On the Characteristic Surfaces of the von Mises Plasticity Equations

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1. Introduction. Characteristic curves are frequently introduced in the two dimensional treatment of the plasticity problem. Such curves are sometimes referred to as slip lines¹ (*lignes de glissement*) and they have also been considered as the curves of separation of elastic and plastic states.² In view of these interpretations it would seem reasonable to carry out an investigation of the possibility of the existence of characteristic surfaces for general three dimensional plastic flow but we have found no reference to such a study in the literature.

As the basis of the following discussion of characteristic surfaces we have chosen the plasticity equations of von Mises, *first* because these equations have been widely used and *second* because of their inherent mathematical simplicity. To express the von Mises equations we first define the so called *stress deviation* s of the stress tensor σ by writing³

$$(1) \quad s_{\alpha\beta} = \sigma_{\alpha\beta} + p\delta_{\alpha\beta}, \quad p = -\frac{1}{3}\sigma_{\alpha\alpha}$$

so that $s_{\alpha\alpha} = 0$. The von Mises plasticity equations assert the proportionality of the tensor s and the *rate of strain tensor* ϵ , i.e.

$$(2) \quad s_{\alpha\beta} = \lambda\epsilon_{\alpha\beta}, \quad \epsilon_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha})$$

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¹ H. GEIRINGER, *Fondements Mathématiques de la Théorie des Corps Plastique Isotropes*, Mémorial des Sciences Mathématiques No. 86, Paris, Gauthier-Villars, (1937), p. 33. Also R. HILL, *The Mathematical Theory of Plasticity*, Oxford Eng. Sci. Ser., (1950), p. 134.

² See W PRAGER and P. G. HODGE JR., *Theory of Perfectly Plastic Solids*, J. Wiley and Sons, New York, (1951), p. 145.

³ All indices (subscripts or superscripts) can have the values 1, 2, 3, and it is to be understood that the repetition of an index in any term denotes a summation over these values unless the contrary is stated or is otherwise evident from the context. Since rectangular coordinates are assumed throughout the discussion we need not distinguish between subscripts and superscripts in the application of the summation convention.

where the v_α are the velocity components and the proportionality factor λ is, in general, a function of position and time. It will be assumed that the function λ is *positive* and *finite*. Further equations occurring in this theory are as follows⁴

$$(3) \quad \sigma_{\alpha\beta,\beta} = 0 \quad (\text{equations of equilibrium}),$$

$$(4) \quad v_{\alpha,\alpha} = 0 \quad (\text{equation of incompressibility}).$$

Two yield conditions are considered—the von Mises and the Tresca. They are given by the following equations

$$(5) \quad s_{\alpha\beta} s_{\alpha\beta} = 2k^2 \quad (\text{von Mises yield condition}),$$

$$(6) \quad s_1 - s_3 = 2k \quad (\text{Tresca yield condition}),$$

where k is a *positive* constant depending on the material and s_1, s_2, s_3 denote the principal values of the tensor s with the ordering $s_1 \geq s_2 \geq s_3$. Equations (2), (3), (4) together with either (5) or (6) constitute a system of $5 + 3 + 1 + 1 = 10$ relations in the 10 variables $\sigma_{\alpha\beta}, v_\alpha$ and λ . We shall investigate the characteristic surfaces of the basic equations (2), (3) and (4) under each of the above yield conditions.

When the von Mises yield condition (5) is assumed it is shown (§5) that real characteristic directions (normals to the characteristic surface elements) can exist if, and only if, the intermediate principal value s_2 (or the corresponding principal value ϵ_2 of the rate of strain tensor) is equal to zero. If the condition $s_2 = 0$ is satisfied there will be precisely two real characteristic directions at each point, and the characteristic elements which they determine will be identical with the surface elements of maximum and minimum shearing stress. The requirement that $s_2 = 0$, which appears somewhat unnatural, does not arise when the von Mises yield condition is replaced by the Tresca condition (6). Otherwise the result is the same, i.e. there exist two, and only two, real characteristic directions at each point of the medium and the characteristic surface elements thereby determined are the surface elements of maximum and minimum shearing stress (§6). In a sense this result is somewhat remarkable. For, in general, the real characteristic directions (if they exist) of a system of differential equations will lie on the so called characteristic cone and hence will be infinite in number at each point. The fact that there are only two characteristic surface elements at each point and that they are the same as the elements of maximum and minimum shearing stress (or rate of strain) points to the qualitative validity of the above plasticity equations in certain essential physical aspects.

⁴ Equations of equilibrium (3) rather than equations of motion are used since it is customary to consider that for slow plastic flow the inertial effects can be neglected. However the inertial terms involve derivatives of lower order and hence the omission of these terms is without influence in the determination of the characteristic surfaces.

We now raise the following question. *Do the above two families of characteristic surface elements unite to form two congruences of surfaces in the space?* Here the result is disappointing. The requirement, in the form of differential equations in the velocity vector v , for the existence of the congruences of characteristic surfaces is given in §7 under the assumption of the von Mises and also under the assumption of the Tresca yield condition. These differential equations are not consequences of the above plasticity equations (including the yield condition) and hence the condition that characteristic surfaces exist in the space will, in general, result in an over-determination of the differential equations for the velocity. It is doubtful of course if this over-determined system is admissible.

The non-existence of characteristic surfaces in general casts doubt as to the basic validity of the usual interpretation of characteristics as slip lines in plane flow, which must logically be considered as a special case of the general three dimensional problem. If we wish to interpret characteristic surfaces as slip surfaces (and hence to retain the interpretation of characteristic curves as slip lines in plane flow) we must be prepared to modify the above plasticity equations. Now it is questionable if any of the theories of plasticity which have so far been devised will yield definitely determined families of characteristic surface elements, analogous to those of the von Mises theory, possessing the property that these elements will combine to form actual surfaces. If the characteristic directions lie on a cone at each point of the medium the characteristic surfaces will be determined by a differential condition; this would resolve the immediate difficulty but the solution of the required differential equation for the characteristic surfaces will depend on a set of extraneous boundary conditions with the result that these surfaces may not have a definite determination. It is not our intention to consider these matters in the present paper. However it is our feeling that the questions here raised should be the object of a serious investigation preferably by one of the more experienced workers in the field of plasticity.

2. The reduced system of differential equations. Using the von Mises or the Tresca yield condition it is possible to eliminate the deviation s from the plasticity equations and thus to obtain a system of differential equations in the quantity p and the components v_α of the velocity vector. This system will be referred to as *the reduced system of differential equations*. To derive the reduced system we first eliminate the proportionality factor λ from (2) by means of (5) or (6). Thus

$$(7) \quad s_{\alpha\beta} s_{\alpha\beta} = \lambda^2 \epsilon_{\alpha\beta} \epsilon_{\alpha\beta} = 2k^2 \quad (\text{von Mises yield}),$$

$$(8) \quad s_1 - s_3 = \lambda(\epsilon_1 - \epsilon_3) = 2k \quad (\text{Tresca yield}),$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ are the principal values of the rate of strain tensor ϵ . Equation (8) follows immediately from the fact that $s_i = \lambda\epsilon_i$ ($i = 1, 2, 3$). Furthermore

$\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$ since $s_1 \geq s_2 \geq s_3$ by the above ordering and since λ is positive and finite by the assumption in §1. Before proceeding further it will be helpful to make the following

Remark. Under the von Mises yield condition the inequality $\epsilon_{\alpha\beta}\epsilon_{\alpha\beta} > 0$ is a consequence of (7) and the conditions $k > 0, \lambda > 0$ stated in §1. Similarly the inequality $\epsilon_1 - \epsilon_3 > 0$ follows from (8) under the Tresca yield condition.

Now $\epsilon_1 - \epsilon_3 = 0$ implies $\epsilon_1 = \epsilon_2 = \epsilon_3$ on account of the condition $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$. But the equality of the ϵ_i and the form $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ of the incompressibility condition gives $\epsilon_i = 0$ for $i = 1, 2, 3$ and hence $\epsilon_{\alpha\beta} = 0$ from the relation (11). It follows that $\epsilon_1 - \epsilon_3 > 0$ also under the yield condition of von Mises. Similarly we see that under the yield condition of Tresca the inequality $\epsilon_{\alpha\beta}\epsilon_{\alpha\beta} > 0$ must likewise be satisfied. Hence both inequalities $\epsilon_{\alpha\beta}\epsilon_{\alpha\beta} > 0$ and $\epsilon_1 - \epsilon_3 > 0$ are valid at each point irrespective of whether the yield condition of von Mises or Tresca is employed.

An interesting consequence of the above inequalities is the result that the plastic flow in any region cannot arise solely from a rigid motion of the medium. For the condition $\epsilon_{\alpha\beta} = 0$, which is seen to be excluded, is precisely the condition for the flow to be of the rigid type.⁵

From (2), (7) and (8) we now have⁶

$$(9) \quad s_{\alpha\beta} = \frac{\sqrt{2} k \epsilon_{\alpha\beta}}{\sqrt{\epsilon_{ij} \epsilon_{ij}}} \quad (\text{von Mises yield}),$$

$$(10) \quad s_{\alpha\beta} = \frac{2k \epsilon_{\alpha\beta}}{\epsilon_1 - \epsilon_3} \quad (\text{Tresca yield}).$$

By expressing the von Mises yield condition (5) in terms of the invariants s_i we are led to a form of the equations (9) more closely resembling the form (10) which results from the Tresca yield condition. From the form of the equations (1) and (2) the tensors σ , s and ϵ will determine the same principal directions at any point. Let us denote by h_i ($i = 1, 2, 3$) the three mutually perpendicular unit vectors giving the three principal directions and by h_i^α the components of these vectors. Furthermore let y^i be the coordinates of a rectangular system having origin at a point P such that the positive y^i axis has the direction of the vector h_i at P . It is then well known that $\sigma_{\alpha\beta} = s_{\alpha\beta} = \epsilon_{\alpha\beta} = 0$ at P for $\alpha \neq \beta$ relative to the y system and also that $\sigma_{\alpha\alpha} = \sigma_\alpha$, $s_{\alpha\alpha} = s_\alpha$ and $\epsilon_{\alpha\alpha} = \epsilon_\alpha$ under this same condition, where the σ_i are the principal values of the σ tensor. Hence we can write

$$(11) \quad \sigma_{\alpha\beta} = \sigma_\alpha \delta_{\alpha\beta}, \quad s_{\alpha\beta} = s_\alpha \delta_{\alpha\beta}, \quad \epsilon_{\alpha\beta} = \epsilon_\alpha \delta_{\alpha\beta}$$

at the origin of the y system. Also it can easily be shown that⁷

$$(12) \quad \sigma_{i,\gamma} = \sigma_{\alpha\beta,\gamma} h_i^\alpha h_i^\beta; \quad s_{i,\gamma} = s_{\alpha\beta,\gamma} h_i^\alpha h_i^\beta; \quad \epsilon_{i,\gamma} = \epsilon_{\alpha\beta,\gamma} h_i^\alpha h_i^\beta,$$

⁵ See T. Y. THOMAS, *On the stability of viscous fluids*, Seminar Reports in Mathematics (Los Angeles), Univ. of Calif. Pub. in Math., N.S., vol. 2, (1944), p. 23.

⁶ Cp. W. PRAGER and P. G. HODGE JR., *loc. cit.*, p. 31.

⁷ See T. Y. THOMAS, *Surfaces of maximum shearing stress*, Jour. of Math. and Physics, vol. 24, (1944), p. 170.

these relations being independent of the rectangular coordinate system employed. At the origin of the y system the first of the equation (12) reduces to $\sigma_{i,\gamma} = \sigma_{ii,\gamma}$ since $h_i^\alpha = \delta_i^\alpha$ at the origin and there are corresponding relations obtained from the other equations of the set (12). On account of (11) we readily obtain the two following equivalent forms of the von Mises yield condition (5), namely

$$(13) \quad s_1^2 + s_2^2 + s_3^2 = 2k^2 \quad (\text{von Mises yield condition}),$$

$$(14) \quad (s_1 - s_2)^2 + (s_1 - s_3)^2 + (s_2 - s_3)^2 = 6k^2 \quad (\text{von Mises yield condition}).$$

In fact the form (13) results immediately by evaluating (5) at the origin of the y system and (14) is determined from (13) and the relation $s_{\alpha\alpha} = 0$. Since the left members of (13) and (14) are symmetric functions of the invariants s_1, s_2, s_3 some simplification of analysis is obtained in certain applications and hence the von Mises yield condition is usually preferred to the unsymmetrical form (6) of the Tresca condition by most writers. Using the form (14) of the von Mises yield condition we obtain, in place of (9), the following equivalent equations

$$(15) \quad s_{\alpha\beta} = \frac{\sqrt{6}k\epsilon_{\alpha\beta}}{\sqrt{(\epsilon_1 - \epsilon_2)^2 + (\epsilon_1 - \epsilon_3)^2 + (\epsilon_2 - \epsilon_3)^2}} \quad (\text{von Mises yield}).$$

Let us now differentiate the equations (9) and (10) with respect to x^β and sum on the repeated index β . The left members of the resulting equations then become equal to $p_{,\alpha}$ on account of (1) and (3). After some manipulation of the right members we obtain the following equations

$$(16) \quad p_{,\alpha} = \frac{k}{A} v_{\alpha,\beta\beta} - \frac{2k}{A^3} \epsilon_{\alpha\beta} \epsilon_{ij} v_{i,j\beta} \quad (\text{von Mises yield}),$$

$$(17) \quad p_{,\alpha} = \frac{kv_{\alpha,\beta\beta}}{\epsilon_1 - \epsilon_3} - \frac{2k\epsilon_{\alpha\beta} q_{\mu\nu} v_{\mu,\nu\beta}}{(\epsilon_1 - \epsilon_3)^2} \quad (\text{Tresca yield}),$$

where we have put

$$(18) \quad A^2 = \epsilon_{ij} \epsilon_{ij}; \quad q_{\mu\nu} = h_1^\mu h_1^\nu - h_3^\mu h_3^\nu.$$

In obtaining the right members of these equations use is made of the equation of incompressibility (4); in addition, equations of the type (12) are used in deriving the right member of (17). Equations (16) contain as independent variables the four quantities p, v_α and similarly the equations (17) involve these same quantities as independent variables since the ϵ_i and the h_i^α can be regarded as functions of the components $\epsilon_{\alpha\beta}$ of the rate of strain tensor. *Equations (4) and (16) constitute the reduced system under the von Mises yield condition and equations (4) and (17) the reduced system under the Tresca yield condition.* It is seen that each reduced system is comprised of four equations for the determination of the four independent variables p and v_α .

Let $p(x)$ and $v_\alpha(x)$ be a solution of one of the reduced systems of equations. Then from (7) or (8) we determine λ , from (2) the tensor s and finally from (1) the tensor σ . It is easily seen that these determinations provide a solution of the plasticity equations (2), (3), (4) as equations for the determination of the variables $\sigma_{\alpha\beta}$, v_α and λ in conjunction with the yield condition under consideration.

3. Characteristic conditions. Let $\phi(x^1, x^2, x^3) = \text{const.}$ define a regular surface in space such that $\phi_{,\alpha}\phi_{,\alpha} > 0$ at each point of the surface. As a consequence of this inequality at least one of the derivatives $\phi_{,\alpha}$ does not vanish at any point of the surface and therefore the above equation $\phi = \text{const.}$ can be solved for that one of the variables x^α which corresponds to the non-vanishing derivative in accordance with the usual existence theorem. If P is an arbitrary point on the surface we can choose the coordinate axes so that $\phi_{,3} \neq 0$ at P and thus we can represent the surface in the neighborhood of P by an equation of the form

$$(19) \quad x^3 = f(x^1, x^2).$$

Now consider the values of the function p over the surface $\phi = \text{const.}$ On the part of the surface for which the representation (19) is valid the function p is a function of the variables x^1, x^2 so that

$$(20) \quad \frac{dp}{dx^\beta} = p_{,\beta} + p_{,3}f_{,\beta}, \quad \text{or} \quad p_{,\beta} = \frac{dp}{dx^\beta} - p_{,3}f_{,\beta} \quad (\beta = 1, 2),$$

where the derivative dp/dx^β denotes *surface differentiation* and the letter p appearing in the symbol of this derivative denotes the function p expressed as a function of the surface coordinates x^1, x^2 . By (20) the partial space derivatives $p_{,\beta}$ ($\beta = 1, 2$) are given in terms of the partial space derivative $p_{,3}$. An analogous consideration applies to the components v_α of the velocity vector. Thus we have⁸

$$(21) \quad \frac{dv_\alpha}{dx^\beta} = v_{\alpha,\beta} + v_{\alpha,3}f_{,\beta} \quad (\alpha = 1, 2, 3; \beta = 1, 2),$$

$$(22) \quad \frac{dv_{\alpha,\beta}}{dx^\gamma} = v_{\alpha,\beta\gamma} + v_{\alpha,\beta 3}f_{,\gamma} \quad (\alpha, \beta = 1, 2, 3; \gamma = 1, 2).$$

If we first let β have the value 3 in (22) and then restrict β to the values 1, 2 in this equation, so that we can interchange the indices β and γ , we can write

$$(23) \quad v_{\alpha,\gamma 3} = \frac{dv_{\alpha,3}}{dx^\gamma} - v_{\alpha,33}f_{,\gamma} \quad (\alpha = 1, 2, 3; \gamma = 1, 2),$$

⁸ Cp. J. HADAMARD, *Leçons sur la Propagation des Ondes*, A. Hermann, Paris, (1903), p. 264.

$$(24) \quad v_{\alpha,\beta\gamma} = \frac{dv_{\alpha,\gamma}}{dx^\beta} - v_{\alpha,\gamma\beta} f_{,\beta} \quad (\alpha = 1, 2, 3; \beta, \gamma = 1, 2).$$

Combining (23) and (24) we thus have

$$(25) \quad v_{\alpha,\beta\gamma} = \frac{dv_{\alpha,\gamma}}{dx^\beta} - \frac{dv_{\alpha,\beta}}{dx^\gamma} f_{,\beta} + v_{\alpha,\beta\beta} f_{,\beta} f_{,\gamma} \quad (\alpha = 1, 2, 3; \beta, \gamma = 1, 2).$$

Hence, if the components v_α and their first derivatives $v_{\alpha,\beta}$ are assigned over the surface $\phi = \text{const.}$, i.e. are given as functions of the above surface coordinates x^1, x^2 subject to the conditions (21), the second derivatives $v_{\alpha,\beta\gamma}$ and $v_{\alpha,\gamma\beta}$ for $\alpha = 1, 2, 3; \beta, \gamma = 1, 2$ can be found by (23) and (25) when the derivatives $v_{\alpha,\beta\beta}$ are known.

To find the values of $p_{,\beta}$ and $v_{\alpha,\beta\beta}$ which are not determined over the surface by the above assignments of p, v_α and $v_{\alpha,\beta}$ we must have recourse to one of the reduced systems of equations derived in §2. In general the elimination of the derivatives in the left members of (20), (23) and (25) from the reduced system will result in a set of algebraic equations for the determination of the four quantities $p_{,\beta}$ and $v_{\alpha,\beta\beta}$ such that the determinant Δ of the coefficients of these quantities will not vanish at any point of the surface. In this case the determination of the remaining derivatives is reduced to the solution of a set of algebraic equations. *However in particular the determinant Δ may vanish at each point of the surface so that the solution of the equations for the quantities $p_{,\beta}$ and $v_{\alpha,\beta\beta}$ is algebraically indeterminate over the surface. The surface is then called a characteristic surface relative to the assigned data.* A modification of the assigned data on a characteristic surface will of course, in general, invalidate the characteristic conditions.

The problem of evaluating Δ , as above considered, appears to be somewhat complicated, especially for the case of the reduced system based on the Tresca yield condition. This problem however can be materially simplified by the device of choosing the coordinate axes so that one of the coordinate planes is tangent to the surface $\phi = \text{const.}$ at the above point P . Let us think of the coordinate system as chosen so that the origin is at the point P and the x^1, x^2 plane tangent to the surface at this point. Then the above representation (19) applies and moreover $f_{,\beta} = 0$ ($\beta = 1, 2$) at P . Hence with reference to the point P , i.e. at the origin of the coordinate system under consideration, the term in $p_{,\beta}$ drops out of (20) and similarly the terms containing $v_{\alpha,\beta\beta}$ are removed from (23) and (25). It follows that in order to determine the quantities $p_{,\beta}$ and $v_{\alpha,\beta\beta}$ at the origin of coordinates we have merely to solve the reduced system for these quantities without the necessity of first making the substitutions (20), (23) and (25). Let us now apply this procedure to the determination of the quantities $p_{,\beta}$ and $v_{\alpha,\beta\beta}$ by the reduced system associated with the von Mises yield condition, namely the system composed of the equations (4) and (16). Expanding (16) so as to exhibit the terms $p_{,\beta}$ and $v_{\alpha,\beta\beta}$ for solution we have

$$\begin{aligned}
 (26) \quad \frac{A^3}{k} p_{,1} &= A^2 v_{1,33} - 2\epsilon_{13} \epsilon_{i3} v_{i,33} + \dots, \\
 \frac{A^3}{k} p_{,2} &= A^2 v_{2,33} - 2\epsilon_{23} \epsilon_{i3} v_{i,33} + \dots, \\
 \frac{A^3}{k} p_{,3} &= A^2 v_{3,33} - 2\epsilon_{33} \epsilon_{i3} v_{i,33} + \dots.
 \end{aligned}$$

Conditions on the quantities $v_{\alpha,33}$ due to (4) are of course obtained by differentiating this equation with respect to x^3 , which leads to the relation

$$(27) \quad v_{1,13} + v_{2,23} + v_{3,33} = 0.$$

Let us now arrange the quantities $p_{,3}$ and $v_{\alpha,33}$ for definiteness in the order: $v_{1,33}$, $v_{2,33}$, $v_{3,33}$ and $p_{,3}$. Then the determinant of the coefficients of these quantities in (26) and (27) can be written, to within an unimportant factor, as follows:

$$\begin{vmatrix}
 (A^2 - 2\epsilon_{13}^2) & -2\epsilon_{13} \epsilon_{23} & -2\epsilon_{13} \epsilon_{33} & 0 \\
 -2\epsilon_{23} \epsilon_{13} & (A^2 - 2\epsilon_{23}^2) & -2\epsilon_{23} \epsilon_{33} & 0 \\
 -2\epsilon_{33} \epsilon_{13} & -2\epsilon_{33} \epsilon_{23} & (A^2 - 2\epsilon_{33}^2) & 1 \\
 0 & 0 & 1 & 0
 \end{vmatrix}.$$

Equating this determinant to zero we immediately obtain

$$\begin{vmatrix}
 (A^2 - 2\epsilon_{13}^2) & -2\epsilon_{13} \epsilon_{23} \\
 -2\epsilon_{23} \epsilon_{13} & (A^2 - 2\epsilon_{23}^2)
 \end{vmatrix} = 0,$$

and on expanding we find,

$$(28) \quad A^2 - 2\epsilon_{13}^2 - 2\epsilon_{23}^2 = 0 \quad (\text{von Mises yield}).$$

The equation corresponding to (28), under the assumption of the Tresca yield condition, can now be obtained rather quickly. Thus expanding (17) we have

$$(29) \quad \begin{cases} \frac{(\epsilon_1 - \epsilon_3)^2}{k} p_{,1} = (\epsilon_1 - \epsilon_3) v_{1,33} - 2\epsilon_{13} q_{\mu 3} v_{\mu,33} + \dots, \\ \frac{(\epsilon_1 - \epsilon_3)^2}{k} p_{,2} = (\epsilon_1 - \epsilon_3) v_{2,33} - 2\epsilon_{23} q_{\mu 3} v_{\mu,33} + \dots, \\ \frac{(\epsilon_1 - \epsilon_3)^2}{k} p_{,3} = (\epsilon_1 - \epsilon_3) v_{3,33} - 2\epsilon_{33} q_{\mu 3} v_{\mu,33} + \dots. \end{cases}$$

Equations (27) and (29) now lead to the following determinant

$$\begin{vmatrix} (\epsilon_1 - \epsilon_3 - 2\epsilon_{13} q_{13}) & -2\epsilon_{13} q_{23} & -2\epsilon_{13} q_{33} & 0 \\ -2\epsilon_{23} q_{13} & (\epsilon_1 - \epsilon_3 - 2\epsilon_{23} q_{23}) & -2\epsilon_{23} q_{33} & 0 \\ -2\epsilon_{33} q_{13} & -2\epsilon_{33} q_{23} & (\epsilon_1 - \epsilon_3 - 2\epsilon_{33} q_{33}) & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix},$$

the vanishing of which expresses the characteristic condition at the origin of the above special coordinate system. We thus obtain immediately

$$\begin{vmatrix} (\epsilon_1 - \epsilon_3 - 2\epsilon_{13} q_{13}) & -2\epsilon_{13} q_{23} \\ -2\epsilon_{23} q_{13} & (\epsilon_1 - \epsilon_3 - 2\epsilon_{23} q_{23}) \end{vmatrix} = 0,$$

and from this relation we find the equation corresponding to (28), namely

$$(30) \quad 2\epsilon_{13} q_{13} + 2\epsilon_{23} q_{23} = \epsilon_1 - \epsilon_3 \quad (\text{Tresca yield}).$$

Equations (28) and (30) express the condition relative to the above special coordinate system, whose x^1, x^2 coordinate plane is tangent to the surface $\phi = \text{const.}$ at an arbitrary point, for this surface to be a characteristic under the assumption of the von Mises yield condition and the Tresca yield condition respectively. However the problem of characteristic surfaces which we are considering is the problem of determining surfaces $\phi = \text{const.}$ which will be characteristic for an assignment of data resulting from a solution of the plasticity equations. On this account the forms (28) and (30) of the characteristic condition would appear at first sight to be of questionable value since the surface $\phi = \text{const.}$ and hence the coordinate system on the basis of which these conditions are expressed is not known in advance of the solution of the problem itself. But we shall show in the next section how the general invariant formulation of the characteristic directions can be obtained immediately from (28) and (30) by a simple application of the principles of tensor algebra, and in so doing we shall have avoided all of the troublesome computation of the usual procedure for obtaining this result.

4. Invariant formulation of the characteristic directions. Relative to the coordinate system used in §3 to express the characteristic conditions (28) and (30) the unit vector w normal to the surface $\phi = \text{const.}$ has the components $w_1, w_2, w_3 = 0, 0, 1$ at an arbitrary point P of this surface. Hence we see that

$$\epsilon_{13}^2 + \epsilon_{23}^2 = \epsilon_{\alpha 3} \epsilon_{\alpha 3} - \epsilon_{33}^2 = \epsilon_{\alpha \beta} \epsilon_{\alpha \gamma} w_\beta w_\gamma - (\epsilon_{\alpha \beta} w_\alpha w_\beta)^2,$$

and

$$\epsilon_{13} q_{13} + \epsilon_{23} q_{23} = \epsilon_{\alpha 3} q_{\alpha 3} - \epsilon_{33} q_{33} = \epsilon_{\alpha \beta} q_{\alpha \gamma} w_\beta w_\gamma - (\epsilon_{\alpha \beta} w_\alpha w_\beta)(q_{\mu \nu} w_\mu w_\nu).$$

Substituting these expressions into (28) and (30) we obtain the desired formulations for the characteristic directions w as follows

$$(31) \quad \epsilon_{\alpha\beta} \epsilon_{\alpha\gamma} w_\beta w_\gamma - (\epsilon_{\alpha\beta} w_\alpha w_\beta)^2 = \frac{1}{2} A^2 \quad (\text{von Mises yield}),$$

$$(32) \quad \epsilon_{\alpha\beta} q_{\alpha\gamma} w_\beta w_\gamma - (\epsilon_{\alpha\beta} w_\alpha w_\beta)(q_{\mu\nu} w_\mu w_\nu) = \frac{1}{2}(\epsilon_1 - \epsilon_3) \quad (\text{Tresca yield})$$

in which the quantities A^2 and $q_{\mu\nu}$ are given by (18).

5. Algebraic determination of the characteristic directions under the yield condition of von Mises. Consider the condition (31) in the y coordinate system defined in §2; we shall seek to determine the values of the components of the characteristic unit vector w at the origin of this system. But at the origin of the y system we have

$$h_i{}^\alpha = \delta_i{}^\alpha; \quad \epsilon_{\alpha\beta} = \epsilon_\alpha \delta_{\alpha\beta}; \quad A^2 = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2.$$

Making these substitutions in (31) and expanding we now obtain

$$(33) \quad \epsilon_1^2 w_1^2 + \epsilon_2^2 w_2^2 + \epsilon_3^2 w_3^2 - (\epsilon_1 w_1^2 + \epsilon_2 w_2^2 + \epsilon_3 w_3^2)^2 = \frac{1}{2}(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2).$$

Since w is a unit vector we have

$$w_3^2 = 1 - w_1^2 - w_2^2,$$

and we can therefore eliminate the components w_3 from (33) by this substitution. This gives

$$\begin{aligned} & (\epsilon_1^2 - \epsilon_3^2)w_1^2 + (\epsilon_2^2 - \epsilon_3^2)w_2^2 + \epsilon_3^2 - [(\epsilon_1 - \epsilon_3)w_1^2 + (\epsilon_2 - \epsilon_3)w_2^2 + \epsilon_3]^2 \\ & \quad = \frac{1}{2}(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2). \end{aligned}$$

Let us now expand the bracket expression in this equation and then arrange the terms to obtain the following quadratic expression in w_1^2 , namely

$$(34) \quad \begin{aligned} & (\epsilon_1 - \epsilon_3)^2 w_1^4 + (\epsilon_1 - \epsilon_3)[2(\epsilon_2 - \epsilon_3)w_2^2 - (\epsilon_1 - \epsilon_3)]w_1^2 \\ & \quad + [(\epsilon_2 - \epsilon_3)^2(w_2^4 - w_2^2) + \frac{1}{2}(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2)] = 0. \end{aligned}$$

Now we can consider only real roots w_1^2 of the quadratic (34); hence writing the usual condition for the reality of these roots we find

$$(35) \quad \begin{aligned} & (\epsilon_1 - \epsilon_3)^2 \{ [2(\epsilon_2 - \epsilon_3)w_2^2 - (\epsilon_1 - \epsilon_3)]^2 - 4[(\epsilon_2 - \epsilon_3)^2(w_2^4 - w_2^2) \\ & \quad + \frac{1}{2}(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2)] \} \geq 0. \end{aligned}$$

By the *Remark* in §2 the quantity $(\epsilon_1 - \epsilon_3)^2$ is positive and hence can be cancelled from the inequality (35). Further simplification, involving the expansion of the bracket-expression and collection of terms, now gives the following reality condition,

$$(36) \quad -4(\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)w_2^2 - (\epsilon_1 + \epsilon_3)^2 - 2\epsilon_2^2 \geq 0.$$

But each term in (36) is non-positive in consequence of the ordering $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$ and hence must vanish separately in order for the inequality to be satisfied. Hence

$$(37) \quad (\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)w_2^2 = 0; \quad \epsilon_1 + \epsilon_3 = 0; \quad \epsilon_2 = 0.$$

Now the last equation (37), i.e. $\epsilon_2 = 0$, implies the second equation of the set in consequence of the form $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ of the incompressibility condition. Furthermore it follows that the first equation becomes

$$(38) \quad \epsilon_1 \epsilon_3 w_2^2 = 0.$$

But the vanishing of either of the invariants ϵ_1 or ϵ_3 implies the vanishing of the other and thus contradicts the *Remark* in §2. Hence (38) gives $w_2 = 0$, and hence equation (34) becomes

$$(\epsilon_1 - \epsilon_3)^2 w_1^4 - (\epsilon_1 - \epsilon_3)^2 w_1^2 + \frac{1}{2}(\epsilon_1^2 + \epsilon_3^2) = 0.$$

But the roots of this quadratic equation are equal and are given by $w_1^2 = \frac{1}{2}$. Hence $w_1 = \pm 1/\sqrt{2}$ and also $w_3 = \pm 1/\sqrt{2}$ since $w_2 = 0$. There are thus two characteristic directions w at each point which may be taken to have the components $(1/\sqrt{2}, 0, 1/\sqrt{2})$ and $(1/\sqrt{2}, 0, -1/\sqrt{2})$ at the origin of the y system under consideration. Since it is well known that these directions are normal to the surface elements of maximum and minimum shearing stress we have now proved the following result. *Under the assumption of the von Mises yield condition there exist at each point in the plastic flow two, and only two, real characteristic surface elements, which are identical with the surface elements of maximum and minimum shearing stress, provided that the invariant $\epsilon_2 = 0$. If ϵ_2 does not vanish real characteristic surface elements do not exist.*

Remark. It is immediately seen from the above values $(1/\sqrt{2}, 0, 1/\sqrt{2})$ and $(1/\sqrt{2}, 0, -1/\sqrt{2})$ of the components of the characteristic directions w relative to the y system that these vectors are given by

$$(39) \quad w_\alpha = \frac{1}{\sqrt{2}} (h_1^\alpha + h_3^\alpha) \quad \text{and} \quad w_\alpha = \frac{1}{\sqrt{2}} (h_1^\alpha - h_3^\alpha)$$

in an arbitrary coordinate system. For the vectors w given by (39) are unit vectors and their components have the above numerical values of the components of the characteristic directions at the origin of the y system.

6. Algebraic determination of the characteristic directions under the yield condition of Tresca. The process of determining the characteristic directions under the yield condition of Tresca is similar to that employed in §5 so that most of the preceding discussion can be eliminated. We first observe that at the origin of the y system $q_{11} = 1$, $q_{33} = -1$ while the remaining $q_{\mu\nu}$ are equal to zero. Hence evaluating (32) at the origin of the y system we obtain

$$\epsilon_1 w_1^2 - \epsilon_3 w_3^2 - (\epsilon_1 w_1^2 + \epsilon_2 w_2^2 + \epsilon_3 w_3^2)(w_1^2 - w_3^2) = \frac{1}{2}(\epsilon_1 - \epsilon_3).$$

As before we now eliminate the quantity w_3^2 and are thus led to the following quadratic equation

$$(40) \quad 2(\epsilon_1 - \epsilon_3)w_1^4 + [2(\epsilon_2 - \epsilon_3)w_2^2 + (\epsilon_1 - \epsilon_3)w_2^2 - 2(\epsilon_1 - \epsilon_3)]w_1^2 + [(\epsilon_2 - \epsilon_3)(w_2^4 - w_2^2) + \frac{1}{2}(\epsilon_1 - \epsilon_3)] = 0.$$

We next impose the condition for the roots w_1^2 of (40) to be real; after some reduction this inequality can be given the form

$$(41) \quad \{[2(\epsilon_2 - \epsilon_3) - (\epsilon_1 - \epsilon_3)]^2 w_2^2 - 4(\epsilon_1 - \epsilon_3)^2\} w_2^2 \geq 0.$$

Let us now assume that $w_2 \neq 0$, which permits us to remove the factor w_2^2 from (41). Furthermore

$$2(\epsilon_2 - \epsilon_3) - (\epsilon_1 - \epsilon_3) = (\epsilon_1 - \epsilon_3) - 2(\epsilon_1 - \epsilon_2).$$

Hence the inequality (41) becomes

$$[(\epsilon_1 - \epsilon_3) - 2(\epsilon_1 - \epsilon_2)]^2 w_2^2 \geq 4(\epsilon_1 - \epsilon_3)^2,$$

or

$$(42) \quad \left[1 - 2\left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 - \epsilon_3}\right)\right]^2 w_2^2 \geq 4.$$

But

$$0 \leq \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 - \epsilon_3} \leq 1,$$

and hence

$$-1 \leq 1 - 2\left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 - \epsilon_3}\right) \leq 1.$$

Then, since $0 \leq w_2^2 \leq 1$, it is clear that the left member of (42) must actually be less than 4 so that this inequality is not satisfied. *Hence the above assumption that $w_2 \neq 0$ is not valid and consequently w_2 must be equal to zero.*

Putting $w_2 = 0$ in (40) and cancelling the non-vanishing factor $\epsilon_1 - \epsilon_3$ we now have

$$4w_1^4 - 4w_1^2 + 2 = 0,$$

from which it follows that $w_1^2 = \frac{1}{2}$. Hence $w_1 = \pm 1/\sqrt{2}$ and hence also $w_3 = \pm 1/\sqrt{2}$. The following result has now been proved. *Under the assumption of the yield condition of Tresca there exists at each point of the plastic flow two, and only two, real characteristic surface elements, which are identical with the surface elements of maximum and minimum shearing stress.* As before the unit vectors w

for the characteristic directions are given in general coordinates by the relations (39), but the previous condition $\epsilon_2 = 0$ for the existence of the characteristic surface elements is not necessary under the yield condition of Tresca.

7. Existence of characteristic surfaces. It is well known⁹ that a vector field $w(x)$ will be normal to a congruence of surfaces in three dimensional space if, and only if, the following condition is satisfied

$$(43) \quad w_1(w_{2,3} - w_{3,2}) + w_2(w_{3,1} - w_{1,3}) + w_3(w_{1,2} - w_{2,1}) = 0.$$

Hence when we substitute the vectors w given by (39) into (43) we obtain the necessary and sufficient conditions for the two families of characteristic surface elements to combine to form two congruences of characteristic surfaces *under the assumption of the yield condition of Tresca*. This substitution gives the required conditions immediately in terms of the derivatives of the components h_i^α of the principal directions. We shall now show how the conditions can be expressed in terms of derivatives of the components of the rate of strain tensor or, what amounts to the same thing, in the derivatives of the components v_α of the velocity vector.¹⁰

Consider the following relations,

$$(44) \quad h_i^\alpha h_j^\alpha = \delta_{ij},$$

which express the condition that the vectors h_i are unit vectors and mutually orthogonal. If we transform (44) to a y coordinate system, differentiate with respect to y^k , and evaluate at the origin of the system, we are led to the equations

$$(45) \quad h_j^{i,k} + h_i^{j,k} = 0.$$

The conditions expressed by (45) are *invariant*. In fact the quantities appearing in (45) can be interpreted as *scalar differential invariants of the plastic flow*. In (45) and in the following equations of this section we have used Latin indices to denote quantities of scalar character and have reserved Greek indices to signify components of vector or tensor character. It would be possible to derive an explicit formula for the scalars $h_j^{i,k}$ expressing them in terms of the flow quantities used in the preceding work but such a formula will not be needed in our discussion.

Let us now insert the vector components w_α given by (39) into (43) and evaluate at the origin of the y system; when use is made of the relations (45)

⁹ See, for example, T. LEVI-CIVITA, *The Absolute Differential Calculus*, Blackie and Son, London, (1929), p. 29.

¹⁰ Conditions of this type have been obtained without recourse to the formula (39) for the components of the characteristic directions w by T. Y. THOMAS, *Surfaces of maximum shearing stress*, *loc. cit.*, p. 172. However the derivation of the conditions is simplified considerably by use of the equation (39).

we then obtain the following equations

$$(46) \quad h_1^2{}_{,3} + h_3^2{}_{,3} + 2h_3^1{}_{,2} + h_2^1{}_{,1} + h_2^3{}_{,1} = 0,$$

$$(47) \quad h_1^2{}_{,3} + h_2^3{}_{,3} + 2h_3^1{}_{,2} + h_1^2{}_{,1} + h_2^3{}_{,1} = 0.$$

Equation (46) is an invariant condition for the existence of one of the congruences of characteristic surfaces and equation (47) gives the corresponding condition for the other congruence of these surfaces under the yield condition of Tresca. Some simplification is obtained by adding and subtracting the above relations. Thus we find

$$(48) \quad h_1^2{}_{,3} + 2h_3^1{}_{,2} + h_2^3{}_{,1} = 0, \quad h_2^1{}_{,1} + h_3^2{}_{,3} = 0,$$

as the conditions for the simultaneous existence of the two congruences of characteristic surfaces.

Now form the relations

$$(49) \quad \epsilon_{\alpha\beta} = \sum_i \epsilon_i h_i^\alpha h_i^\beta,$$

which are obtained immediately from the usual equations defining the principal directions h_i of the rate of strain tensor ϵ . Differentiation of (49), followed by evaluation at the origin of the y system, will lead (after a slight change of indices) to a set of equations which can be written

$$(50) \quad \epsilon_{ij,k} = \epsilon_{i,k} \delta_{ij} + (\epsilon_j - \epsilon_i) h_j^i{}_{,k}$$

where the $\epsilon_{ij,k}$ ($= \epsilon_{ji,k}$) and $\epsilon_{i,k}$ are scalar differential invariants. Hence

$$(51) \quad h_j^i{}_{,k} = \frac{\epsilon_{ij,k}}{\epsilon_i - \epsilon_j} \quad (i \neq j).$$

When we substitute (51) into (48) the following relations

$$(52) \quad \frac{\epsilon_{12,3}}{\epsilon_1 - \epsilon_2} - 2 \frac{\epsilon_{13,2}}{\epsilon_1 - \epsilon_3} + \frac{\epsilon_{23,1}}{\epsilon_2 - \epsilon_3} = 0, \quad \frac{\epsilon_{12,1}}{\epsilon_1 - \epsilon_2} + \frac{\epsilon_{23,3}}{\epsilon_2 - \epsilon_3} = 0,$$

are obtained. As written (52) is another scalar formulation, corresponding to (48), of the conditions for the existence of the two congruences of characteristic surfaces under the yield condition of Tresca. However a simple modification of (52) will provide us with an expression of these conditions of conventional tensor character in terms of the derivatives of the rate of strain components $\epsilon_{\alpha\beta}$. In fact from (52) we have immediately

$$(53) \quad \left\{ \begin{aligned} & \left[\frac{h_1^\alpha h_2^\beta h_3^\gamma}{\epsilon_1 - \epsilon_2} - \frac{2h_1^\alpha h_3^\beta h_2^\gamma}{\epsilon_1 - \epsilon_3} + \frac{h_2^\alpha h_3^\beta h_1^\gamma}{\epsilon_2 - \epsilon_3} \right] \epsilon_{\alpha\beta,\gamma} = 0, \\ & \left[\frac{h_1^\alpha h_2^\beta h_1^\gamma}{\epsilon_1 - \epsilon_2} + \frac{h_2^\alpha h_3^\beta h_3^\gamma}{\epsilon_2 - \epsilon_3} \right] \epsilon_{\alpha\beta,\gamma} = 0, \end{aligned} \right.$$

in which the $\epsilon_{\alpha\beta,\gamma}$ are now the usual partial derivatives of the $\epsilon_{\alpha\beta}$ with respect to the coordinates x^γ . The validity of these relations follows from the fact *first*, that they are of tensor character and *second*, that they reduce to (52) at the origin of the y system. *The above relations (53) in which $\epsilon_{\alpha\beta}$ is thought of as given by*

$$(54) \quad \epsilon_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha})$$

express the conditions in terms of the velocity components v_α and their derivatives for the two families of characteristic surface elements to unite to form two congruences of actual surfaces under the assumption of the yield condition of Tresca.

In the case of the von Mises yield condition we must add the requirement $\epsilon_2 = 0$ to the equations (53) to obtain the conditions for the existence of the two congruences of characteristic surfaces. Then, since $\epsilon_3 = -\epsilon_1 \neq 0$, equations (53) assume the simpler form

$$(55) \quad \begin{cases} (h_1^\alpha h_2^\beta h_3^\gamma - h_1^\alpha h_3^\beta h_2^\gamma + h_2^\alpha h_3^\beta h_1^\gamma) \epsilon_{\alpha\beta,\gamma} = 0, \\ (h_1^\alpha h_2^\beta h_1^\gamma + h_2^\alpha h_3^\beta h_3^\gamma) \epsilon_{\alpha\beta,\gamma} = 0. \end{cases}$$

Now if we substitute (54) into (55), evaluate at the origin of the y system, and make use of the equation of incompressibility (4) we obtain

$$(56) \quad v_{2,13} = 0; \quad v_{2,11} + v_{2,33} = v_{2,22}.$$

But from the equations

$$\epsilon_2 = \epsilon_{\alpha\beta} h_2^\alpha h_2^\beta = 0; \quad \epsilon_{2,\gamma} = \epsilon_{\alpha\beta,\gamma} h_2^\alpha h_2^\beta = 0,$$

it follows that at the origin of the y system we must have $v_{2,2} = 0$ and $v_{2,2i} = 0$ for $i = 1, 2, 3$. Hence taking $i = 2$ in these latter equations we see that the right member of the second equation (56) must vanish so that this equation can be given the contracted form $v_{2,ii} = 0$. Combining these results it follows that the required conditions for the characteristic surfaces are given by

$$v_{2,2} = 0; \quad v_{2,13} = 0; \quad v_{2,2i} = 0; \quad v_{2,ii} = 0$$

relative to the origin of the y system. These conditions admit the following invariant formulation

$$(57) \quad \begin{cases} v_{\alpha,\beta} h_2^\alpha h_2^\beta = 0; & v_{\alpha,\beta\gamma} h_2^\alpha h_1^\beta h_3^\gamma = 0, \\ v_{\alpha,\beta\gamma} h_2^\alpha h_2^\beta = 0; & v_{\alpha,\beta\beta} h_2^\alpha = 0. \end{cases}$$

Hence the necessary and sufficient condition for the two possible families of characteristic surface elements to exist and combine to form two congruences of characteristic surfaces under the yield condition of von Mises is that the equation (57) be satisfied.

Several general comments concerning the conditions derived in this section in relation to the problem of plastic flow have been made in §1. We shall not attempt further discussion of this problem.