Uniqueness of Axially Symmetric Flows with Free Boundaries

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Introduction. Many of the notable successes in the hydrodynamics of plane flows rest on the power of function-theoretic methods that are inapplicable to three-dimensional flow problems. In particular, the free boundary problems of hydrodynamics, which have been successfully studied in the plane by methods of conformal mapping, have resisted solution in the case of axial symmetry. To date the only noteworthy theoretical results on the latter problem seem to be Levinson's formula for the asymptotic shape of the axisymmetric infinite cavity [1]¹ and Friedrichs' proof by a variational method of local uniqueness of the jet flow from a channel [2].

In papers marking important advances in the plane free boundary problem, Lavrentieff [3] and Serrin [4] reduced the previously formidable uniqueness problem to comparatively simple geometric considerations centering about some hydrodynamic comparison theorems. These theorems show the qualitative dependence of the flow speed, at a fixed boundary point of a flow region, on variations of the region preserving certain properties of the flow in the large. The simplicity of the comparison theorems and of the methods based on them suggest an approach to the axially symmetric free boundary problem. The purpose of this paper is to prove the analogue of two of these theorems for axially symmetric flows (Theorems 1 & 2), and by means of them to establish, under certain broad assumptions, uniqueness in the large for two of the classical axisymmetric free boundary problems, namely, the problem of the Helmholtz infinite cavity or wake (Theorem 3) and the problem of jet flow out of a channel or vessel (Theorem 4). While further applications suggest themselves, we re-

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¹ For additional results on the asymptotic shape of the cavity, see F. SCHEID, *Amer. Jour. Math.* **72**, 485-501 (1950), and M. GUREVICH, *Prikl. Mat. i. Mekh.* **11**, 97-104 (1947).

strict the discussion to these two problems, which are the subject of the major portion of the literature on plane flows with free boundaries.²

Although the emphasis in this paper is on their application to free boundary problems, comparison theorems of the type proved here should prove useful in many other qualitative problems of hydrodynamics as well. Further generalizations, in particular to gas flows, will be the subject of a later study.

Lavrentieff's comparison theorems for axially symmetric flows. In the following we shall consider axially symmetric potential flows of an ideal incompressible fluid. It suffices to discuss the flows in the meridian half-plane, in which x and y will designate the axial and radial variables, respectively. These flows are described by a stream function $\psi(x, y)$ satisfying the equation

$$(1) \quad \psi_{xx} + \psi_{yy} - \psi_y/y = 0.$$

The most important property of this equation for our subsequent work is the fact that the maximum principle holds: a non-constant solution of (1) can achieve neither a maximum nor minimum in the interior of its region of definition.

To fix our ideas, the boundary curves of the flows considered here are assumed piecewise smooth. A *regular* boundary point is a point, not on the axis of symmetry, at which the boundary curve has a tangent circle lying entirely in the flow region.

FIG. 1

We now prove

Theorem 1. *Let D and $\bar{D}$ be flow regions for two axially symmetric flows having the same uniform non-zero velocity at infinity; let D and $\bar{D}$ be bounded by the single streamlines γ and $\bar{\gamma}$, extending to $x = \pm \infty$. If $D \subset \bar{D}$ and if γ and $\bar{\gamma}$ have a regular*

² For a survey of methods and results on these problems, see A. WEINSTEIN, *L'Enseignement Mathématique* **35**, 107-125 (1936), and *Proc. First Symposium on Applied Mathematics*, Amer. Math. Soc. (1949), 1-18. The assertion in the latter paper that LAVRENTIEFF's method is not suited to the problem of jet flow is shown to be unfounded by Theorem 4 of the present paper and by unpublished results of J. SERRIN on plane flows.

point P in common (Fig. 1), then the respective flow speeds q and $\bar{q}$ satisfy at P the inequality

$$q(P) \leq \bar{q}(P);$$

the equality holds if and only if $D = \bar{D}$ and the two flows are identical.

Proof. Let ψ and $\bar{\psi}$ be the stream functions of the flows in D and $\bar{D}$, and let them be so normalized that $\psi = 0$ on γ , $\bar{\psi} = 0$ on $\bar{\gamma}$. Since $D \subset \bar{D}$, $\bar{\psi} \geq 0$ on γ ,³ so that $\Omega \equiv \bar{\psi} - \psi \geq 0$ on γ . We show first that $\Omega \geq 0$ in the interior of D , the equality (by the maximum principle) holding either everywhere or nowhere in D .

In polar coordinates r, θ , the fact that the two flows approach the same uniform velocity at infinity can be stated in terms of the transverse and radial components of the velocity difference in the form

$$\begin{aligned} \frac{1}{r^2 \sin \theta} \frac{\partial \Omega}{\partial \theta} &= \epsilon_1(r, \theta), \\ \frac{1}{r \sin \theta} \frac{\partial \Omega}{\partial r} &= \epsilon_2(r, \theta), \quad ((r, \theta) \in D), \end{aligned}$$

where $\epsilon_1(r, \theta), \epsilon_2(r, \theta) \rightarrow 0$ as $r \rightarrow \infty$, uniformly in θ . By direct integration along $r = \text{const.}$, we have for the value of $\Omega(r, \theta)$ in D

$$\Omega(r, \theta) = \int_{\gamma_r}^{(r, \theta)} \epsilon_1 r^2 \sin \theta d\theta + \Omega(\gamma_r),$$

where γ_r is the first point at which the path $r = \text{constant}$ meets γ , and is in the same quadrant as (r, θ) .⁴ Hence

$$(2) \quad |\Omega(r, \theta) - \Omega(\gamma_r)| \leq \epsilon r^2 (1 - |\cos \theta|),$$

where

$$\epsilon = \epsilon(r) = \max_{\theta} |\epsilon_1(r, \theta)| \rightarrow 0 \text{ as } r \rightarrow \infty.$$

From this growth condition on Ω and the fact that $\Omega \geq 0$ on γ , we shall be able to conclude that $\Omega \geq 0$ in D . (We note the similarity to the Phragmén-Lindelöf principle for harmonic functions.)

For any fixed $\alpha > 0$, consider the solution of (1) defined by

$$u_\alpha(r, \theta) \equiv \alpha r^2 \sin^2 \theta + \Omega(r, \theta), \quad ((r, \theta) \in D).$$

³ That $\bar{\psi} > 0$ in D follows from the uniformity of the flow at infinity.

⁴ For definiteness, if $0 < \theta \leq \pi/2$, take γ_r to have a smaller argument than θ , and if $\pi/2 < \theta < \pi$, take γ_r to have a larger argument than θ ; this fixes γ_r uniquely. However, for large r , since γ becomes horizontal, there can be no ambiguity.

From (2) we have

$$\begin{aligned} u_\alpha(r, \theta) &\geq \alpha r^2 \sin^2 \theta - \epsilon r^2 (1 - |\cos \theta|) + \Omega(\gamma_r) \\ &\geq r^2 \left(\alpha - \frac{\epsilon}{1 + |\cos \theta|} \right) \sin^2 \theta \\ &\geq r^2 (\alpha - \epsilon) \sin^2 \theta \\ &> 0, \text{ for sufficiently large } r. \end{aligned}$$

Thus $u_\alpha(r, \theta) \geq 0$ on γ , and for all sufficiently large r , from which it follows by the maximum principle that

$$u_\alpha(r, \theta) \geq 0 \text{ in } D.$$

Consequently in the limit as $\alpha \rightarrow 0$ we obtain

$$(3) \quad \Omega = \bar{\psi} - \psi \geq 0 \text{ in } D,$$

and from this, by the maximum principle, either $\bar{\psi} \equiv \psi$, or else $\Omega > 0$ at all interior points of D .

At the point P common to γ and $\bar{\gamma}$ we have $\Omega(P) = 0$. This in conjunction with (3) implies that

$$\frac{\partial \Omega}{\partial n}(P) \geq 0,$$

where n is the normal to γ drawn inward to D . Thus

$$q(P) = \frac{1}{y_P} \frac{\partial \psi}{\partial n}(P) \leq \frac{1}{y_P} \frac{\partial \bar{\psi}}{\partial n}(P) = \bar{q}(P).$$

To complete the proof of the theorem it remains to show that the equality holds if and only if $D = \bar{D}$, $\psi \equiv \bar{\psi}$.

If $D = \bar{D}$, it is clear that $\psi \equiv \bar{\psi}$ and $q(P) = \bar{q}(P)$. If $D \neq \bar{D}$ and $\Omega \not\equiv 0$, consider a circle C lying entirely in D , with circumference above the x -axis, and touching γ at P ; such a circle can be found since P is a regular boundary point. Let $G(z, \zeta)$ be the Green's function in this circle for equation (1). The existence of the Green's function for equations of type (1) is well known. Let the singularity ζ of $G(z, \zeta)$ be taken at some fixed point in C , and let C' be another circle about this point and inside C . Since $\Omega > 0$ on C' , there is a constant $\epsilon > 0$ for which

$$(4) \quad \Omega - \epsilon G \geq 0$$

on C' , and since $G = 0$ on C , this inequality holds also on C . By the maximum principle it follows that the inequality (4) prevails in and on the annulus be-

tween C and C' , while at the same time $\Omega(P) - \epsilon G(P) = 0$. Consequently

$$\frac{\partial \Omega}{\partial n}(P) \geq \epsilon \frac{\partial G}{\partial n}(P) > 0.$$

That $\frac{\partial G}{\partial n}(P) > 0$ follows from a known theorem⁵ which asserts that the normal derivative of the Green's function of an elliptic differential equation of suitably general type, in which (1) is included, is positive on the boundary. Therefore

$$\bar{q}(P) - q(P) = \frac{1}{y_P} \left[\frac{\partial \bar{\psi}}{\partial n}(P) - \frac{\partial \psi}{\partial n}(P) \right] = \frac{1}{y_P} \frac{\partial \Omega}{\partial n}(P) > 0,$$

and the theorem is proved.

We remark that the general theorems quoted above on the existence and normal derivative of Green's functions can be avoided in the present problem with the assistance of known particular solutions of (1). Indeed, consider in space the vortex flow with axial symmetry, for which the stream function can be given explicitly⁶. The center of the vortex can be chosen so close to P that one of the streamlines C is tangent to γ at P and lies entirely within D . The stream function of the vortex flow is to within a factor precisely the Green's function of (1) for the region enclosed by C , provided, of course, that we take the value of the stream function as zero on the streamline C . From the explicit formula for the stream function we see readily that its normal derivative on C , in particular at P , is greater than zero, or, in other words, that the flow speed on the streamline is non-zero. The same argument as before then shows that $\frac{\partial \Omega}{\partial n}(P) > 0$.

The restriction that P not lie on the axis of symmetry was required only in the second part of the proof, where the stated theorems on Green's functions assume the equation (1) is nonsingular. This restriction can undoubtedly be lifted with the aid of results on differential equations whose coefficients are permitted to become singular, or by means of explicit solutions unknown to the author.

A similar theorem can be stated for flows in a strip domain. By a *strip domain* we shall mean a region bounded by two nonintersecting curves in the meridian plane, each having its endpoints at $x = \pm\infty$ (Fig. 2). There are no further limitations on the behavior of the curves at infinity, in contrast with the boundary curves of Theorem 1. The flows in these regions are now not necessarily uniform at infinity.

Theorem 2. *Let D and $\bar{D}$ be strip domains bounded by the curves γ , Γ and $\bar{\gamma}$, $\bar{\Gamma}$ respectively, and suppose $D \subset \bar{D}$. Let axially symmetric flows through D and $\bar{D}$*

⁵ See LICHTENSTEIN, *Enc. der Math. Wiss.*, Bd. 2, Teil 3, 2 Hälfte (II C12), p. 1290, 1310.

⁶ See, for example, LAMB, *Hydrodynamics*, 6th edition, p. 237.

be defined by respective stream functions ψ and $\bar{\psi}$ such that $\psi = \bar{\psi} = 0$ on γ , $\psi = Q$ on Γ , $\bar{\psi} = Q$ on $\bar{\Gamma}$, and $0 \leq \psi, \bar{\psi} \leq Q$, Q being a positive constant; that is, the two flows have the same flux. If P is a regular point common to Γ and $\bar{\Gamma}$, and if R is any regular point on γ then the respective flow speeds $q, \bar{q}$ satisfy at P and R the inequalities,

$$q(P) \leq \bar{q}(P), \quad q(R) \geq \bar{q}(R);$$

for either equality to hold it is necessary and sufficient that $D = \bar{D}$ and $\psi \equiv \bar{\psi}$.

FIG. 2

Proof. Let $\Omega \equiv \psi - \bar{\psi}$; then $\Omega(x, y)$ is a solution of (1) in D such that $\Omega = 0$ on γ , $\Omega \geq 0$ on Γ , and $|\Omega| \leq Q$ in D . We show first that $\Omega \geq 0$ throughout D . Let us suppose to the contrary that $\Omega < 0$ somewhere in D , say for convenience, and without loss of generality, on the line $x = 0$. For any $\alpha > 0$, the solution of (1) given by

$$v_\alpha(x, y) \equiv \alpha x + \Omega(x, y)$$

is positive in D for sufficiently large positive x and also on the boundary of D . Since v_α is negative at some point on $x = 0$, we then have by the maximum principle

$$v_\alpha(x, y) \geq \min_y v_\alpha(0, y) = \min_y \Omega(0, y) < 0, \quad (x \geq 0).$$

In the limit, therefore, as $\alpha \rightarrow 0$,

$$\Omega(x, y) \geq \min_y \Omega(0, y), \quad ((x, y) \in D, x \geq 0).$$

Similarly, by considering negative α we reach the same conclusion for all points $(x, y) \in D$, $x \leq 0$. In other words, $\Omega(x, y)$ achieves its minimum value in D on the line $x = 0$, therefore on the boundary of D , where, however, $\Omega \geq 0$; this contradicts the original assumption that Ω takes negative values in D .

If $\Omega \not\equiv 0$, we have as in the proof of Theorem 1

$$q(R) - \bar{q}(R) = \frac{1}{y_R} \frac{\partial \Omega}{\partial n}(R) > 0,$$

the normal being directed into D , while at P

$$q(P) - \bar{q}(P) = \frac{1}{y_P} \frac{\partial \Omega}{\partial n}(P) < 0,$$

the normal being directed out of D . These inequalities prove the theorem.

If, in the definition of strip domain, the boundary curves are only required to be unbounded at both ends, but do not have to extend to $x = \pm\infty$, the inequality $\Omega \geq 0$ (and therefore Theorem 2) can still be proved without difficulty, but the argument must be modified.

In most of the special cases of interest, for example, if the flow is uniform at $x = \pm\infty$, or even if the flow speed is only bounded, the condition $0 \leq \psi \leq Q$ need not be assumed but can be proved to hold if

$$\psi = 0 \text{ on } \gamma, \quad \psi = Q \text{ on } \Gamma.$$

Uniqueness of the infinite cavity. We consider now the axisymmetric analogue of the classical Helmholtz flow with infinite cavity or wake. Let C be a fixed arc in the x, y plane with one endpoint on the axis of symmetry at the point $(a, 0)$, and let a steady axially symmetric flow be defined with the following properties:

- (a) C is a streamline arc;
- (b) a free streamline Γ , on which the flow speed is equal to the constant U , detaches from C at its endpoint and extends to infinity in the positive x direction (Fig. 3a);
- (c) the flow is everywhere regular inside the region bounded by the curve made up of the axis of symmetry ($x < a$) and the arcs C and Γ , and the flow approaches uniformly at infinity a velocity of magnitude U in the positive x direction;
- (d) the arc $C + \Gamma$ is starlike with respect to the origin, that is, every straight line from the origin intersects $C + \Gamma$ in at most a straight segment or a single point; also, Γ is convex downward as $x \rightarrow \infty$.

The first three conditions define an axially symmetric flow with infinite cavity past the profile C . The fourth condition is a limitation, for the sake of proof, on the class of admissible flows. If C is assumed starlike with $a \leq 0$, then (d) is implied by the physically important *cavitation condition*:

- (d') *The flow speed nowhere exceeds U .*

This condition reflects the physical requirement that the pressure in the fluid cannot decrease below a certain fixed value, the vapor pressure of the fluid, and hence by Bernoulli's law that the flow speed cannot exceed a corresponding maximum value. Actually we shall apply (d') only to a neighborhood of the free streamline. To see that (d') implies (d), assuming starlikeness of C , we observe that the free streamline Γ is convex downward everywhere as a consequence of (d') in conjunction with the irrotationality condition,

$$\frac{\partial q}{\partial n} = \kappa q,$$

where q = flow speed, n is the normal to Γ drawn into the flow, and κ is the curvature of Γ . (Indeed the free streamline must be strictly convex, that is, it contains no straight line segment, for otherwise the flow would contain a boundary arc on which the flow speed and direction are constant; the flow must then coincide with a uniform flow in the large.)

A flow satisfying conditions (a)–(d) or (a)–(d') will be called a *flow with infinite cavity* past the starlike obstacle C . With this definition we now prove,

Theorem 3. *An axially symmetric flow with infinite cavity past a starlike obstacle is unique.*

FIG. 3a

FIG. 3b

Proof. Let Γ and $\bar{\Gamma}$ be the free streamlines for two flows with infinite cavity past the starlike obstacle C (Fig. 3a), and designate by D and $\bar{D}$ the respective flow regions. Consider now Γ and $\bar{\Gamma}$ to be represented in the form $r = r(\theta)$ and $r = \bar{r}(\theta)$; since the two curves are starlike, such representations are possible. Then form the ratio $r(\theta)/\bar{r}(\theta)$. At the endpoint of C , where $\theta = \theta_0$, we have $r(\theta_0)/\bar{r}(\theta_0) = 1$. As $\theta \rightarrow 0$ and $r \rightarrow \infty$ on Γ , while $\bar{r} \rightarrow \infty$ on $\bar{\Gamma}$, either

(i) Γ lies above $\bar{\Gamma}$ at infinity, in which case

$$\liminf_{\theta \rightarrow 0} r(\theta)/\bar{r}(\theta) \geq 1;$$

(ii) Γ lies below $\bar{\Gamma}$ at infinity, in which case

$$\limsup_{\theta \rightarrow 0} r(\theta)/\bar{r}(\theta) \leq 1;$$

or

(iii) Γ and $\bar{\Gamma}$ intersect infinitely often at infinity, in which case $\lim_{\theta \rightarrow 0} r(\theta)/\bar{r}(\theta)$ exists and equals 1.

The proof of the latter assertion is based on Levinson's formula [1] for the asymptotic shape of the free streamlines in the axially symmetric cavity:⁷

$$y = \frac{Cx^{\frac{1}{2}}}{(\log x)^{\frac{1}{2}}} [1 + \epsilon(x)], \quad \epsilon(x) \rightarrow 0 \text{ as } x \rightarrow \infty.$$

Applying this formula to Γ and $\bar{\Gamma}$ we obtain

$$\left(\frac{r}{\bar{r}}\right)^{\frac{1}{2}} \left(\frac{\log r + \log \cos \theta}{\log \bar{r} + \log \cos \theta}\right)^{\frac{1}{2}} = 1 + \epsilon(\theta), \quad \epsilon(\theta) \rightarrow 0 \text{ as } \theta \rightarrow 0.$$

As $\theta \rightarrow 0$,

$$\lim_{\theta \rightarrow 0} \frac{r^{\frac{1}{2}}(\log r)^{\frac{1}{2}}}{\bar{r}^{\frac{1}{2}}(\log \bar{r})^{\frac{1}{2}}} = 1,$$

from which follows

$$\lim_{\theta \rightarrow 0} r(\theta)/\bar{r}(\theta) = 1.$$

Hence, either

$$(i) \liminf_{\theta \rightarrow 0} r(\theta)/\bar{r}(\theta) \geq 1, \text{ or } (ii) \limsup_{\theta \rightarrow 0} r(\theta)/\bar{r}(\theta) \leq 1.$$

Both these cases can be treated in identical fashion. In case (i), we have that the minimum of $r(\theta)/\bar{r}(\theta)$ for $\theta \geq \theta_0$ is some positive quantity $\rho \leq 1$, and that this minimum is achieved at some value of θ corresponding to points P and $\bar{P}$ on Γ and $\bar{\Gamma}$. Let ψ be the stream function corresponding to Γ and D . We construct a new flow with stream function

$$\Psi(x, y) = \frac{1}{\rho^2} \psi(\rho x, \rho y);$$

this flow is obtained from the former one by a similarity transformation of the flow region D in the ratio $1/\rho$ and has the property that the velocity at corresponding points is preserved. Thus the flow defined by Ψ is also a flow with infinite cavity with velocity U at infinity. If it is proved that $\Psi \equiv \bar{\psi}$, then $\rho = 1$ and of course $\bar{\psi} \equiv \psi$. Because of the starlike character of $C + \bar{\Gamma}$, the similarity transformation takes C into a similar curve lying in the closure of $\bar{D}$. The free streamline Γ is transformed into a free streamline which, from the particular

⁷ Proof of this formula assumes convexity of the free streamline at infinity, as well as a regularity condition not explicitly stated in our assumptions.

choice of ratio $1/\rho$, has a point of contact in common with $\bar{\Gamma}$ at $\bar{P}$, but otherwise does not cross $C + \bar{\Gamma}$ (Fig. 3b). Thus the image of D under the similarity transformation is a flow region contained in $\bar{D}$ and having a common regular boundary point with $\bar{D}$ at $\bar{P}$. Therefore Theorem 1 applies, and either the two flow speeds at $\bar{P}$ are unequal, or the flows are identical. The former possibility is excluded by the fact that the flow speeds are the same on the two free streamlines. In case (ii) the proof follows as above, the roles of barred and unbarred quantities being reversed. Thus the theorem is proved.⁸

We note that the preceding proof rests in an apparently essential way on Levinson's deep result on the asymptotic shape of the axisymmetric cavity. Since this was required only for the proof that $\lim_{\theta \rightarrow 0} r(\theta)/\bar{r}(\theta) = 1$ in case the free streamlines Γ and $\bar{\Gamma}$ intersect infinitely often at infinity, one might hope to dispense with the asymptotic formula and to prove the existence of the limit directly from convexity or other simple regularity conditions on the free streamlines. However, it is possible to show by counterexamples, for which I am indebted to W. Gustin, that two convex curves lying everywhere above the x -axis and possessing horizontal tangents at infinity can intersect infinitely often, and yet satisfy the simultaneous inequalities

$$\liminf_{\theta \rightarrow 0} r(\theta)/\bar{r}(\theta) = a < 1, \quad \limsup_{\theta \rightarrow 0} r(\theta)/\bar{r}(\theta) = b > 1,$$

where a and b may be arbitrary positive numbers. A proof not requiring the asymptotic shape of the free streamlines is much to be desired, but is lacking even in the simpler plane case.

The preceding ideas can be exploited to achieve further qualitative results on axisymmetric cavities. In the plane, the analogues of these theorems and of Theorem 3 can be proved in sharper form and under weaker restrictions than is now possible in the axially symmetric case (Serrin [4, 5], Lavrentieff [3]). The basis for these stronger results are certain theorems connecting geometric properties of the given obstacle with those of the free streamline⁹, and the latter theorems in turn depend strongly on properties of the hodograph plane that are not so well understood in the case of axial symmetry.

Uniqueness of the flow from an orifice. As a second type of free boundary problem, we consider the flow out of an infinite axially symmetric channel or vessel. Theorem 2 plays the same role here in the proof of uniqueness as Theorem 1 in the cavity problem. A *flow from an orifice* will be a flow in a strip domain having as one of its two bounding streamlines the axis of symmetry, the other boundary curve consisting of a fixed semi-infinite arc which is the channel or vessel wall, and of a free streamline that detaches from the end of the channel at the orifice

⁸ The possibility of using starlike boundaries and similarity transformations in these problems was observed independently by both SERRIN and myself.

⁹ For example, starlikeness of the obstacle implies the same property for the free streamline (SERRIN [5], Theorem 20.1).

(Fig. 4a). It is further assumed that as $x \rightarrow +\infty$ the flow becomes uniform and the free streamline approaches a horizontal asymptote, and also that the stream function grows from zero on the axis of symmetry to the positive value Q on the outer streamline, Q being the flux through the orifice. Finally, it is supposed that the channel wall and free streamline together form a curve that is starlike with respect to some point on the axis of symmetry, say the origin. Then we have

Theorem 4. *There can be at most one axially symmetric flow from an orifice with prescribed flux.*

FIG. 4a

FIG. 4b

Proof. Let Γ and $\bar{\Gamma}$ be the free streamlines belonging to two jet flows from the channel C and detaching from C at its endpoint A . We designate by ψ and $\bar{\psi}$ the stream functions for the two flows, such that $\psi = \bar{\psi} = 0$ for $y = 0$, while $\psi = Q$ on Γ , $\bar{\psi} = Q$ on $\bar{\Gamma}$. It is required to show that $\psi \equiv \bar{\psi}$ and $\Gamma = \bar{\Gamma}$. Let Γ and $\bar{\Gamma}$, since they are starlike, be represented as $r = r(\theta)$ and $r = \bar{r}(\theta)$, and be asymptotic to the lines $y = h$ and $y = \bar{h}$ (Fig. 4a). Consider the three cases

$$(i) \ h > \bar{h}, \quad (ii) \ h < \bar{h}, \quad (iii) \ h = \bar{h}.$$

Since at A we have $r(\theta) = \bar{r}(\theta)$, it follows that in all three cases

$$\min_{\theta} r(\theta)/\bar{r}(\theta) = \rho \leq 1.$$

Take case (i). Then the above minimum is achieved at some non-zero value of θ . Hence a similarity transformation from the origin of ratio $1/\rho$ transforms

the flow region D belonging to the flow ψ into a region D' containing $\bar{D}$ in its interior, and such that the image Γ' of Γ has a point of contact with Γ (Fig. 4b). In the regions D' and $\bar{D}$ the flows defined by the stream functions $\psi'(x, y) \equiv \psi(\rho x, \rho y)$ and $\bar{\psi}(x, y)$ satisfy the conditions of Theorem 2, from which we conclude that either the flow speed on Γ' exceeds that on $\bar{\Gamma}$ or the flows are identical. The former is of course impossible, since for the same flux the flow speed on the wider jet must be smaller than that on the narrower jet.

Case (ii) can be treated in the same way, the roles of the barred and unbarred quantities being reversed.

In case (iii) we observe that the free streamlines can be represented in the form,

$$\Gamma: r \sin \theta = h[1 + \epsilon_1(\theta)],$$

$$\bar{\Gamma}: \bar{r} \sin \theta = h[1 + \epsilon_2(\theta)],$$

where $\epsilon_1(\theta), \epsilon_2(\theta) \rightarrow 0$ as $\theta \rightarrow 0$. From this it follows that $\lim_{\theta \rightarrow 0} r(\theta)/\bar{r}(\theta) = 1$, and hence the minimum $\rho \leq 1$ of $r(\theta)/\bar{r}(\theta)$ is achieved at some non-zero value of θ . The argument now proceeds as in case (i): if $\rho < 1$, then the same contradiction prevails; while if $\rho = 1$, then the flow speeds on the two jets Γ' and $\bar{\Gamma}$ are the same, and application of Theorem 2 to the two stream functions ψ' and $\bar{\psi}$ again yields the result that the two flows are identical. The theorem is thus proved.

We remark that if the flux is not prescribed, then the velocity fields of two such flows differ only by a constant factor, and therefore the geometry of the flows is the same. In particular, the shape of the free streamline is unique. A similar remark could be made about the cavity flow, namely, that the flow pattern and the shape of the free streamline are unique, independently of the velocity at infinity.

Further extensions from the plane to axially symmetric case can undoubtedly be carried through and should provide some qualitative results, as well as methods, of considerable interest.

REFERENCES

- (1) N. LEVINSON, *On the asymptotic shape of the cavity behind an axially symmetric nose moving through an ideal fluid*, *Annals of Mathematics* **47**, 704-730 (1946).
- (2) K. FRIEDRICHS, *Über ein Minimumproblem für Potentialströmungen mit freiem Rande*, *Mathematische Annalen* **109**, 60-82 (1933-34).
- (3) M. LAVRENTIEFF, *On certain properties of univalent functions and their application to wake theory* (in Russian), *Mat. Sbornik* **46**, 391-458 (1938).
- (4) J. SERRIN, *Uniqueness theorems for two free boundary problems*, to appear in the *American Journal of Mathematics*.
- (5) J. SERRIN, *Existence theorems for some hydrodynamical free boundary problems*, *Journal of Rational Mechanics and Analysis* **1**, 1-48 (1952).