(\nabla\psi^*)^2 > (\nabla\psi)^2.$$

On the other hand, if we translate $\bar{D}$ vertically until it just includes $\bar{D}^*$, with at least one point P_2 of tangency of the boundaries, then by the same reasoning

$$(\nabla\psi)^2 > (\nabla\psi^*)^2$$

at P_2 . It follows that both P_1 and P_2 cannot occur on the free boundaries L' and L'^* of $\bar{D}$ and $\bar{D}^*$, since the velocities $(\nabla\psi)^2$ and $(\nabla\psi^*)^2$ are constant there. Hence, in their original position, one region must include the other, say $\bar{D} \subset \bar{D}^*$. But then by translation of $\bar{D}$ we obtain

$$\lambda > \lambda^*,$$

where λ and λ^* are the cavitation constants of D and D^* .

The reasoning degenerates if D and D^* are the same, but it can then be used to show that there is a unique extremal flow for each λ . Finally, we conclude that the flow region D^* for λ^* includes the flow region D for λ when $\lambda > \lambda^*$. Thus the extremal flow region D diminishes monotonically as λ increases.

It is easily deduced from the unique dependence of D on λ that D also depends continuously on λ . For given a convergent sequence of λ 's, one can show that the complex potentials of the corresponding free stream flows converge to the complex potential of a free stream flow with the limiting value of λ as cavitation constant. Thus we obtain continuous dependence of the point of separation of a free streamline on λ .

Let ψ_0 denote the stream function of the flow past C alone. Then in an extremal flow region D , we have $\psi_0 > \psi$, by the maximum principle. If L' has a point of tangency P with C , then at that point

$$(\nabla\psi_0)^2 > (\nabla\psi)^2 = \lambda.$$

Thus for values

$$\lambda > \max_c (\nabla\psi_0)^2$$

the free streamline L' lies entirely above C , and the remainder of L consists of four segments with $x = \pm k$.

Suppose now that C consists of the two symmetric arcs C_1 and C_2 together with two horizontal segments joining their end-points. If we translate $\bar{D}$ vertically until L' lies entirely below this segment, except for one point P of tangency, then at that point

$$\lambda = (\nabla\psi)^2 > (\nabla\psi_0)^2.$$

Hence for

$$\lambda < \min_c (\nabla\psi_0)^2$$

the extremal arcs L coincide with C . It follows that there is a value of λ for which the free streamlines L' themselves join the end-points of C_1 to the end-points of C_2 , since for large λ the free stream L' lies above C .

Thus we are able to show through the extremal problem (6.2) the existence of a cavitation flow past the plates C_1 and C_2 with free surfaces joining the edges of the plates. We remark that by allowing the right-hand plate to move off to infinity, we can obtain in the limit the classical Helmholtz flow past the first plate C_1 above with an infinite cavity which separates from C_1 at its end-points.

It is also worthwhile to notice that a limit process can be made to obtain cavitation flows past smooth curves C by approximation with such flows past nearby analytic curves. The generalization to less restricted fixed boundaries can also be made directly with the variational method by suitable modification of formula (5.7), but we do not go into these further extensions of the theory here.

It is natural to inquire at this point whether the discussion of the existence of cavitation flow given here can be extended to the case of circulating flow about an object B' due to a vortex at infinity which was taken up formally in section 2. That this can be done for objects B' which are unchanged by symmetrization in the x -axis and y -axis will be easily verified by the reader. For the existence of a body B which includes B' , has a fixed area, and has a minimum logarithmic capacity is an easy consequence of the theory of normal families of analytic functions, and the extremal body must be invariant under symmetrization in the x -axis or y -axis, since this process diminishes capacity while leaving area unchanged. Thus the extremal body B is bounded by rectifiable curves, the variational formula (5.7) can again be used to deduce analyticity of the free boundary of B , and finally, we are justified in using formula (2.22), which led us before to the conclusion that the free boundary of the extremal body is a constant pressure free surface for the flow.

7. Free stream flow with gravity. It is of interest to inquire whether the procedure of the previous section for discussing cavitation flow can be extended to the treatment of free surface flow in a gravity field. We make such an extension here, and thus place the minimum problem (2.16) on a firm mathematical basis. The physical model to which we turn in this connection will yield results of special interest even when the gravitational constant g vanishes. We shall obtain, in fact, cavitation flow past an object in a channel, or water tunnel, and the flow from a symmetric nozzle without gravity. Those details of the development which do not differ in an essential way from the corresponding steps in section 6 will be discussed more briefly.

Let C be a simple closed analytic curve in the channel $-h < y < h$ which is unchanged by symmetrization in the x -axis or y -axis, and let C_1 and C_2 be, respectively, the arcs of C lying to the left and to the right of the strip $-k < x < k$. Let L be a pair of curves which join the end-points of C_1 to the end-points of C_2 and which lie outside C and inside the rectangle $-h < y < h$, $-k \leq x \leq k$. We shall have occasion also to denote by L just one of these curves. Let B denote the body bounded by C_1 , C_2 and L , and let ψ be the stream function of the flow past B in the channel $-h < y < h$ which has uniform velocity 1 in the direction of the x -axis near infinity. We shall consider here the region of flow D lying above the x -axis in the channel, and we shall assume that $\psi = 0$ on the x -axis. Thus we investigate the flow in the tunnel $0 < y < h$ with an obstruction B . It is clear that $\psi = h$ for $y = h$.

The virtual mass of the flow through D is

$$M = \iint_D (\nabla\psi - \nabla y)^2 dx dy.$$

Green's theorem yields for M the relation

$$M = \int (\psi - y) \frac{\partial}{\partial n} (\psi - y) ds,$$

where the integral is extended over the line $y = h$, the part of the x -axis bounding D , and the upper half $\bar{C}$ of the curve $C_1 + C_2 + L$. The integrand vanishes on all but the last curve, and hence

$$\begin{aligned} M &= \int_{\bar{C}} y \frac{\partial y}{\partial n} ds - \int_{\bar{C}} \left\{ y \frac{\partial \psi}{\partial n} - \psi \frac{\partial y}{\partial n} \right\} ds \\ &= -\frac{1}{2} V + \lim_{x \rightarrow \infty} \int_{-x}^x h \left\{ \frac{\partial \psi(x, h)}{\partial y} - 1 \right\} dx, \end{aligned}$$

where V is the area of B . We set

$$\alpha = \lim_{x \rightarrow \infty} \int_{-x}^x h \left\{ \frac{\partial \psi(x, h)}{\partial y} - 1 \right\} dx$$

and obtain

$$(7.1) \quad \alpha = M + \frac{1}{2} V.$$

Since M decreases under symmetrization of B in the x -axis or the y -axis by section 4, we find from (7.1) that α also decreases with such a symmetrization.

The upper curve L bounds together with C a certain cavity W in the upper half-plane. We define

$$A = \iint_W dx dy,$$

$$E = \iint_W gy dx dy.$$

Clearly A is invariant under symmetrization of B in the x -axis or y -axis. The potential energy E is invariant under symmetrization in the y -axis, but E decreases under symmetrization in the x -axis, since this process can bring W nearer the x -axis, thus diminishing the integrand gy .

Given a positive number λ , we seek to choose the variable curve L in such a way that

$$(7.2) \quad \frac{1}{2}\alpha + E - \lambda A = \text{minimum},$$

in analogy with (2.16). In this problem there is no less of generality if we assume that the competing shapes for B are invariant under symmetrization in the x -axis or the y -axis, because α and E decrease under such symmetrizations, while A remains unchanged. Thus an extremal curve L for (7.2) exists and is monotonic in each quadrant, since the mappings by the complex potentials $\zeta = \phi + i\psi$ in the flow regions D form a normal family of schlicht functions and since the virtual mass M becomes infinite if L rises to the line $y = h$. The last conclusion is based on the fact that as L approaches $y = h$, a sink with infinite virtual mass is generated on $y = h$ because a uniform amount of fluid must pass between L and $y = h$.

If L has no arcs in common with the lines $x = \pm k$, we take Γ to be the piecewise analytic curve consisting of the line $y = h$, the two infinite intervals of the x -axis which bound the flow region D , and the upper half of the analytic curve C . In the case where L has two segments in common with $x = \pm k$, we replace C in this definition by the curve made up of these two segments, two adjacent horizontal segments, and appropriate arcs of C . The region R enclosed by the infinite curve Γ contains the extremal flow region D and it contains those portions of the extremal cavity W which border on the arcs L' of L which are distinct from C and from $x = \pm k$.

It is not possible to apply the variational formula (5.7) directly to the infinite curve Γ , because the terms of order ϵ become infinite at the far ends of the curve. However, if we let $\zeta_0(z) = \phi_0 + i\psi_0$ be the complex potential of the flow in the channel R bounded by Γ which is uniform in the horizontal direction at large distances, then (5.7) can be replaced by a transformation

$$(7.3) \quad z^* = z + \frac{\bar{\epsilon}P'(z; w, t) - \epsilon Q'(z; w, t)}{ip'(z)\zeta_0'(z)} + O(\epsilon^2)$$

which takes Γ into itself and has an ϵ term which is bounded in R outside a neighborhood of $z = t$. The behavior of (7.3) is studied in precisely the same way as we studied (5.7), with formula (5.5) replaced on Γ by

$$(7.4) \quad ip'(z)\zeta_0'(z)z'^2 = \text{real}.$$

We choose t in the cavity W and note that the conformal mapping (7.3) takes the extremal flow region D into a new flow region D^* which is admissible for (7.2). Since t lies in W , the modified virtual mass α^* of the flow through D^* is

identical with α . Thus from the extremal property of D we have under the variation (7.3)

$$(7.5) \quad \delta E - \lambda \delta A = 0.$$

In analogy with (5.9), we obtain here

$$(7.6) \quad \delta A = 2 \operatorname{Re} \left\{ \frac{\epsilon}{ip'(t) \xi'_0(t)} \iint_W \frac{dx dy}{(z-t)^2} + \epsilon \iint_W l_1(t; w, z) dx dy \right\} + O(\epsilon^2),$$

where l_1 is a regular function of t in R . Similarly

$$\begin{aligned} \delta E = 2 \operatorname{Re} \left\{ \epsilon g \iint_W y \left[\left(\frac{d}{dz} \frac{P'(z; w, t)}{ip'(z) \xi'_0(z)} \right) - \frac{d}{dz} \frac{Q'(z; w, t)}{ip'(z) \xi'_0(z)} \right] dx dy \right\} \\ + \operatorname{Re} \left\{ \epsilon i g \iint_W \left[\left(\frac{P'(z; w, t)}{ip'(z) \xi'_0(z)} \right) + \frac{Q'(z; w, t)}{ip'(z) \xi'_0(z)} \right] dx dy \right\} + O(\epsilon^2), \end{aligned}$$

or

$$(7.7) \quad \delta E = \operatorname{Re} \left\{ \frac{g\epsilon}{ip'(t) \xi'_0(t)} \iint_W \left[\frac{2y}{(z-t)^2} + \frac{i}{z-t} \right] dx dy \right. \\ \left. + \epsilon \iint_W l_2(t; w, z) dx dy \right\} + O(\epsilon^2),$$

where the improper integral is taken in the sense of the Cauchy principal value, and where l_2 is a regular function of t in R .

Combining (7.5), (7.6) and (7.7) and using the knowledge that ϵ is an arbitrarily small complex number, we obtain the final variational formula

$$(7.8) \quad g \iint_W \left[\frac{2y}{(z-t)^2} + \frac{i}{z-t} \right] dx dy - 2\lambda \iint_W \frac{dx dy}{(z-t)^2} = F'_1(t),$$

where

$$F'_1(t) = -ip'(t) \xi'_0(t) \iint_W \{ l_2(t; w, z) - \lambda l_1(t; w, z) \} dx dy$$

is an analytic function of t in R .

Green's theorem gives

$$\iint_W \frac{i dx dy}{z-t} = \frac{1}{2} \int_\gamma \frac{\bar{z} dz}{z-t} - \pi i \bar{t},$$

where γ is the boundary of W and consists of arcs of L and C . We define a function $A(z, \bar{z})$ by the formula

$$A(z, \bar{z}) = \frac{1}{2}(x^2 + y^2) + ixy - 2i \int_k^z y(x) dx,$$

where $y = y(x)$ is the parametric representation of L . We note that vertical segments on L for $-k < x < k$ are excluded by the two-dimensional analogue of (2.17); cf. also (7.13). Hence the function A is continuously differentiable for $-k \leq x \leq k$, and for z on L we have

$$A = \int \bar{z} dz.$$

Hence by integration by parts

$$\iint_w \frac{i dx dy}{z - t} = \frac{1}{2} \int_L \frac{A(z, \bar{z}) dz}{(z - t)^2} + F_2'(t) - \pi i \bar{t},$$

where $F_2'(t)$ is analytic in R .

We can now rewrite (7.8) in the form

$$(7.9) \quad 2g \iint_w \frac{y dx dy}{(z - t)^2} - 2\lambda \iint_w \frac{dx dy}{(z - t)^2} + \frac{g}{2} \int_L \frac{A dz}{(z - t)^2} - \pi g i \bar{t} = F'(t)$$

with

$$F'(t) = F_1'(t) - g F_2'(t).$$

We set

$$I = 2g \iint_w \frac{y dx dy}{z - t} - 2\lambda \iint_w \frac{dx dy}{z - t} + \frac{g}{2} \int_L \frac{A dz}{z - t} - F(t).$$

Since F is analytic, we obtain by Poisson's equation

$$\frac{\partial I}{\partial \bar{t}} = -2\pi g \frac{t - \bar{t}}{2i} + 2\lambda\pi,$$

whereas by (7.9)

$$\frac{\partial I}{\partial t} = \pi g i \bar{t}.$$

Thus by integration of these partial differential equations

$$(7.10) \quad I = 2\pi\lambda\bar{t} - \frac{\pi g i \bar{t}^2}{2} + \pi g i t \bar{t} + \text{const.}$$

in W .

Our next remark is that by Green's theorem

$$\frac{1}{2i} \int \frac{A dz}{z - t} = \iint_w \frac{\partial A}{\partial \bar{z}} \frac{dx dy}{z - t} + \pi \delta(t) A(t, \bar{t}),$$

where $\delta(t) = 1$ for t in W and $\delta(t) = 0$ for t outside W . Thus the integral on the left jumps by $\pi A(t, \bar{t})$ as t crosses L , since the integral on the right is continuous across L . The integral $I(t)$ is an analytic function of t in the region of flow D , and it now follows from (7.10) that this analytic function assumes on the arcs L' of L distinct from $x = \pm k$ and separating W from D the continuous boundary values

$$(7.11) \quad I(t) = 2\pi\lambda\bar{t} - \frac{\pi g i \bar{t}^2}{2} + \pi g i t \bar{t} - \pi g i \int^t \bar{z} dz + \text{const.}$$

This is a consequence of the continuity of the first two integrals in the definition of I across L' and our analysis of the function $A(t, \bar{t})$. The integral on the right in (7.11) is extended along L from a fixed point t_0 to the variable point t .

In order to deduce the analyticity of L' from the identity (7.11), we consider in D the ordinary differential equation

$$(7.12) \quad I = 2\pi\lambda w - \frac{\pi g i w^2}{2} + \pi g i t w - \pi g i \int^t w dz + \text{const.}$$

for the determination of the analytic function $\int^t w dz$. By (7.11), $\int^t \bar{z} dz$ is a solution of (7.12) on L' . We choose $\int^t w dz$ to be the analytic solution of (7.12) in D which assumes at t_0 on L' the initial value 0. This solution can be analyzed by the standard method of successive approximations on each curve Λ in D obtained by vertical translation of L . It is clear that the solution is unique and depends continuously on the curve Λ , and in particular it approaches the known solution $\int^t \bar{z} dz$ on L' as Λ approaches L . Since $\int^t w dz$ satisfies (7.12) and tends to $\int^t \bar{z} dz$ on L' , where (7.11) holds, we find that the analytic function $w(t)$ has continuous boundary values

$$w(t) = \bar{t}, \quad \text{on } L'.$$

These steps are justified because L is already known to be a rectifiable curve.

We consider the analytic functions

$$f(t) = w(t) - t,$$

$$g(t) = w(t) + t$$

as functions of the complex potential $\zeta = \phi + i\psi$ of the flow through D . On the segments of the ϕ -axis corresponding to L' we have

$$\operatorname{Re} f(t) = \operatorname{Im} g(t) = 0,$$

and therefore f and g are analytic on those segments, by the Schwarz reflection principle. Hence

$$t = \frac{1}{2}\{g - f\}$$

is analytic there, and we conclude that L' consists of analytic arcs.

Once in possession of the analyticity of L' , we can proceed with formal variations as in section 2. Let D^* be the region obtained from D by shifting the boundary L' a distance $\delta n = \epsilon \rho(s)$ units along the outer normal. In D^* there is a flow admissible for the extremal problem (7.2), and we associate with it energies α^* , E^* and A^* . By Green's theorem

$$\begin{aligned}\alpha^* - \alpha &= \lim_{z \rightarrow \infty} \int_{-z}^z h \left\{ \frac{\partial \psi^*(x, h)}{\partial y} - \frac{\partial \psi(x, h)}{\partial y} \right\} dx \\ &= \iint_D \nabla \psi \nabla \psi^* dx dy - \iint_{D^*} \nabla \psi \nabla \psi^* dx dy \\ &= - \int_{L'} (\nabla \psi)^2 \delta n ds + O(\epsilon^2).\end{aligned}$$

Thus the variational formulas for α , E and A with the shift δn of L' are

$$\begin{aligned}\delta \alpha &= - \int_{L'} (\nabla \phi)^2 \delta n ds + O(\epsilon^2), \\ \delta E &= - \int_{L'} gy \delta n ds + O(\epsilon^2), \\ \delta A &= - \int_{L'} \delta n ds + O(\epsilon^2).\end{aligned}$$

From the extremal problem (7.2) we have

$$\frac{1}{2} \delta \alpha + \delta E - \lambda \delta A = 0,$$

whence

$$\int_{L'} \{ \frac{1}{2} (\nabla \phi)^2 + gy - \lambda \} \delta n ds + O(\epsilon^2) = 0.$$

From the arbitrariness of δn , we conclude that on L'

$$(7.13) \quad \frac{1}{2} (\nabla \phi)^2 + gy = \lambda,$$

and thus it is proved that L' is a constant pressure free streamline for the flow through D in the gravity field with gravitational constant g .

We note that differentiation of (7.11) with respect to arc length yields

$$I'(t)t' = 2\pi\lambda\bar{t}' - \pi g i \bar{t}' + \pi g i t \bar{t}',$$

or

$$I'(t)t'(s)^2 = 2\pi\lambda - 2\pi g \operatorname{Im}(t).$$

Comparison of this identity with (7.13) shows that

$$I'(t) = \pi \zeta'(t)^2.$$

It is possible to derive (7.13) from (7.11) directly by a more exact investigation of the variational integral I .

We have obtained here a one-parameter family of free stream flows for each value of the gravitational constant g . We show, as in section 6, that for each λ there is just one such flow, and that the region of flow diminishes monotonically as λ increases. For let D and D^* be two flow regions with parameters λ and λ^* , but with the same value of g . We magnify D^* about the origin until its lower boundary curve $C_1 + C_2 + L^*$ is just contained in D , and we magnify the stream function ψ^* by the same factor so as to obtain always uniform velocity 1 at infinity. The level curve $\tilde{\psi}^* = h$ of the magnified stream function $\tilde{\psi}^*$ is asymptotic to the line $y = h$, and since B^* and ψ^* are symmetrized, by virtue of the minimum problem (7.2) and section 4, we see that this level curve ascends in the second quadrant and descends in the first quadrant. Hence $\psi > \tilde{\psi}^*$ in the region common to D and the magnification $\tilde{D}^*$ of D^* . It follows that

$$(\nabla\psi)^2 > (\nabla\tilde{\psi}^*)^2$$

at the point where the boundaries of D and D^* touch. Thus at this point on $C_1 + C_2 + L$ and the corresponding point on $C_1 + C_2 + L^*$ we have

$$(\nabla\psi)^2 > (\nabla\psi^*)^2$$

A similar inequality in the other direction can be obtained if we magnify D until $C_1 + C_2 + L$ is just included in D^* . Hence we arrive at a contradiction unless one of the flow regions contains the other. We obtain now the unique, monotonic dependence of the flow region on λ precisely as in section 6.

If we take for C in the strip $-k \leq x \leq k$ a horizontal segment joining the upper end-point of C_1 to the upper end-point of C_2 , we find that for a particular value of λ the free stream L' is a curve joining these end-points. This follows from the fact that for large values of λ the streamline L' lies entirely above C , while for small λ it coincides with C . This special flow past C_1 and C_2 with just the single free streamline L' leaving from the end of C_1 and returning at the end of C_2 merits particular attention. If we allow the righthand curve C_2 to approach infinity, we obtain as the limit of this flow a flow past the curve C_1 only with an infinite free streamline L' .

If we take $g = 0$ for the above flow and make a reflection in the x -axis, we obtain the cavitation flow past a prescribed object in the channel $-h < y < h$. On the other hand, if we again cut the flow off along the x -axis, but reflect now across the line $y = h$, we obtain the flow from a nozzle with fixed boundaries consisting of the lines $y = 0$, $y = 2h$, and the curve C_1 with its reflection in the line $y = h$. Thus a wide variety of flows can be studied through the basic minimum energy principle embodied in (7.2).

8. Existence of cavitation flow by the minimax procedure. We shall solve the problem of the existence of cavitation flows in this section by a method, quite independent of sections 2, 6 and 7, which is based on the principle of maximum least velocity stated in section 3.

We denote by C a convex arc in the second quadrant which rises from the x -axis and has everywhere a tangent with positive slope. We let C_1 be the pointed curve obtained by combining C with its reflection in the x -axis, and we let C_2 be the reflection of C_1 in the y -axis. We denote by L_1 a concave arc joining the upper end-point of C_1 to the upper end-point of C_2 , and we let L_2 be the reflected image of L_1 . The curves C_1 , C_2 , L_1 and L_2 form together the boundary of an object B whose nose is pointed and concave.

We consider the flow past the object B with a complex potential

$$\zeta = \phi + i\psi = z + \frac{a}{z} + \dots$$

The conformal mapping by the derivative

$$w = \frac{d\zeta}{dz}$$

takes the portion D of the region of flow which lies in the upper half-plane into a region R of the hodograph plane which has the property that each ray from the origin intersects it in at most one line segment. For if C and L_1 are smooth curves, then

$$\frac{\partial}{\partial \phi} \arg z'(\zeta) = \operatorname{Im} \frac{z''(\zeta)}{z'(\zeta)}$$

is non-negative on C , by the convexity of C , and is non-positive on L_1 , by the concavity of L_1 . Thus

$$\frac{\partial}{\partial \phi} \arg w = \operatorname{Im} \frac{w'(\zeta)}{w(\zeta)} \leq 0$$

on C and

$$\frac{\partial}{\partial \phi} \arg w = \operatorname{Im} \frac{w'(\zeta)}{w(\zeta)} \geq 0$$

on L_1 , since

$$\arg w = -\arg z'(\zeta).$$

Therefore L_1 maps by the transformation $w(z)$ into a starlike curve and the arcs of C_1 and C_2 in the upper half-plane map into another starlike curve. These two starlike curves bound, together with a segment of the real axis, the region R in the hodograph plane. Thus R does intersect each ray from the origin in just one segment, as stated, and this follows even when C and L_1 are not smooth, since in that case we can approximate C and L_1 arbitrarily well by smooth curves and obtain the result in the limit.

We consider C_1 and C_2 to be fixed, while L_1 and L_2 are allowed to vary. We denote by α the minimum of the velocity

$$|w| = \left| \frac{d\zeta}{dz} \right|$$

on L_1 , and we choose L_1 so that α is a maximum. In order to verify the existence of an extremal curve L_1 of this type, we note that the analytic functions ζ and w associated with the flows past competing curves L_1 form normal families of schlicht functions in the upper half of the z -plane. Thus from a sequence of such functions corresponding to curves L_1 for which α approaches its maximum we can obtain a convergent subsequence. This yields in turn an extremal pair of curves L_1 and L_2 for our problem.

Consider now the region R in the hodograph plane which is the image of the upper half D of the extremal flow region under the mapping

$$w = \frac{d\zeta}{dz}.$$

The arc L_1 corresponds to a closed set l on the boundary of R which lies outside the circle $|w| = \alpha$ and within the sector $-\theta \leq \arg w \leq \theta$, where θ is the angle of inclination of the tangent to C at its upper end-point.

We prove that l covers the arc $-\theta \leq \arg w \leq \theta$ of the circle $|w| = \alpha$. If this were not the case, there would exist an open arc γ on $|w| = \alpha$, $-\theta \leq \arg w \leq \theta$, which is not covered. If the end-points of γ are in l , then their images in the z -plane cut off an arc Γ of L_1 which does not degenerate to a segment. Hence there is an arc Γ_0 included in Γ such that for some $\epsilon > 0$

$$(8.1) \quad \lim_{z \rightarrow \Gamma_0} \left| \frac{d\zeta}{dz} \right| \geq \alpha + \epsilon.$$

We can even choose Γ_0 not to be a segment.

We now make a variation of L_1 by replacing a small arc of Γ_0 by a segment δ , and we also make a symmetric variation of L_2 below the x -axis. This variation yields a larger flow region with a new boundary curve L_1^* . By the maximum

principle, the stream function ψ^* of the new flow is larger than the stream function ψ in D . But since ψ and ψ^* both vanish on the portion of L_1 which is unchanged, we see that

$$(8.2) \quad \left| \frac{d\zeta^*}{d\zeta} \right| > 1$$

there, where ζ^* denotes the complex potential of the new flow.

From (8.1), we find that if the arc δ is sufficiently small

$$(8.3) \quad \lim_{z \rightarrow \delta} \left| \frac{d\zeta^*}{dz} \right| \geq \alpha + \frac{\epsilon}{2}.$$

For the quantity on the left represents the distance from the origin in the hodograph plane of a ray corresponding to δ , and this ray must withdraw outside the circle $|w| = \alpha + \frac{1}{2}\epsilon$ as δ shrinks to a point. Combined with (8.2), this proves that the minimum velocity α^* on L_1^* is larger than α . Since L_1 maximizes α , we arrive at a contradiction, and we conclude that Γ must degenerate to a segment.

Since Γ is a segment, the arc γ does not have both its end-points in l . If we take γ as large as possible, we see that it must either include the entire arc $-\theta < \arg w < \theta$ of $|w| = \alpha$ or else consist of an arc $-\theta < \arg w < -\theta_0$ or an arc $\theta_0 < \arg w < \theta$ of $|w| = \alpha$. The first possibility is excluded, since it again implies the existence on L_1 of an arc Γ_0 satisfying (8.1) which is not a segment. It remains only to show that $\theta_0 < \theta$ is impossible.

If $\theta_0 < \theta$, then since R is intersected by each ray from the origin in at most one line segment, we find that l consists of the arc $-\theta_0 \leq \arg w \leq \theta_0$ of $|w| = \alpha$ and two infinite rays along $\arg w = \theta_0$ and $\arg w = -\theta_0$. These two rays correspond to symmetric segments Γ_1 and Γ_2 at the ends of L_1 . We make a variation of L_1 which consists in raising the arc of L_1 between the midpoints of Γ_1 and Γ_2 a small amount, while the remainder of L_1 is replaced by two segments joining the upper end-points of C_1 and C_2 to the shifted curve. We make a symmetric variation of L_2 below the x -axis, and thus we obtain a new flow region D^* and a new curve L_1^* in the upper half-plane. If we translate D^* vertically downwards until it just includes D , we find that the translated stream function ψ^* satisfies

$$\psi^* > \psi$$

in D , by the maximum principle. Thus

$$(\nabla\psi^*)^2 > \alpha^2$$

all along L_1^* for a small variation and we obtain a contradiction of the existence of the segments Γ_1 and Γ_2 .

Thus $\theta_0 = \theta$, and l consists of the circular arc $-\theta \leq \arg w \leq \theta$ on $|w| = \alpha$ alone. The remainder of the boundary of R consists of a starlike curve inside

$|w| = \alpha$ corresponding to C_1 and C_2 and a segment of the real axis corresponding to the x -axis in the z -plane.

We conclude that L_1 and L_2 are free streamlines for the flow past C_1 , C_2 , L_1 and L_2 in the sense that velocity is a constant there,

$$(8.4) \quad \left| \frac{d\zeta}{dz} \right| \equiv \alpha.$$

Hence the principle of maximum least velocity yields the existence of a cavitation flow past the plates C_1 , C_2 .

A wide variety of further physical models could be treated by the same method. Note also that by letting C_2 approach infinity to the right, we can obtain in the limit the Helmholtz flow past C_1 alone with an infinite cavity.

9. Finite wakes. In this section we develop a theory of the free stream flow past a body followed by a wake of gas at pressure higher than the pressure at infinity. The method is essentially the same as that in section 8, except that here we minimize the maximum velocity β on the free surface.

Let C denote a simple convex arc which is symmetric in the x -axis and whose tangent turns through more than 180 degrees as we proceed from its upper end-point P_1 to its lower end-point P_2 . We suppose that the tangents at P_1 and P_2 intersect on the real axis to the right of C and that they bound with C a convex body B . We assume that the velocity of the symmetric flow past B with uniform velocity 1 at infinity has on these tangents a maximum value less than one.

We denote by L_1 a curve which is convex upwards and joins P_1 to a variable point Q on the x -axis, and we denote by L_2 the reflection of L_1 in the x -axis, which joins P_2 to Q . The curves L_1 and L_2 together bound a wake behind C which ends in a cusp on the real axis at Q .

Consider the flow past C , L_1 and L_2 with a complex potential

$$\zeta = \phi + i\psi = z + \frac{a}{z} \cdots,$$

and denote the maximum of the velocity

$$|w| = \left| \frac{d\zeta}{dz} \right|$$

on L_1 and L_2 by β . We shall only be concerned with curves L_1 and L_2 for which $\beta < 1$. This class of curves is not empty, since it contains the tangents to C at P_1 and P_2 .

The conformal mapping by the velocity

$$w = \frac{d\zeta}{dz}$$

takes the portion D of the region of flow past C , L_1 and L_2 lying in the upper half-plane into a region R of the hodograph plane with the property that every ray from the origin intersects it in at most one line segment. For on the upper arc of C

$$\frac{\partial}{\partial \phi} \arg w = -\frac{\partial}{\partial \phi} \arg z'(\zeta) = -\operatorname{Im} \frac{z''(\zeta)}{z'(\zeta)} \geq 0,$$

since C is convex downwards, and on L_1

$$\frac{\partial}{\partial \phi} \arg w = -\frac{\partial}{\partial \phi} \arg z'(\zeta) = -\operatorname{Im} \frac{z''(\zeta)}{z'(\zeta)} \leq 0,$$

by the convexity of L_1 upwards, so that C maps into a starlike curve on the boundary of R and L_1 maps into another starlike curve bounding R . Thus R is bounded by two starlike curves and a segment of the real axis and therefore has the desired property. This reasoning does not apply directly if C and L_1 are not smooth, but the result can then be obtained by approximating C and L_1 by smooth curves and passing to the limit.

We take C to be fixed, and we choose the variable curves L_1 and L_2 and the variable point Q on the x -axis so that the maximum β of the velocity on L_1 of the flow past C , L_1 and L_2 is a minimum. Such a flow for which β is a minimum exists, because the functions ζ and w associated with the flows corresponding to a sequence of curves L_1 for which β tends to its minimum value form normal families of schlicht functions in the upper half of the z -plane. Thus a convergent subsequence of these functions exists, and hence a minimum curve L_1 can be obtained. The extremal curve does not extend to infinity, since if it did then β would be unity, and thus Q is located at a finite position on the x -axis.

Consider the region R of the hodograph plane which is the image of the extremal flow region D . On the boundary of R there is a closed set l corresponding to L_1 in the physical plane. The set l lies inside the circle $|w| \leq \beta$ and inside the sector $-\theta \leq \arg w \leq 0$, where θ is the angle of inclination of the tangent to C at P_1 . Suppose that l does not cover the arc $-\theta \leq \arg w \leq 0$ of $|w| = \beta$. Then there is an open arc γ of $-\theta \leq \arg w \leq 0$, $|w| = \beta$ which is not in l . The portion of l which projects radially onto the closure of γ corresponds to an arc Γ of L_1 . If Γ is not a segment, there is an arc Γ_0 included in Γ , different from a segment, and such that

$$(9.1) \quad \overline{\lim}_{z \rightarrow \Gamma_0} \left| \frac{d\zeta}{dz} \right| \leq \beta - \epsilon,$$

for some $\epsilon > 0$.

We make a variation of L_1 by replacing an arc of Γ_0 by a small segment δ , with a symmetric change in L_2 . This gives a varied region of flow D^* in the upper half-plane which is included in D and has a new boundary arc L_1^* . The varied

stream function ψ^* is smaller than ψ , by the maximum principle, and hence

$$(9.2) \quad \left| \frac{d\zeta^*}{d\zeta} \right| < 1$$

on the arcs of L_1 which are not varied, since ψ and ψ^* both vanish there. The image of δ in the hodograph plane is a ray from the origin which must recede within the circle $|w| = \beta - \frac{1}{2}\epsilon$ as δ shrinks to a point. Thus for a small enough segment

$$(9.3) \quad \overline{\lim}_{\delta \rightarrow 0} \left| \frac{d\zeta^*}{dz} \right| \leq \beta - \frac{1}{2}\epsilon.$$

Formulas (9.2) and (9.3) together show that the maximum velocity β^* associated with L_1^* is smaller than β , which contradicts the extremal character of L_1 . It follows that Γ must be a segment or must not exist.

If the end-points of γ are in l , then Γ cannot be a segment. But since R intersects each ray from the origin in at most one segment, the point $w = \beta e^{-i\theta}$ is in l , and thus if γ is chosen to be as large as possible it can only consist of an arc $-\theta_0 < \arg w < 0$ of $|w| = \beta$. Since Γ is a segment, in this case the curve L_1 ends to the right in a segment inclined at an angle θ_0 with the real axis and meeting the real axis at Q . We can replace L_1 near Q by a small segment δ inclined at a positive angle smaller than θ_0 , and we can repeat the reasoning just given to obtain (9.2) and (9.3) for this additional variation. We again contradict the extremal character of L_1 , and the conclusion is that γ does not exist.

It follows that l consists of precisely the arc $-\theta \leq \arg w \leq 0$ of the circle $|w| = \beta$. Hence

$$(9.4) \quad \left| \frac{d\zeta}{dz} \right| \equiv \beta$$

on L_1 and L_2 , and L_1 and L_2 are free streamlines for the flow past C . This is therefore a flow past an object C followed by a finite wake ending in a cusp on the real axis. We have thus obtained a simple proof of the existence of such a flow based on the extremal problem

$$(9.5) \quad \beta = \max_{L_1} \left| \frac{d\zeta}{dz} \right| = \text{minimum.}$$

We note that the above procedure gives a very simple estimate of the possible points of separation P_1 and P_2 of a finite wake. Let C_0 be a closed convex curve of which C is only an arc. Suppose that the uniform flow

$$\zeta = z + \frac{a}{z} + \dots$$

past C_0 has a velocity $v_0 < 1$ on $C_0 - C$. We compare the stream function ψ_0 of this flow with the stream function ψ of the flow past the object B bounded by

C and the tangents to C at P_1 and P_2 . In the exterior of B , $\psi_0 > \psi$ by the maximum principle. Thus at P_1

$$v^2 = (\nabla\psi)^2 \leq (\nabla\psi_0)^2 = v_0^2 < 1,$$

since $\psi = \psi_0 = 0$ on C . The velocity v of the flow past B decreases monotonically on the segments from P_1 and P_2 to Q , as can be seen by a consideration of the geometry of the image of the flow in the hodograph plane. Thus $v < 1$ on these tangents, and our hypotheses on C are satisfied.

If we now solve the extremal problem (9.5) for C , we obtain two symmetric free streamlines L_1 and L_2 leaving C_0 at P_1 and P_2 . Since L_1 is convex upward, it does not intersect C_0 , and hence P_1 and P_2 are possible points of separation for a wake following C_0 . Thus our estimate on the location of P_1 and P_2 merely states that they must lie in a position on C_0 where $v_0 < 1$.

10. Axially symmetric free surface flow. Since most of the material of the preceding two sections depends rather on the topological properties of conformal mappings than on the analytical properties, it would seem feasible to attempt generalizing the proof of existence of cavitation flow based on the principle of maximum least velocity to the case of axially symmetric flows. However, a serious difficulty is encountered because the basic topological questions which arise for the quasi-conformal mapping by the velocity vector of an axially symmetric flow remain unsettled. Thus more progress is obtained by generalizing the procedures developed around the principle of minimum virtual mass.

We consider a figure B_0 in the (x, y) -plane which is symmetrized in the x -axis and y -axis. We denote by C the boundary of B_0 , by C_1 the arc of C in the half-plane $x < -k$, and by C_2 the arc of C in the half-plane $x > k$, where $k > 0$. Let L_1 and L_2 be two arcs which, together with C_1 and C_2 , bound an object B containing B_0 , and assume that L_1 and L_2 lie in the strip $-k \leq x \leq k$. We consider the axially symmetric flow past the object B' obtained by rotating B about the x -axis. We attempt to choose L_1 and L_2 so that for fixed volume V of B' the virtual mass

$$M = \pi \iint_D \frac{1}{y} (\nabla\psi - \frac{1}{2}\nabla y^2)^2 dx dy$$

of the flow is a minimum. Here D is the region of flow and ψ is the stream function.

Since, by section 4, M decreases when B is symmetrized in the y -axis, or in the x -axis relative to 3-dimensional space, we can restrict the competing shapes for B to be invariant under such symmetrization. Hence the curves L_1 and L_2 can be assumed to be monotonic in each quadrant. Also, it is easily seen that M increases more rapidly than V if B rises to infinity, whence the competing shapes can be assumed to lie in a bounded region.

Since the class of bounded functions of bounded variation is compact, by a

theorem of Helly, we find that an extremal object B exists such that M is a minimum for fixed V . A simpler formulation of the problem even states that B can be chosen with rectifiable boundaries L_1 and L_2 so that

$$(10.1) \quad M - \lambda V = \text{minimum}$$

for any fixed $\lambda > 0$.

The extremal curves L_1 and L_2 for (10.1) can be considered to yield a generalized solution of the free surface problem for axially symmetric flow. If we could show that the rectifiable curves L_1 and L_2 are analytic, then by section 2 we would obtain an exact solution.

While we are unable to present here a proof of the analyticity of L_1 and L_2 , it is perhaps worth while to mention how far one can get with methods suggested by formula (5.7).

We denote by R the exterior of a large circle S containing B , and we set $\psi = \sqrt{y}u$. Thus

$$\begin{aligned} \frac{1}{\pi} M = & \iint_R \frac{1}{y} (\nabla \psi - \tfrac{1}{2} \nabla y^2)^2 dx dy + \int_S \left\{ 2\psi - \frac{1}{2} \frac{u^2}{y} \right\} dx \\ & + \iint_{D-R} \left\{ (\nabla u)^2 + \frac{3}{4y^2} u^2 \right\} dx dy + \iint_{D-R} y dx dy. \end{aligned}$$

We can now apply the Dirichlet principle to u , which is equivalent to the Dirichlet principle for ψ . This shows by (10.1) that when u and D are defined by the extremal object B ,

$$(10.2) \quad \iint_{D-R} \left\{ (\nabla u)^2 + \frac{3}{4y^2} u^2 \right\} dx dy + (\lambda + 1) \iint_{D-R} y dx dy = \text{minimum}$$

among all functions u with given boundary values on S equal to $\psi y^{-\frac{1}{2}}$ and with zero boundary values on C_1 , C_2 , L_1 and L_2 , and among all curves L_1 , L_2 . A better formulation even states that if Γ is any curve in D bounding with an arc l of L_1 a certain region Ω , then

$$(10.3) \quad \iint_{\Omega} \left\{ (\nabla u)^2 + \frac{3}{4y^2} u^2 + (\lambda + 1)y \right\} dx dy = \text{minimum}$$

in the class of all functions u which vanish on l and have on Γ values $\psi y^{-\frac{1}{2}}$, and in the class of all curves l connecting the end-points of Γ .

If we make a variation

$$z^* = z + \frac{\epsilon}{z - z_0}$$

of Ω near l which is continued smoothly to become the identity transformation near Γ , we can induce a variation of u and obtain a variational identity from

(10.3) which states that certain integrals over Ω vanish. A penetrating discussion of the dependence of these integrals on z_0 could be made along the lines of Fatou's theorem, and it would be possible to show that the velocity $(\nabla\psi)^2/y^2$ of the extremal flow tends almost everywhere to $\lambda + 1$ on the free curves L_1, L_2 . We do not push this discussion further, however, since the only satisfactory result would involve proving analyticity of l from (10.3).

Although existence of axially symmetric free surface flow has not yet been shown by (10.1), it is still possible to make an approximation to the free surface using this minimum principle. For example, one could take for C_1 and C_2 a pair of symmetric vertical segments and for L_1, L_2 polynomial curves of a fixed finite degree. The volume V could be immediately computed, and the virtual mass might be approximated by an expression obtained from a suitable set of orthogonal functions over C_1, C_2, L_1 , and L_2 [1]. Minimization of (10.1) with respect to the coefficients of the polynomial defining L_1, L_2 would then yield an approximation of the free surface. Whether such a procedure gives accurate results is a question well worth investigating with explicit calculations.

11. Remarks on water entry of flat missiles. The motion of an object striking a water surface vertically or obliquely is divided into several stages. We shall be concerned here with the initial, or impact, phase of the motion, which covers the period from contact with the water surface up to the instant when a cavity begins to form behind the object. During this phase the wetted area on the surface of the entering object increases with time, and the fluid pressure acting on this wetted area retards the object through a force of impact. The water rises, and this creates a splash around the sides of the object. The determination of the splash contour is a difficult free boundary problem which will be decisive in studying the entry of the object.

For the sake of mathematical simplicity, we shall assume here that the object is fixed and that the water rises at uniform velocity to meet the fixed surface. At a point of this surface the normal component of the velocity of the fluid vanishes, while on the free surface of the splash contour each fluid particle has an acceleration which is normal to the surface. The latter condition arises from the assumption that the pressure along the free surface is constant and that the pressure gradient is in the normal direction. We shall only consider the cases in which the flow pattern at any time t is obtained from that at some fixed time t_0 by a similarity transformation, which implies that the entering object must be a cone, a wedge or a plate [2, 10, 13].

Let time t be measured from the instant when the horizontal water surface first meets the vertex of the object. At time $t > 0$ the surface S_t of the water will have assumed a certain splash configuration. Let any surface particle be described by the coordinate s which represents the distance of this particle from the vertex of the object at the initial time $t = 0$, and let $\mathbf{r}(s, t)$ denote the radius vector extending from the vertex to the particle s at time t in general. We write $r(s, t) = |\mathbf{r}(s, t)|$ and we conclude from the similarity of the splash contours

S_t and S_τ that

$$\frac{r(s, t)}{r(\sigma, \tau)} = \frac{t}{\tau},$$

where σ denotes the surface particle which lies at time τ on the extension of the radius vector $\mathbf{r}(s, t)$. On the other hand, since the particle s and the particle σ have similar trajectories with increasing time, we obtain

$$\frac{r(s, t)}{r(\sigma, \tau)} = \frac{s}{\sigma}.$$

Thus, setting

$$\lambda = \frac{\tau}{t} = \frac{\sigma}{s},$$

we have the homogeneity relation

$$(11.1) \quad \lambda \mathbf{r}(s, t) = \mathbf{r}(\lambda s, \lambda t).$$

Since the pressure gradient on the free surface is normal, we obtain there

$$(11.2) \quad \mathbf{r}_{tt} \cdot \mathbf{r}_s = 0.$$

Also, by differentiation of (11.1) with respect to λ we have for $\lambda = 1$

$$(11.3) \quad \mathbf{r} = t\mathbf{r}_t + s\mathbf{r}_s.$$

These are the two basic boundary conditions on the free surface. Differentiating (11.3) with respect to t and s , we obtain

$$\mathbf{r}_t = \mathbf{r}_t + t\mathbf{r}_{tt} + s\mathbf{r}_{st},$$

$$\mathbf{r}_s = \mathbf{r}_s + s\mathbf{r}_{ss} + t\mathbf{r}_{st},$$

whence by (11.2)

$$2\mathbf{r}_s \cdot \mathbf{r}_{ss} = \frac{\partial}{\partial s} \mathbf{r}_s^2 = 0.$$

Therefore $\mathbf{r}_s^2$ is constant on each free surface, and if $\mathbf{r}_s^2 = 1$ at $s = \infty$ we obtain

$$(11.4) \quad \mathbf{r}_s^2 = 1$$

over the whole free boundary.

The identity (11.4) shows that the parameter s is actually arc length along the free surface. With this interpretation of s , condition (11.3) is all that is required along the free boundary.

Since the flow is incompressible and irrotational, there is at each time t a velocity potential ϕ . We have on the free boundary

$$\begin{aligned} r_t &= \nabla \phi, \\ r &= \nabla \frac{1}{2} r^2, \end{aligned}$$

and therefore, taking $t = 1$ without loss of generality, we can write (11.3) in the form

$$(11.5) \quad \nabla \phi = \nabla \frac{1}{2} r^2 - s \mathbf{r}_s.$$

Thus for either plane flow or axially symmetric flow, that is, for either an entering wedge or an entering cone, we can formulate our boundary value problem in terms of the function

$$\Phi = \frac{1}{2} r^2 - \phi$$

as follows. We ask that Φ satisfy in the flow region the partial differential equation

$$(11.6) \quad \Delta \Phi = 2,$$

that it satisfy on the entire boundary

$$(11.7) \quad \frac{\partial \Phi}{\partial n} = 0,$$

and we ask that the free boundary be such that

$$(11.8) \quad (\Delta \Phi)^2 = 2\Phi$$

there. Indeed, one easily sees from the definition of Φ that (11.7) is valid on the edge of the cone or wedge, and (11.7) and (11.8) can be integrated along the free surface to give

$$\Phi = \frac{1}{2} s^2$$

there, in agreement with (11.5). The conditions (11.6), (11.7) suggest an analogy between the present problem and the study of torsional rigidity, which leads to (11.6), but with (11.7) replaced by the corresponding first boundary condition $\Phi = 0$. It is also natural to ask for a generalization of the minimum principle of section 2 which would yield (11.8) on the free surface in addition to (11.7). A beginning can be made in this direction by investigating the change of the improper integral

$$\iiint \{(\nabla \Phi)^2 + 4\Phi\} dx dy dz$$

over the flow region when the free boundary is shifted. If we interpret this integral as a modified virtual mass, (11.8) suggests that the free surface gives sta-

tionary energy, but a rigorous formulation requires the introduction of an impact pressure term and leads to numerous difficulties.

We proceed to discuss in some detail the case of plane flow against an entering wedge. If

$$\zeta = \phi + i\psi$$

denotes the complex potential of the flow and the origin is taken to lie at the vertex of the wedge, then (11.5), or (11.3), can be written

$$(11.9) \quad \zeta'(z) = \bar{z} - s\bar{z}'(s),$$

where z and r are to be identified and where $z'(s)$ and r , both denote the unit tangent vector along the free surface. If 2α is the interior angle of the wedge, we have on the fixed boundary

$$(11.10) \quad \arg \zeta'(z) = \pm \frac{\pi}{2} \pm \alpha,$$

by (11.7).

The condition (11.9) has an interesting geometrical interpretation. It states that the conformal mapping by the velocity $\zeta'(z)$ takes the free boundary into the reflection in the real axis of its evolute, with corresponding points in the mapping identified with corresponding points on the curve and evolute. The fact that the normal to the evolute is the tangent to the free boundary can be exploited. We define the curvature $k(s)$ of the free boundary by the relation

$$z''(s) = ik(s)z'(s)$$

and we differentiate (11.9) with respect to s to obtain

$$\zeta''(z)z'(s) = \bar{z}'(s) - \bar{z}'(s) + isk(s)\bar{z}'(s),$$

or

$$(11.11) \quad \zeta''(z)z'(s)^2 = isk(s).$$

We set

$$F(z) = \int \sqrt{\frac{\zeta''(z)}{i}} dz$$

and find

$$F = \int \sqrt{sk} ds = \text{real}$$

on the free boundary. By (11.10), the argument of $F(z)$ is also known on the fixed boundary, and hence $F(z)$ can be calculated explicitly in terms of the conformal mapping of the region of flow onto the upper half-plane.

If we could find another expression in terms of ζ which could be calculated in this way, we would be able to obtain the free surface in a manner analogous to the procedure of Kirchhoff for obtaining cavitation flows past flat objects. Although such a method is not at present available, we can make an interesting attempt in this direction based on an idea due to Lewy [7] already used in section 7.

We consider the arc length s on one arc L of the free boundary as an analytic function $w(z)$,

$$(11.12) \quad s = w(z).$$

We find immediately that on L

$$(11.13) \quad w'(z) = \frac{ds}{dz} = \bar{z}'(s),$$

while also

$$\bar{z} = \int \bar{z}' ds = \int w'(z)^2 dz$$

there. Hence in terms of w , (11.9) can be written

$$\zeta'(z) = \int w'(z)^2 dz - w(z)w'(z).$$

Differentiating with respect to z , we have

$$(11.14) \quad \zeta''(z) = -w(z)w''(z).$$

Therefore w can be continued analytically throughout the region of flow as a solution of the ordinary differential equation (11.14). Conversely, it is possible to prove by means of (11.14) that the free boundary is composed of analytic curves.

On the arc L of the free boundary, we know by (11.12) that w is real and we know by (11.13) that w' has unit modulus. If we could obtain similar information about the behavior of w and w' on the fixed boundary and on the remainder of the free boundary from (11.10) and (11.14), we could find w' as a function of w by conformal mapping. The unknown functions z and ζ' are expressed by the formulas

$$(11.15) \quad z = \int \frac{dw}{w'},$$

$$(11.16) \quad \zeta' = \int w' dw - ww',$$

and thus we would obtain a complete determination of the splash contour for the wedge.

Although the determination of w' as a function of w is beyond our ability here, it is worth while to notice that (11.15) and (11.16) generate solutions of the free boundary problem (11.9) whenever w' represents as a function of w a conformal mapping of a segment of the real axis onto an arc of the unit circle. Such a construction of flows satisfying (11.9) is of qualitative interest even when all requirements are not fulfilled.

As an example, let us take for the region in the w' -plane the truncated sector

$$-\beta < \arg w' < \beta, \quad |w'| > 1,$$

and let us take as the region in the w -plane the half-plane

$$\operatorname{Im} w < 0,$$

with the segment $-1 \leq w \leq 1$ corresponding to $\arg w' = \pm\beta$ and with $w = \infty$ corresponding to $w' = 1$. Then the conformal map from the w -plane to the w' -plane is

$$w' = \left\{ \frac{i}{w} + \sqrt{1 - \frac{1}{w^2}} \right\}^{-2\beta/\pi},$$

and by (11.15) and (11.16) we obtain

$$(11.17) \quad z = \int_0^w \left\{ \frac{i}{w} + \sqrt{1 - \frac{1}{w^2}} \right\}^{2\beta/\pi} dw,$$

$$(11.18) \quad \zeta' = \frac{2\beta}{\pi} \int_0^w \left\{ \frac{i}{w} + \sqrt{1 - \frac{1}{w^2}} \right\}^{-(2\beta+\pi)/\pi} \left\{ \frac{1}{w\sqrt{w^2-1}} - \frac{i}{w} \right\} dw.$$

The fixed surface of the flow in the z -plane is a wedge with interior angle

$$2\alpha = \pi - 2\beta,$$

and the free surface consists of two concave arcs which leave the wedge tangentially upwards and rise to a horizontal position at infinity. The flow is symmetric in the y -axis and (11.9) is satisfied on the free boundary, but this solution leaves something to be desired, since the velocity at infinity is infinite.

We obtain another interesting example if we map the lower half of the w -plane slit along a segment of the imaginary axis onto the sector $-\alpha < \arg w' < \alpha$, $|w'| < 1$ in the w' -plane in such a way that the slit in the w -plane corresponds to the rays $\arg w' = \pm\alpha$ and the point $w = \infty$ corresponds to $w' = 1$. Integrating to determine z and ζ' , we find that the fixed boundary in the z -plane is again an entering wedge with interior angle 2α , but the free surface consists this time of two symmetric convex arcs which leave the wedge at right angles and descend to a horizontal position at infinity. The solution is again unsatisfactory because of poor behavior at infinity.

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