Spinor Space and Line Geometry II

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Synopsis. In the first paper with the same title [1], which will be referred to as SI, we discussed the representation and the mapping of the ideal space L_3 of the centered vector space R_4 of the theory of special relativity onto the spinor space S_3 . It turned out that a congruence K as well as a linear complex Γ played a very important role. In the first part of this paper (§§1–5) we shall use these line configurations in S_3 as “absolute” configurations in S_3 , where now spinor lines (and not spinors) will be considered as elements. Thus a line possesses an invariant, a couple of lines possess a couple of invariants. These invariants lead to a “line metric” dealt with in the two last sections of the first part. In the second part (§§6, 7) we use the biaxial involution in S_3 dealt with in SI for a geometric interpretation of Dirac equations. Moreover, this biaxial involution in S_3 is a map of a reflection in L_3 and this reflection leads in a natural way to Schrödinger’s relativistic wave equation.

While in the present paper we deal with the spinor space S_3 from the point of view of its lines, in the next paper we will discuss the analysis of S_3 from the point of view of its spinors.

0. Generalities

Our considerations are based on results of SI. In particular we shall use its symbolism as well as its language. For the reader’s convenience we repeat that the congruence K is the intersection of the complexes

$$(0.1) \quad \Gamma^{ab}(1, 1, 0, 0, 0, 0), \quad * \Gamma^{ab}(-1, 1, 0, 0, 0, 0),$$

and that its axes are

$$(0.2) \quad {}^1\phi^{ab}(0, 1, 0, 0, 0, 0), \quad {}^2\phi^{ab}(1, 0, 0, 0, 0, 0).$$

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The role played by K , Γ and $\ast\Gamma$ in the theory of the spinor space S_3 was dealt with in SI.

Let Λ^{ab} (Λ_{ab}) be the point (plane) Plücker coordinates of a spinor line Λ and Ω^{ab} (Ω_{ab}) the corresponding components of another spinor line Ω . Then by $\Lambda.\Omega$ we understand the product

$$(0.3) \quad \begin{aligned} \Lambda.\Omega &= \Omega.\Lambda \equiv \frac{1}{2}\Lambda^{ab}\Omega_{ab} = \frac{1}{2}\Omega^{ab}\Lambda_{ab} \\ &= \Lambda_{12}\Omega_{34} + \Lambda_{23}\Omega_{14} + \Lambda_{31}\Omega_{24} + \Omega_{12}\Lambda_{34} + \Omega_{23}\Lambda_{14} + \Omega_{31}\Lambda_{24}. \end{aligned}$$

Unless otherwise stated we assume that no spinor line dealt with meets ${}^1\phi$, ${}^2\phi$. Hence if Λ is an arbitrary spinor line in S_3 we have

$$(0.4) \quad ({}^1\phi.\Lambda)({}^2\phi.\Lambda) \neq 0$$

unless otherwise stated.

If we refer to results of SI we denote them always by I. Thus by “equation I(4.10)” (Theorem I(5.2)) we understand the equation (4.10) (the Theorem (5.2)) of SI.

I. Spinor line metric

§1. Invariants of a spinor line

Lemma (1.1). *Let Λ and $\Omega \neq \Lambda$ be two conjugate polar lines with respect to Γ . Then the spinor lines ${}^1\phi$, ${}^2\phi$, Λ and Ω are quadratically dependent, i.e. they are rulings of the same regulus R .*

Proof. Let R be the regulus containing ${}^1\phi$, ${}^2\phi$, Λ . Let Π be an arbitrary ruling of the conjugate regulus R' . Since Π meets ${}^1\phi$, ${}^2\phi$ it is a ruling of the congruence $K({}^1\phi, {}^2\phi)$ and consequently is also a ruling of Γ . Since Λ and Ω are conjugate polar lines with respect to Γ and the ruling Π of Γ meets Λ it also meets Ω . This result holds for any arbitrary ruling Π of R' and consequently Ω belongs to R .

Note. Since we assumed $\Lambda \neq \Omega$ these two spinor lines are skew, and moreover

$$(1.1) \quad (\Lambda.\Gamma)(\Omega.\Gamma) \neq 0.$$

In the following theorem we denote by $(\Lambda\Omega, {}^1\phi^2\phi)$ the cross ratio of the four spinor lines Λ , Ω , ${}^1\phi$, ${}^2\phi$ taken on R .

Definition (1.1). *The cross ratio $(\Lambda\Omega, {}^1\phi^2\phi)$ will be denoted by $\theta(\Lambda)$*

$$(1.2) \quad \theta(\Lambda) = (\Lambda\Omega, {}^1\phi^2\phi)$$

and referred to as “the invariant of the spinor line Λ ”.

Note. According to this definition we have

$$(1.3) \quad \theta(\Lambda)\theta(\Omega) = 1.$$

Theorem (1.1). *The invariant $\theta(\Lambda)$ is given by*

$$(1.4) \quad \theta(\Lambda) = - \frac{{}^2\phi \cdot \Gamma {}^1\phi \cdot \Lambda}{{}^1\phi \cdot \Gamma {}^2\phi \cdot \Lambda}.$$

It does not depend on

a) *factors of proportionality of ${}^1\phi$, ${}^2\phi$, Γ , Λ , or*

b) *choice of coordinate system in S_3 .*

Proof. Denote by k ($= 1, 2, 3, 4$) the spinor vertex of the coordinate tetrahedron whose k -th coordinate is different from zero. According to I(1.3) we might as well assume

$$(1.5a) \quad {}^1\phi \equiv 34, \quad {}^2\phi \equiv 12.$$

Moreover we assume without loss of generality that the ruling Π of R' , mentioned in the proof of Lemma (1.1), is the edge 24:

$$(1.5b) \quad \Pi \equiv 24,$$

and that the edge 13 is another ruling, say Σ , of R' :

$$(1.5c) \quad \Sigma \equiv 13.$$

Because Λ meets Σ as well as Π we have according to (1.5b, c)

$$(1.6a) \quad \Lambda^{13} = \Lambda^{24} = 0,$$

and similarly

$$(1.6b) \quad \Omega^{13} = \Omega^{24} = 0.$$

If we denote by ${}^\tau\lambda$ (${}^\tau\omega$) the common point of Π and Λ (Ω) we obtain from (1.6)

$$(1.7) \quad {}^\tau\lambda(0, \Lambda^{21}, 0, \Lambda^{41}), \quad {}^\tau\omega(0, \Omega^{21}, 0, \Omega^{41}).$$

Hence the cross ratio (${}^\tau\lambda {}^\tau\omega, 42$) of the four points ${}^\tau\lambda$, ${}^\tau\omega$, 4, 2 on Π is

$$(1.8a) \quad ({}^\tau\lambda {}^\tau\omega, 42) = \frac{\Lambda^{21} \Omega^{41}}{\Omega^{21} \Lambda^{41}}.$$

This cross ratio is obviously equal to the cross ratio $(\Lambda\Omega, {}^1\phi {}^2\phi)$ taken on R :

$$(1.8b) \quad (\Lambda\Omega, {}^1\phi {}^2\phi) = \frac{\Lambda^{21} \Omega^{41}}{\Omega^{21} \Lambda^{41}}.$$

If we denote by ψ the edge 23:

$$(1.5d) \quad \psi \equiv 23,$$

we have

$$\frac{\Omega^{41}}{\Lambda^{41}} = \frac{\psi.\Omega}{\psi.\Lambda}, \quad \frac{\Lambda^{21}}{\Omega^{21}} = \frac{{}^1\phi.\Lambda}{{}^1\phi.\Omega}.$$

Therefore

$$(1.9) \quad (\Lambda\Omega, {}^1\phi^2\phi) = \frac{{}^1\phi.\Lambda}{{}^1\phi.\Omega} \frac{\psi.\Omega}{\psi.\Lambda},$$

and the equation (1.4) follows from (1.2), (1.9) and

$$(1.10) \quad \Omega = 2\Gamma(\Gamma.\Lambda) - \Lambda(\Gamma.\Gamma),$$

$$(1.11) \quad \psi.\Gamma = 0,$$

$$(1.12) \quad \frac{2\Gamma}{(\Gamma.\Gamma)^\dagger} = \left(\frac{{}^1\phi}{{}^1\phi.\Gamma} + \frac{{}^2\phi}{{}^2\phi.\Gamma} \right) (\Gamma.\Gamma)^\dagger.$$

(The equation (1.10) defines Ω as a conjugate polar line to Λ with respect to Γ , the equation (1.11) describes ψ as a ruling of Γ while (1.12) is an obvious expression for Γ based on I(1.3) and (1.5).) The statement b) follows from Definition (1.1), while a) is an immediate consequence of (1.4).

§2. Geometric interpretation of $\theta(\Lambda)$

In the next two theorems we shall use the symbol $\Gamma(\mu, \lambda)$ to describe a complex in a linear set:

$$(2.1) \quad \Gamma(\mu, \lambda) \equiv \lambda \frac{{}^1\phi}{{}^1\phi.\Gamma} + \mu \frac{{}^2\phi}{{}^2\phi.\Gamma}.$$

Hence μ, λ may be thought of as homogeneous coordinates of a complex in the set (2.1):

$$\Gamma(\mu, \lambda) = \Gamma(\rho\mu, \rho\lambda), \quad (\rho \neq 0).$$

Lemma (2.1). *Any spinor line Λ belongs to exactly one of the complexes (2.1).*

The proof follows at once from (2.1) and

$$(2.2) \quad 0 = \Lambda.\Gamma(\mu, \lambda) = \lambda \frac{{}^1\phi.\Lambda}{{}^1\phi.\Gamma} + \mu \frac{{}^2\phi.\Lambda}{{}^2\phi.\Gamma}.$$

The following theorems offer a geometric interpretation of $\theta(\Lambda)$ as well as complete the previous Lemma:

Theorem (2.1a). *An arbitrary spinor line Λ belongs to exactly one complex of the set (2.1), namely to*

$$(2.3a) \quad \Gamma(\theta(\Lambda), 1) = \Gamma(1, \theta^{-1}(\Lambda)).$$

Hence all rulings of the same complex $\Gamma(\mu, \lambda)$ and only such rulings have the same invariant $\frac{\mu}{\lambda} = \theta(\Lambda)$.

The proof follows at once from

$$(2.4) \quad \frac{\mu}{\lambda} = \theta(\Lambda),$$

which is an immediate consequence of (2.2).

Note. From (2.4) and (1.12) we have $\Gamma(1, 1) = \Gamma$. By virtue of Theorem (2.1a) all rulings Λ of Γ have $\theta(\Lambda) = 1$, and in particular

$$\theta(^+\Lambda) = \theta(^-\Lambda) = 1,$$

where $^+\Lambda, ^-\Lambda$ are rulings of Γ which represent a point in L_3 (cf. Theorem I (2.5)).

Later on we shall use the following

Theorem (2.1b). *The complex $\Gamma(\theta(\Lambda), 1)$ is a gradient:*

$$(2.3b) \quad \Gamma(\theta(\Lambda), 1) = -\frac{\partial}{\partial \Lambda} \ln \theta(\Lambda).$$

Proof. Comparing

$$-\frac{\partial}{\partial \Lambda} \ln \theta(\Lambda) = \frac{{}^2\phi}{{}^2\phi.\Gamma} \frac{{}^2\phi.\Gamma}{{}^2\phi.\Lambda} - \frac{{}^1\phi}{{}^1\phi.\Gamma} \frac{{}^1\phi.\Gamma}{{}^1\phi.\Lambda}$$

with (2.1) we obtain (2.3b).

Note. According to our assumption in section 0 we have $({}^1\phi.\Lambda)({}^2\phi.\Lambda) \neq 0$ and consequently

$$\theta(\Lambda) \neq 0, \quad \theta^{-1}(\Lambda) \neq 0.$$

In the next theorem we shall deal also with the case $({}^1\phi.\Lambda)({}^2\phi.\Lambda) = 0$.

Theorem (2.2). *If we regard the theorem (2.1a) as a definition of $\theta(\Lambda)$, then the formula (1.4) is valid also for the case*

$$({}^1\phi.\Lambda) = 0, \quad ({}^2\phi.\Lambda) \neq 0 \quad \text{or} \quad {}^2\phi.\Lambda = 0, \quad {}^1\phi.\Lambda \neq 0.$$

Proof. If ${}^1\phi.\Lambda = 0, {}^2\phi.\Lambda \neq 0$ we have according to (2.2) $\mu = 0$ and by virtue of (2.4) we obtain $\theta(\Lambda) = 0$. On the other hand (1.4) yields $\theta(\Lambda) = 0$ for ${}^1\phi.\Lambda = 0$. The proof of the second part is similar.

Note. For a ruling Λ of K the invariant $\theta(\Lambda)$ is not defined, as was to be expected according to Theorem (2.1a): A ruling of K belongs to any complex of the set (2.1).—According to Theorem (2.2) the invariant $\theta(\Lambda)$ may be thought of as an affine coordinate of the complex $\Gamma(\theta(\Lambda), 1) \equiv \Gamma(1, \theta^{-1}(\Lambda))$ in the set (2.1).

In the next considerations we resume the assumption $(^1\phi.\Lambda)(^2\phi.\Lambda) \neq 0$. In order to obtain another interpretation of $\theta(\Lambda)$ we need the following

Lemma (2.2). *Let $\xi, \eta \neq \xi$ be two spinors of a spinor line $\Lambda = \xi\eta$. Then there is only one regulus L with the following properties:*

- 1) *It contains ξ and η .*
- 2) *Its ruling through ξ is a ruling of Γ .*
- 3) *Its conjugate regulus L' belongs to K .*

Proof. Suppose first that Λ be a ruling of Γ . Then the theorem is obvious, for the only regulus L which satisfies the conditions 1)–3) is the regulus containing $^1\phi, ^2\phi, \Lambda$. Suppose now that Λ is not a ruling of Γ . According to the requirement 3) the regulus L must contain $^1\phi, ^2\phi$. According to the requirements 1) and 3) the only ruling Σ of K through η is a ruling of L' . Let ζ be the intersection of Σ and the polar plane of ξ with respect to Γ . (Λ is not a ruling of Γ and consequently the polar plane of ξ does not contain Σ .) Then $\xi\zeta$ is the only ruling of Γ which goes through ξ and meets Σ . According to the requirement 2) the spinor line $\xi\zeta$ is a ruling of L , which is the only regulus containing $^1\phi, ^2\phi, \xi\zeta$. It satisfies obviously all three requirements of our theorem.

Theorem (2.3). *Let $\Omega(*\Omega)$ be a ruling through $\eta(\xi)$ of the regulus mentioned in the previous lemma. Then the cross ratio $(\Omega * \Omega, ^1\phi^2\phi)$ taken on L does not depend on the choice of the spinors $\xi, \eta \neq \xi$ and is equal to $\theta(\Lambda)$:*

$$(2.5) \quad \theta(\Lambda) = (\Omega * \Omega, ^1\phi^2\phi).$$

Proof. Suppose first that Λ be a ruling of Γ . Then

$$\Lambda = \Omega \equiv * \Omega,$$

and consequently

$$(\Omega * \Omega, ^1\phi^2\phi) = 1.$$

On the other hand we have $\theta(\Lambda) = 1$. (Cf. Note following Theorem (2.1a).) Hence (2.5) holds. Suppose now $\Lambda.\Gamma \neq 0$ and use the same coordinate system as in the proof of Theorem (1.1). From (1.5c) for Σ (mentioned in the proof of Lemma (2.2)) we obtain $\eta^2 = \eta^4 = 0$ and consequently

$$(2.6) \quad \Lambda^{12}:\Lambda^{34} = \eta^1\xi^2:\eta^3\xi^4.$$

On the other hand the polar plane ξ_a of the spinor ξ with respect to Γ has the coordinates $\xi_a = \Gamma_{ab}\xi^b$, where

$$\xi_1:\xi_2:\xi_3:\xi_4 = \xi^2:-\xi^1:\xi^4:-\xi^3,$$

and consequently its intersection with Σ is

$$\zeta(\xi^4, 0, -\xi^2, 0).$$

The cross ratio of the four spinors

$$\eta(\eta^1, \eta^3), \quad \zeta(\xi^4, -\xi^2), \quad 3(0, 1), \quad 1(1, 0)$$

is obviously equal to the cross ratio $(\Omega^*\Omega, {}^1\phi^2\phi)$ taken on L :

$$(\eta\xi, 3\ 1) = -\frac{\eta^1\xi^2}{\eta^3\xi^4} = (\Omega^*\Omega, {}^1\phi^2\phi),$$

and this equation together with (2.6) yields

$$(2.7) \quad (\Omega^*\Omega, {}^1\phi^2\phi) = -\frac{\Lambda^{12}}{\Lambda^{34}}.$$

Still using the same coordinate system we may put

$$(2.8) \quad \sqrt{2}\Lambda^{12} = \frac{{}^1\phi.\Lambda}{{}^1\phi.\Gamma} (\Gamma.\Gamma)^{\frac{1}{2}}, \quad \sqrt{2}\Lambda^{34} = \frac{{}^2\phi.\Lambda}{{}^2\phi.\Gamma} (\Gamma.\Gamma)^{\frac{1}{2}},$$

so that the right hand members do not depend on the factors of proportionality of ${}^1\phi, {}^2\phi, \Gamma$. Then we have from (2.7) and (2.8)

$$(\Omega^*\Omega, {}^1\phi^2\phi) = -\frac{{}^1\phi.\Lambda}{{}^2\phi.\Lambda} \frac{{}^2\phi.\Gamma}{{}^1\phi.\Gamma} = +\theta(\Lambda).$$

The invariant $\theta(\Lambda)$ being independent of the choice of $\xi, \eta \neq \xi$ on Λ , the same must be true for the cross ratio $(\Omega^*\Omega, {}^1\phi^2\phi)$.

Note. The invariant $\theta(\Lambda)$ associated with a spinor line Λ may in some cases be associated with spinors. Suppose that ξ describes a curve $\xi = \xi(t)$ and put

$$\Lambda^{ab} = \xi^a \frac{d\xi^b}{dt} - \xi^b \frac{d\xi^a}{dt} :$$

then $\theta(\Lambda(t))$ may be thought of as associated with the spinor defined by t .

§3. Invariants of two spinor lines

Lemma (3.1). *Let Λ and Ω be two different spinor lines. The four spinor lines $\Lambda, \Omega, {}^1\phi, {}^2\phi$ are met at least by two spinor lines (which may coincide).*

Proof. Suppose first $\Lambda.\Omega \neq 0$. Then $\Lambda, {}^1\phi, {}^2\phi$ and Ω are either quadratically dependent, or not. In the first case the locus of lines which meet $\Lambda, \Omega, {}^1\phi, {}^2\phi$ is a regulus (conjugate to the regulus containing $\Lambda, \Omega, {}^1\phi, {}^2\phi$). In the second case the quadrics $Q(\Lambda, {}^1\phi, {}^2\phi)$ and $Q(\Omega, {}^1\phi, {}^2\phi)$ intersect in ${}^1\phi, {}^2\phi$ and two lines different from ${}^1\phi, {}^2\phi$ which meet $\Lambda, \Omega, {}^1\phi, {}^2\phi$. (These two lines coincide if the two quadrics mentioned above are tangent to each other along a line different from ${}^1\phi, {}^2\phi$). Suppose now $\Lambda.\Omega = 0$. Then there is a ruling of K through the point $\Lambda\Omega$ (in the plane $\Lambda\Omega$) which meets $\Lambda, \Omega, {}^1\phi, {}^2\phi$. These two rulings coincide if the ruling of K through the point $\Lambda\Omega$ is incident with the plane.

In the following theorem we denote by Π and Σ the lines which meet Λ , Ω , ${}^1\phi$ and ${}^2\phi$, introduce the notation

$${}^2\sigma, {}^1\sigma, {}^2\pi, {}^1\pi, {}^\tau\lambda, {}^\tau\omega, {}^\sigma\lambda, {}^\sigma\omega$$

for the points

$${}^2\phi\Sigma, {}^1\phi\Sigma, {}^2\phi\Pi, {}^1\phi\Pi, \Lambda\Pi, \Omega\Pi, \Lambda\Sigma, \Omega\Sigma,$$

and put

$$\psi \equiv {}^2\pi {}^1\sigma.$$

Theorem (3.1). *The spinor lines Λ , Ω have two invariants:*

$$(3.1) \quad \begin{aligned} I_{\Pi}(\Lambda, \Omega) &\equiv ({}^\tau\lambda {}^\tau\omega, {}^1\pi {}^2\pi), \\ I_{\Sigma}(\Lambda, \Omega) &\equiv ({}^\sigma\lambda {}^\sigma\omega, {}^1\sigma {}^2\sigma), \end{aligned}$$

and these invariants may be expressed as follows:

$$(3.2) \quad \begin{aligned} \text{a) } I_{\Pi}(\Lambda, \Omega) &\equiv \frac{{}^1\phi.\Lambda \ \psi.\Omega}{{}^1\phi.\Omega \ \psi.\Lambda}, \\ \text{b) } I_{\Sigma}(\Lambda, \Omega) &\equiv \frac{{}^2\phi.\Omega \ \psi.\Lambda}{{}^2\phi.\Lambda \ \psi.\Omega}. \end{aligned}$$

Proof. Suppose first $\Lambda.\Omega \neq 0$ and use the same coordinate system as in the proofs of Theorems (1.1) and (2.3):

$$\begin{aligned} 1 &\equiv {}^2\sigma, & 2 &\equiv {}^2\pi, & 3 &\equiv {}^1\sigma, & 4 &\equiv {}^1\pi, \\ {}^1\phi &\equiv 34, & {}^2\phi &\equiv 12, & \Sigma &\equiv 13, & \Pi &\equiv 24, \\ \psi &\equiv 23. \end{aligned}$$

Then we have $({}^\tau\lambda {}^\tau\omega, 42) = ({}^\tau\lambda {}^\tau\omega, {}^1\pi {}^2\pi)$, and consequently (1.8a) and (1.9) yield (3.2a). The equation (3.2b) may be obtained in a similar manner.

Suppose now $\Lambda.\Omega = 0$. The proof of Theorem (3.1) will be given simultaneously with the proof of the following

Theorem (3.2). *A necessary and sufficient condition that the spinor lines Λ and Ω meet is: One of the invariants I_{Π} , I_{Σ} , say I_{Π} , is equal to one:*

$$(3.3a) \quad I_{\Pi}(\Lambda, \Omega) = 1.$$

If the condition (3.3a) is satisfied, then

$$(3.3b) \quad I_{\Sigma}(\Lambda, \Omega) = \frac{\theta(\Lambda)}{\theta(\Omega)},$$

Proof. A necessary and sufficient condition for the condition (3.3a) is

$$(3.4) \quad ({}^{\pi}\lambda {}^{\pi}\omega, {}^1\pi {}^2\pi) = 1.$$

A necessary and sufficient condition for (3.4) is that two of the points ${}^{\pi}\lambda, {}^{\pi}\omega, {}^1\pi, {}^2\pi$ coincide. The points ${}^1\pi, {}^2\pi$ cannot coincide, since ${}^1\phi$ and ${}^2\phi$ are skew. Neither of the points ${}^1\pi, {}^2\pi$ can coincide with ${}^{\pi}\lambda({}^{\pi}\omega)$, for according to our assumption in the section 0 we have ${}^1\phi.\Lambda {}^2\phi.\Lambda \neq 0$ (${}^1\phi.\Omega {}^2\phi.\Omega \neq 0$). Therefore, the only remaining possibility for coinciding points is ${}^{\pi}\lambda = {}^{\pi}\omega$. A necessary and sufficient condition for this equation is that Λ and Ω meet in a point on Π . Even in this case we obtain (3.1) in the same manner, but according to (3.4) we have also (3.3a), or, by virtue of (3.2a),

$$\frac{\psi.\Omega}{\psi.\Lambda} = \frac{{}^1\phi.\Omega}{{}^1\phi.\Lambda}.$$

Substituting this value in (3.2b) and remembering (1.4) we obtain (3.3b).

Note (i). Suppose that $\Lambda, \Omega, {}^1\phi$ and ${}^2\phi$ are quadratically dependent:

$$(3.5) \quad \begin{aligned} \text{a) } \Omega &= x_1 {}^1\phi + x_2 {}^2\phi + x_3 \Lambda, \\ \text{b) } x_1 x_2 ({}^1\phi, {}^2\phi) + x_1 x_3 ({}^1\phi, \Lambda) + x_2 x_3 ({}^2\phi, \Lambda) &= 0. \end{aligned}$$

Then we obtain from (3.5) and (3.1)

$$(3.6) \quad I_{\Pi} = I_{\Sigma} \left(= - \frac{x_1 {}^1\phi.\Lambda}{x_2 {}^2\phi.\Lambda} \right).$$

as was to be expected.

Note (ii). According to theorem (3.2) the invariant $I_{\Sigma}(\Lambda, \Omega) \equiv \frac{\theta(\Lambda)}{\theta(\Omega)}$ may be associated in some cases either with the plane $\Lambda\Omega$ or with the point $\Lambda\Omega$: If a spinor surface $\xi(u, v)$ is given and $\Lambda(\xi), \Omega(\xi)$ are the tangential lines of its asymptotic lines in the point $\xi = \Lambda\Omega$, then I_{Σ} may be associated either with this point or with the plane $\Lambda\Omega$.

The theorem (3.2) is closely connected with the following

Theorem (3.3). *The invariants $I_{\Pi}(\Lambda, \Omega), I_{\Sigma}(\Lambda, \Omega), \theta(\Lambda)$ and $\theta(\Omega)$ are related by*

$$(3.7) \quad I_{\Pi}(\Lambda, \Omega) I_{\Sigma}(\Lambda, \Omega) = \frac{\theta(\Lambda)}{\theta(\Omega)}.$$

The proof follows at once from (3.2) and (1.4).

Note. The equation (3.3b) is a special case of (3.7) and (3.3a).

Theorem (3.4). *A necessary and sufficient condition that Λ and Ω be rulings of a same complex of the set (2.1) is*

$$I_{\Pi}(\Lambda, \Omega) I_{\Sigma}(\Lambda, \Omega) = 1.$$

The proof follows at once from (3.7) and Theorem (2.1).

§4. Line metric.

Theorem (4.1). *The number $\vartheta \equiv \vartheta(\Lambda, \Omega)$ defined by*

$$(4.1) \quad e^{2\vartheta} \equiv I_{\Pi}(\Lambda, \Omega) I_{\Sigma}(\Lambda, \Omega)$$

has the following properties:

1) *We have*

$$(4.2) \quad \vartheta(\Lambda, \Omega) = \frac{1}{2} \ln \frac{\theta(\Lambda)}{\theta(\Omega)}.$$

2) *Moreover*

$$(4.3) \quad \begin{aligned} \text{a)} \quad & \vartheta(\Lambda, \Omega) = -\vartheta(\Omega, \Lambda), \\ \text{b)} \quad & \vartheta(\Lambda, \Lambda) = 0 \quad (= -\vartheta(\Omega, \Omega)). \end{aligned}$$

3) *If Λ, Ω, Δ are any three spinor lines then*

$$(4.3) \quad \text{c)} \quad \vartheta(\Lambda, \Omega) + \vartheta(\Omega, \Delta) = \vartheta(\Lambda, \Delta).$$

Proof. The equation (4.2) follows from (4.1) and (3.7). The equations (4.3) follow from (4.2). The equations (4.3) justify the following

Definition (4.1). *The invariant $\vartheta(\Lambda, \Omega)$ defined by (4.1) will be termed the projective distance of spinor lines Λ and Ω .*

Note (i). According to Theorem (3.4) two rulings of a same complex of the set (2.1) have their projective distance equal to zero.

Note (ii). The projective distance $\vartheta(\Lambda, \Omega)$ is closely related to the so-called "projective angle" of two complexes (cf. [2] p. 38): Let Γ_1 and Γ_2 be two complexes intersecting in a non-parabolic congruence C whose axes are ${}^1\Delta, {}^2\Delta$. Then

$$(4.4a) \quad \omega(\Gamma_1, \Gamma_2) = \frac{\epsilon}{2} \ln (\Gamma_1 \Gamma_2, {}^1\Delta {}^2\Delta)$$

is the projective angle of Γ_1 and Γ_2 . Here

$$\epsilon = \begin{cases} 1 & \text{if } {}^1\Delta, {}^2\Delta \text{ are } \left\{ \begin{array}{l} \text{real} \\ \text{complex conjugate} \end{array} \right. \\ \sqrt{-1} & \end{cases}$$

and $(\Gamma_1 \Gamma_2, {}^1\Delta {}^2\Delta)$ is the cross ratio of four complexes $\Gamma_1, \Gamma_2, {}^1\Delta, {}^2\Delta$ in the set $({}^1\Delta, {}^2\Delta)$. The projective angle $\omega(\Gamma_1, \Gamma_2)$ has a geometric interpretation: Let P be an arbitrary point, $\pi_1(\pi_2)$ its polar plane with respect to $\Gamma_1(\Gamma_2)$ and ${}^1\varphi \equiv P {}^1\Delta, {}^2\varphi \equiv P {}^2\Delta$ (the case of P on ${}^1\Delta$ or ${}^2\Delta$ is excluded). Then the four planes $\pi_1, \pi_2, \varphi_1, \varphi_2$ belong to the same pencil (whose axis is the ruling of C through P) and moreover

$$(4.4b) \quad \omega(\Gamma_1, \Gamma_2) = \frac{\epsilon}{2} \ln (\pi_1 \pi_2, {}^1\varphi {}^2\varphi),$$

where $(\pi_1\pi_2, {}^1\varphi^2\varphi)$ is the cross ratio of the four planes $\pi_1, \pi_2, {}^1\varphi, {}^2\varphi$. It follows from (4.4) that the cross ratio on the right hand side is independent of the choice of P .

Theorem (4.2). Let $\Gamma_1(\Gamma_2)$ be the only complex in the set (2.1) which contains $\Lambda(\Omega)$. Then the projective angle $\omega(\Gamma_1, \Gamma_2)$ of the complexes Γ_1, Γ_2 is equal to the projective distance $\vartheta(\Lambda, \Omega)$ of Λ and Ω :

$$(4.5) \quad \omega(\Gamma_1, \Gamma_2) = \vartheta(\Lambda, \Omega).$$

Proof. According to Theorem (2.1a) and equation (2.1) we have

$${}^1\phi \equiv \Gamma(0, \rho), \quad {}^2\phi \equiv \Gamma(\sigma, 0), \quad \Gamma_1 \equiv \Gamma(\gamma\theta(\Lambda), \gamma), \quad \Gamma_2 \equiv \Gamma(\nu\theta(\Omega), \nu),$$

where $\rho, \sigma, \gamma, \nu \neq 0$ are arbitrary numbers. Consequently

$$(\Gamma_1\Gamma_2, {}^1\phi^2\phi) = \frac{\theta(\Lambda)}{\theta(\Omega)},$$

and the theorem follows from (4.4a) (for $\epsilon = 1$) and (4.2).

In the next definition we use the projective distance for defining the "linear element" of the projective metric.

Definition (4.2). Let $\Lambda = \Lambda(t)$ be a function of a single parameter. Then

$$(4.6) \quad d\sigma \equiv dt \lim_{t \rightarrow 0} \frac{2\vartheta(\Lambda(0), \Lambda(t))}{t}$$

is termed the "linear element" (at $t = 0$).

Theorem (4.3). Put

$$(4.7) \quad \Psi(\Lambda) \equiv \Psi \equiv \frac{({}^1\phi.\Lambda)^2\phi - ({}^2\phi.\Lambda){}^1\phi}{({}^1\phi.\Lambda)({}^2\phi.\Lambda)}.$$

Then

$$(4.8a) \quad d\sigma = (\Psi.d\Lambda)_{t=0}.$$

Proof. First of all we have

$$\begin{aligned} 2\vartheta(\Lambda(0), \Lambda(t)) &= 2\vartheta(\Lambda(0), \Lambda(0)) + 2t \left(\frac{d}{dt} \vartheta(\Lambda(0), \Lambda(t)) \right)_{t=0} + t^2(\dots) \\ &= 2t \left(\frac{d}{dt} \vartheta(\Lambda(0), \Lambda(t)) \right)_{t=0} + t^2(\dots) \\ &= t \left(\frac{d}{dt} \ln \frac{\theta(\Lambda(0))}{\theta(\Lambda(t))} \right)_{t=0}^* + t^2(\dots) \\ &= -t \left(\frac{d\theta(\Lambda(t))}{dt} \right)_{t=0} \theta^{-1}(\Lambda(0)) + t^2(\dots). \end{aligned}$$

Consequently we have by virtue of (1.4)

$$(4.9) \quad \begin{aligned} d\sigma &= - \left(\frac{d\theta(\Lambda(t))}{dt} \right)_{t=0} \theta^{-1}(\Lambda(0)) \\ &= - \left(\frac{{}^2\phi.\Lambda \quad {}^1\phi.d\Lambda - {}^1\phi.\Lambda \quad {}^2\phi.d\Lambda}{{}^1\phi.\Lambda \quad {}^2\phi.\Lambda} \right)_{t=0}, \end{aligned}$$

and (4.8a) follows from (4.7) and (4.9).

Note. No misunderstanding arises if instead of (4.8a) we use the formula

$$(4.8b) \quad d\sigma = \Psi.d\Lambda,$$

with the understanding that both members of this equation are to be taken for the same value of t .

Theorem (4.4). *The complex Ψ defined by (4.7) has the following properties:*

1) *It belongs to the set (2.1), and in particular*

$$(4.10) \quad \Psi(\Lambda) = \Gamma(\theta(\Lambda), 1).$$

2) *It is the only complex in the set (2.1) which contains Λ :*

$$(4.11) \quad \Psi.\Lambda = 0.$$

3) *It is a gradient*

$$(4.12) \quad \Psi = - \frac{\partial}{\partial \Lambda} \ln \theta(\Lambda).$$

The proof of (4.10) follows at once from (4.7) and (2.1). The statement 2) (3)) follows from the statement 1) and Theorem (2.1a) ((2.1b)).

This theorem will be used in the next section.

§5. Continuation

Theorem (5.1). *The linear element $d\sigma$ as defined by (4.8b) is invariant under the change of the proportionality factor:*

$$(5.1) \quad {}^*\Lambda = \Lambda f.$$

Proof. Using (5.1) and (4.7) we obtain

$${}^*\Psi = \Psi f^{-1}.$$

Combining this equation with

$$d{}^*\Lambda = fd\Lambda + \Lambda df$$

and (4.11) we obtain

$$(5.2) \quad d{}^*\sigma = {}^*\Psi.d{}^*\Lambda = \Psi.d\Lambda = d\sigma,$$

and this equation proves our theorem.

Theorem (4.4) enables us to solve the equation (4.8b), where Λ is no longer supposed to be function of a single parameter t :

Theorem (5.2). *Let Λ be a spinor line which is a function of position in S_3 and let Λ_0 be an arbitrary fixed line. Then the equation (4.8b) admits a solution*

$$(5.3) \quad \sigma(\Lambda_0, \Lambda) \equiv \sigma = 2\vartheta(\Lambda_0, \Lambda)$$

(and this solution is independent of proportionality factors of Λ_0 and Λ).

Proof. We have by virtue of (4.2) and (4.12)

$$2 \frac{\partial}{\partial \Lambda} \vartheta(\Lambda_0, \Lambda) = \frac{\partial}{\partial \Lambda} \ln \frac{\theta(\Lambda_0)}{\theta(\Lambda)} = - \frac{\partial}{\partial \Lambda} \ln \theta(\Lambda) = \Psi,$$

and consequently (5.3) holds according to (4.8b).

Note. The equation (4.8b) (where Λ need not be regarded as a function of one single parameter) may be thought of as defining a *line metric* in S_3 . Therefore it is quite natural to ask about the geodesic and minimal set of this metric.

Definition (5.1). *Any solution $\Lambda = \Lambda(t)$ of the variational problem*

$$(5.4) \quad \delta \int_a^b d\sigma = 0$$

will be termed a geodesic set and in particular if $d\sigma = 0$ it will be called a minimal set.

Theorem (5.3). *A necessary and sufficient condition for a set $\Lambda(t)$, for which $d\sigma$ is defined, to be a minimal one is: all its rulings belong to the same complex of (2.1).*

The proof of this theorem will be given simultaneously with the proof of the following

Theorem (5.4). *Any set $\Lambda(t)$ for which $d\sigma$ is defined is a geodesic one.*

Proof. Put $\dot{\Lambda} = \frac{d\Lambda}{dt}$ and $F(\Lambda, \dot{\Lambda}) \equiv \Psi \cdot \dot{\Lambda}$. Then the Euler equation

$$\frac{\partial F}{\partial \Lambda^{ab}} - \frac{\partial}{\partial t} \frac{\partial F}{\partial \dot{\Lambda}^{ab}} = 0$$

of the variational problem (5.4) reduces to

$$(5.5) \quad \left(\frac{\partial \Psi_{ab}}{\partial \Lambda^{cd}} - \frac{\partial \Psi_{cd}}{\partial \Lambda^{ab}} \right) \dot{\Lambda}^{cd} = 0,$$

and this equation is identically satisfied on account of the statement 3) of Theorem (4.4). Hence any set $\Lambda(t)$ for which $d\sigma$ is defined is a geodesic one. This proves the theorem (5.4). It is now easily seen from (5.3) and theorem (2.1a) that theorem (5.3) holds.

Note. The equation (5.5) is identically satisfied regardless of changes of the proportionality factor (5.1).

II. Dirac and Schrödinger Equations

§6. Auxiliary Theorems

In the following considerations we denote by Ω_b^a the Cartan matrix belonging to a nonisotropic point $\mathbf{v}$ in L_3 and constructed according to $I(0.12)$. Moreover, if A_b^a, B_c^b, C_d^c are three 4×4 matrices we understand by $(ABC)_d^a$ (or simply by ABC) the product

$$(ABC)_d^a = A_b^a B_c^b C_d^c.$$

If A is any nonsingular 4×4 matrix we denote by A^{-1} its inverse matrix. The Cartan matrix Ω_b^a defines a biaxial involution (cf. Theorem I(2.1)). Then the following theorem holds:

Theorem (6.1). *The spinor transformation*

$$(6.1) \quad \xi^a = (S\Omega S^{-1})_b^a \xi^b$$

is a biaxial involution, provided that S be a nonsingular 4×4 matrix.

Proof. If and only if S is a nonsingular 4×4 matrix, its inverse S^{-1} exists. On the other hand the matrices Ω_b^a and $(S\Omega S^{-1})_b^a$ being similar matrices, they have the same "eigenvalues" λ (i.e. the roots of the corresponding secular equation) which satisfy the equation $I(2.3b)$

$$(6.2) \quad \mathbf{v} \cdot \mathbf{v} = \lambda^2.$$

The theorem follows from this fact.

Agreement (6.1). Let Ω_a^b be a Cartan matrix and

$$(6.3) \quad \text{a) } S_a^b \equiv \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{b) } T_A^B \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then the product matrix $(S\Omega S^{-1})$ will be always denoted by

$$(6.4a) \quad {}^*\Omega_a^b = (S\Omega S^{-1})_a^b,$$

and any point which one obtains from a generic point x^A by the transformation Tx will be denoted by

$$(6.4b) \quad {}^*x^A = T_B^A x^B.$$

Note. We have easily from (6.4b)

$$(6.4c) \quad {}^*x \cdot {}^*x = x \cdot x.$$

Hence the transformation (6.4b) preserves the absolute quadric $\mathbf{Q}$ in L_3 .

Theorem (6.2). *The axes ${}^*\Omega$ of the biaxial involution defined by ${}^*\Omega_b^a$ have the following properties:*

1) *They are conjugate polars with respect to Γ .*

- 2) They are rulings of $*\Gamma$.
 3) Their analytic expression is

$$(6.5a) \quad *\Omega^{ab}(-\lambda i\epsilon, -\lambda i\epsilon, *v^{\text{II}}, *v^{\text{I}}, *v^{\text{III}}, *v^{\text{IV}})$$

or

$$(6.5b) \quad *\Omega^{ab}(-\lambda i\epsilon, -\lambda i\epsilon, v^{\text{II}}, v^{\text{I}} - v^{\text{IV}}, -v^{\text{III}}) \\ (\epsilon = + \text{ or } \epsilon = -, \lambda = (\mathbf{v}.\mathbf{v})^{\frac{1}{2}} = (*\mathbf{v}.*\mathbf{v})^{\frac{1}{2}}).$$

Proof. First of all we have according to (6.3a) and (6.4)

$$(6.6) \quad *\Omega_b^a = -i \begin{pmatrix} 0 & 0 & -v^{\text{I}} & v^{\text{IV}} \\ 0 & 0 & v^{\text{III}} & v^{\text{II}} \\ v^{\text{II}} & -v^{\text{IV}} & 0 & 0 \\ -v^{\text{III}} & -v^{\text{I}} & 0 & 0 \end{pmatrix} \\ = -i \begin{pmatrix} 0 & 0 & -*v^{\text{I}} & -*v^{\text{III}} \\ 0 & 0 & -*v^{\text{IV}} & *v^{\text{II}} \\ *v^{\text{II}} & *v^{\text{III}} & 0 & 0 \\ *v^{\text{IV}} & -*v^{\text{I}} & 0 & 0 \end{pmatrix}.$$

The system

$$(6.7) \quad \lambda \xi^a - *\Omega_b^a \xi^b = 0,$$

where λ is given by (6.2), is of rank two and leads at once to (6.5a), which together with (6.4b) yields (6.5b). These formulas at once lead to the statement 2), where the complex $*\Gamma$ is given by I(1.5b). Let Λ be the conjugate polar line of $*^+\Omega$ with respect to Γ :

$$\Lambda = 2\Gamma(\Gamma.*^+\Omega) - *^+\Omega(\Gamma.\Gamma).$$

By virtue of (6.5) and I(1.5a) this equation reduces to

$$(6.8) \quad \Lambda = -2*^-\Omega,$$

which proves the statement 1).

Theorem (6.3). *The biaxial involution*

$$(6.9) \quad '\xi^a = *\Omega_b^a \xi^b$$

is a map of a transformation of lineal elements induced by the transformation

$$(6.10) \quad 'x^A = -x^A(*\mathbf{v}.*\mathbf{v}) + 2*v^A(*\mathbf{v}.\mathbf{x}).$$

The proof follows the line of the proof of theorem I(8.1).

Note. From (6.10) we obtain

$$(6.11a) \quad '\mathbf{x}.\mathbf{x} = (\mathbf{x}.\mathbf{x}) (*\mathbf{v}.*\mathbf{v})^2.$$

Consequently (6.10) reproduces the absolute quadric Q . On the other hand if $\mathbf{x}$ is in the polar plane of $\ast\mathbf{v}$ with respect to Q then $\ast\mathbf{v}.\mathbf{x} = 0$ and consequently (6.10) reduces to

$$(6.11b) \quad 'x^A = -x^A(\ast\mathbf{v}.\ast\mathbf{v}).$$

Hence any point $\mathbf{x}$ of the polar plane of $\ast\mathbf{v}$ (with respect to Q) is a double point of (6.10). Moreover for $\mathbf{x} = \ast\mathbf{v}$ we obtain from (6.10)

$$(6.11c) \quad '\ast v^A = \ast v^A(\ast\mathbf{v}.\ast\mathbf{v})$$

so that $\ast\mathbf{v}$ itself is a double point of (6.10). Since (6.10) is not an identity it can not have more double points. The characteristic of this transformation is obviously -1 and consequently we shall refer to it as a “ $\ast\mathbf{v}$ -involution” or “ $\ast\mathbf{v}$ -reflection.”

Theorem (6.4). *Let $\lambda(\rho)$ be the eigenvalues belonging to (6.9) ((6.10)). Then*

$$(6.12) \quad \rho = \pm\lambda^2.$$

The upper sign refers to the isolated double point $\ast\mathbf{v}$ while the lower sign leads to double points of the plane $\ast\mathbf{v}.\mathbf{x} = 0$.

Proof. First of all we have from (6.11a) and (6.4c) for double points of (6.10)

$$(6.13) \quad (\rho^2 - (\mathbf{v}.\mathbf{v})^2)\mathbf{x}.\mathbf{x} = 0,$$

while (6.11b, c) lead to

$$(6.14) \quad (\rho \pm \mathbf{v}.\mathbf{v})\mathbf{x} = 0.$$

Since we are dealing with a $\ast\mathbf{v}$ -reflection, the equation (6.13) and (6.14) describe all double points of (6.10) and consequently the eigenvalues of (6.10) are defined by

$$\rho = \pm(\mathbf{v}.\mathbf{v}),$$

and this equation together with (6.2) leads to (6.12). The last part of the theorem follows at once from (6.11b, c).

§7. Dirac Equations and Schrödinger Equations

In this section we will be concerned with Dirac equations and Schrödinger equations. First of all we introduce them using the coordinate system x, y, z, t with the metric form

$$(7.1) \quad x^2 + y^2 + z^2 - c^2 t^2.$$

Let us first define Pauli matrices

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

as well as the matrices

$$\xi \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \xi^* = \begin{pmatrix} \bar{\xi}_1^1 \\ \bar{\xi}_2^2 \end{pmatrix}.$$

If we denote by d_x, d_y, d_z, d_t the covariant components of the linear operator used in the theory of quanta, then the Dirac equations are (cf. [3] p. 97, in a slightly different form)

$$(7.2) \quad \begin{aligned} \left(\frac{d_t}{c} \sigma_t + d_x \sigma_x + d_y \sigma_y + d_z \sigma_z \right) \xi + \mu c \xi^* &= 0, \\ \left(\frac{d_t}{c} \sigma_t - d_x \sigma_x - d_y \sigma_y - d_z \sigma_z \right) \xi^* + \mu c \xi &= 0, \end{aligned}$$

where as usual c denotes the velocity of light and μ the mass of the electron. The Schrödinger relativistic wave equation is (cf. [3] p. 95)

$$(7.3) \quad \left(d_x^2 + d_y^2 + d_z^2 - \left(\frac{d_t}{c} \right)^2 \right) \psi + \mu^2 c^2 \psi = 0.$$

In the next theorem we use the contravariant components (with respect to the metric defined by (7.1)) of the linear operator used in (7.2):

$$d^x = d_x, \quad d^y = d_y, \quad d^z = d_z, \quad d^t = -\frac{1}{c^2} d_t.$$

Theorem (7.1). *Introduce the symbolic components*

$$(7.4a) \quad v^I = d^x + id^y, \quad v^{II} = d^x - id^y, \quad v^{III} = d^z + cd^t, \quad v^{IV} = d^z - cd^t;$$

denote by Ω_a^b the Cartan matrix belonging to $\mathfrak{v}(v^I, v^{II}, v^{III}, v^{IV})$ and by $*\Omega_a^b$ the matrix defined by (6.4a). Then the Dirac equations (7.2) reduce to

$$(7.4b) \quad *\Omega_a^b \xi^a + i\mu c \xi^b = 0.$$

Proof. The Dirac equations may be condensed in one:

$$(7.5a) \quad \begin{pmatrix} 0 & 0 & -v^{III} & -v^{II} \\ 0 & 0 & -v^I & v^{IV} \\ v^{IV} & v^{II} & 0 & 0 \\ v^I & -v^{III} & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \bar{\xi}_1^1 \\ \bar{\xi}_2^2 \end{pmatrix} + \mu c \begin{pmatrix} \xi_1 \\ \xi_2 \\ \bar{\xi}_1^1 \\ \bar{\xi}_2^2 \end{pmatrix} = 0.$$

On the other hand we have by virtue of I(2.8c) and I(1.6)

$$(7.6) \quad \begin{pmatrix} \xi_1 \\ \xi_2 \\ \bar{\xi}_1^1 \\ \bar{\xi}_2^2 \end{pmatrix} = \begin{pmatrix} \xi^2 \\ -\bar{\xi}^1 \\ \xi^3 \\ \xi^4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \\ \xi^4 \end{pmatrix}$$

Hence if we denote by V_a^b (P_a^b) the components of the 4×4 matrix in (7.5a) ((7.6)) we may write (7.5a) in an equivalent form:

$$(7.5b) \quad (P^{-1}VP)_b^a \xi^b + \mu c \xi^a = 0.$$

On the other hand a simple computation shows

$$(7.7) \quad (P^{-1}VP)_a^b = -*\Omega_a^b i = \begin{pmatrix} 0 & 0 & +v^I & -v^{IV} \\ 0 & 0 & -v^{III} & -v^{II} \\ -v^{II} & v^{IV} & 0 & 0 \\ v^{III} & v^I & 0 & 0 \end{pmatrix}.$$

Substituting from this equation in (7.5b) we obtain (7.4b).

Theorem (7.2). *The solution ξ^a of (7.4b) satisfies the condition*

$$(7.8) \quad (\mathbf{v} \cdot \mathbf{v} + \mu^2 c^2) \xi^a = 0.$$

Proof. The matrices Ω_a^b and $*\Omega_a^b$ being similar, we obtain by virtue of I(0.13) and (6.4c)

$$(7.9) \quad *\Omega_b^c *\Omega_a^b = \Omega_b^c \Omega_a^b = *\mathbf{v} \cdot *\mathbf{v} \delta_a^c = \mathbf{v} \cdot \mathbf{v} \delta_a^c.$$

Applying the operator $*\Omega_b^c$ on (7.4) we obtain according to (7.9) and (7.4) the equation (7.8).

Theorem (7.3). *Let Ω_a^b be the (symbolic) Cartan matrix belonging to (7.4a). Then the (symbolic) biaxial involution*

$$(7.10a) \quad ' \xi^a = *\Omega_b^a \xi^b$$

has the (symbolic) axes

$$(7.10b) \quad *\Omega^{ab}(\mu c \epsilon, \mu c \epsilon, v^{II}, v^I, -v^{IV}, -v^{III}),$$

which are conjugate polars of Γ belonging to Γ. Moreover the Dirac equations (7.4b) are equations for “eigenspinors” on $*^+\Omega^{ab}$.*

Proof. Comparing (7.8) with (6.2) we are entitled to put

$$(7.11) \quad \lambda^2 = -\mu^2 c^2.$$

Then the first part of Theorem (7.3) follows at once from Theorem (6.2) and equation (7.11). The “eigenspinors” of (7.10a) are given by

$$(7.12) \quad \lambda \xi^a = *\Omega_b^a \xi^b,$$

where λ is the eigenvalue belonging to $*\Omega_a^b$. The last part of Theorem (7.3) follows from (7.12) and (7.11).

Note. It is clear from the geometrical interpretation of Dirac equations and from (7.11) that also the equations

$$(7.13) \quad * \Omega_a^b \xi^a - i \mu c \xi^b = 0$$

(which define the eigenspinors on $*\bar{\Omega}$) must have a physical meaning.

Theorem (7.4). *The biaxial involution (7.10a) is mapped on a (symbolic) $*\mathbf{v}$ -reflection*

$$(7.14) \quad 'x^A = -x^A(\mathbf{v} \cdot \mathbf{v}) + 2 * v^A (* \mathbf{v} \cdot \mathbf{x}).$$

The locus of eigenspinors of (7.10a) defined by Dirac equations (7.4) (as well as by (7.13)) is mapped on the locus of double lineal elements whose supports are given by the Schrödinger relativistic wave equation

$$(7.15) \quad (\mathbf{v} \cdot \mathbf{v} + \mu^2 c^2) \mathbf{x} = 0.$$

Proof. The first part follows at once from Theorem (6.3). The supports of double lineal elements (i.e. lineal elements reproduced by (7.14)) satisfy the equation derived from (6.11b) (and (6.4c))

$$(7.16) \quad (\mathbf{v} \cdot \mathbf{v} + \rho) \mathbf{x} = 0,$$

where according to Theorem (6.4) and equation (7.11)

$$(7.17) \quad \rho = -\lambda^2 = \mu^2 c^2.$$

Substituting ρ from (7.17) to (7.16) one obtains (7.15). On the other hand we have from (7.4a)

$$\mathbf{v} \cdot \mathbf{v} = (d^x)^2 + (d^y)^2 + (d^z)^2 - c^2 (d^t)^2 = d_x^2 + d_y^2 + d_z^2 - \frac{d_t^2}{c^2},$$

and consequently (7.15) may be written

$$(d_x^2 + d_y^2 + d_z^2 - c^{-2} d_t^2 + \mu^2 c^2) \mathbf{x} = 0,$$

and this equation is of the form (7.3).

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