Carathéodory's Temperature Equations

G. WHAPLES

Indiana University, Bloomington, Indiana

Communicated by C. TRUESDELL

Introduction. In his first paper [2] on axiomatization of thermodynamics, C. Carathéodory stated that if a system of equations

$$\begin{aligned} &F_{1,2}(x_1, x_2) = 0 \\ (1) \quad &F_{1,3}(x_1, x_3) = 0 \\ &F_{2,3}(x_2, x_3) = 0 \end{aligned}$$

have the property that any two of them imply the other, then there exist functions f_i such that the graph of the system (1)—*i.e.* the set of all points (x_1, x_2, x_3) whose coördinates satisfy it—is the same as the graph of the system

$$(2) \quad f_1(x_1) = f_2(x_2) = f_3(x_3).$$

In a later paper [3] he used a similar system which will be described soon. Various commentaries by physicists, [1], [4], [5], also use one or the other of these systems, but neither the original papers nor the commentaries contain a proof of this statement.

In these applications each x_i stands for a set of m_i real variables which describe the state of a thermodynamic system S_i ,—in the simplest cases, just the volume and pressure of a gas. The equations $F_{i,j}(x_i, x_j) = 0$ are the conditions that the systems S_i and S_j be in equilibrium, and the condition on the system (1) corresponds to the physical axiom: *if two systems are both in equilibrium with a third, then they are in equilibrium with each other.* The f_i are required to be real valued differentiable functions and $f_i(x_i)$ is, under a preliminary definition which is further restricted later, called the *temperature*.

Clearly systems (1) can occur in any situation—osmosis, for example—which

involves an equilibrium relation satisfying this axiom. We show here (§2) that these systems have a generalization to the case of $n = 2, 3, 4, \dots$ variables x_i , in which the equations $F_{i,j}(x_i, x_j) = 0$ are replaced by binary logical relations, and prove that Carathéodory's statement is true for all such systems if the f_i are certain abstract functions which have as values subsets of $(x_1, x_2, \dots)$ -space. One must then ask whether these abstract f_i can be replaced by the kind of numerical functions required by thermodynamics. We give examples (§3) which show that this is not always the case, even for $F_{i,j}$ of a quite simple form. This proves that assuming an equilibrium relation satisfying the above axiom is not enough to prove existence of a numerical temperature, and that further numerical assumptions about the equilibrium relation are required. We prove (§3) a theorem which provides a physically usable form for such assumptions and (§4) briefly summarize its application to thermodynamics.

1. R -systems. Let $n \geq 2$, let $X_1, X_2 \dots X_n$ be any sets, and suppose that for each $i, j = 1 \dots n$ with $i \neq j$ there is given a binary logical relation $R_{i,j}$ such that $R_{i,j}(x_i, x_j)$ is either true or false for each $x_i \in X_i$ and $x_j \in X_j$, and¹ $R_{i,j}(x_i, x_j) \leftrightarrow R_{j,i}(x_j, x_i)$; so that there are really only $n(n-1)/2$ relations. Let P be the set of all ordered n -tuples $(x_1, x_2 \dots x_n)$, with $x_i \in X_i$, call its elements *points*, call x_i the *i -coördinate* of $(x_1, x_2, \dots x_n)$, and define the graph G of the system of relations $(R_{i,j})$ to be the set of all $(x_1, x_2, \dots x_n) \in P$ such that $R_{i,j}(x_i, x_j)$ is true for all $i \neq j$, $i, j = 1 \dots n$. From now on we shall use $\{x_i, x_j\}$ as abbreviation for $R_{i,j}(x_i, x_j)$, retaining $R_{i,j}$ to denote the relation. This omission of the function symbol could lead to ambiguity if the X_i were not pairwise disjoint. It is easy to see that the assumption of this disjointness is no real restriction of generality and that it justifies our abbreviation; so we shall make it throughout this section.

Such a system of relations shall be called an *R -system of order n* , if it satisfies either of the conditions

$$\text{A. } n \geq 2 \text{ and for all } i \neq j \neq k \neq l \neq i \text{ and all } x_i \in X_i, x_j \in X_j, x_k' \in X_k, x_l' \in X_l, \\ (3) \quad \{x_i, x_j\} \ \& \ \{x_j, x_k'\} \ \& \ \{x_i, x_l'\} \rightarrow \{x_k', x_l'\},$$

or:

$$\text{B. } n \geq 3 \text{ and, for all } i \neq j \neq k \neq i \text{ and all } x_i \in X_i, x_j \in X_j, x_k \in X_k, \\ (4) \quad \{x_i, x_j\} \ \& \ \{x_j, x_k\} \rightarrow \{x_i, x_k\}.$$

For $n = 2$ the only possibilities for i, j, k, l are 1, 2, 1, 2 or 2, 1, 2, 1, and both conditions are equivalent, so we see that an *R -system of order 2* is defined by

$$\text{B}_2. \text{ For all } x_1, x_1' \in X_1, x_2, x_2' \in X_2, \\ (5) \quad \{x_1, x_2\} \ \& \ \{x_2, x_1'\} \ \& \ \{x_1, x_2'\} \rightarrow \{x_1', x_2'\}.$$

¹ We shall use "&" for "and", "→" for "implies", and "↔" for "is equivalent to" and an expression " $A \ \& \ B \ \& \ C \rightarrow D$ " shall be interpreted as " $(A \ \& \ B \ \& \ C) \rightarrow D$ ".

When P is the euclidean plane, this has the meaning: if G contains three vertices of any rectangle with sides parallel to the coördinate axes, then G also contains the fourth vertex. The name R -system was suggested by this property. Condition (5) is exactly that used by Carathéodory, in [3], to define the other system mentioned in the introduction, which we now see to be an R -system of order 2.

It is also easy to see that in system (1) the condition that each two of the equations imply the other is just our condition 3. For $n \geq 3$ 3 gives the more practical definition of an R -system; but condition $\mathcal{R}$ is also interesting because it is of the same form for all n . Neither of $\mathcal{R}$ or 3 implies the other; but as we shall see, both conditions have the same meaning with respect to the graph G .

Now suppose we have any R -system; let G be its graph. Call two points p and q of G *immediately R -connected* if their i -coördinates are the same for at least one i , and *R -connected* if there exists a finite sequence of points of G , with p as its first and q as its last point and with each two adjacent points immediately R -connected.

Lemma 1. *If $(x_1^1, x_2^1 \cdots x_n^1)$ and $(x_1^2, x_2^2 \cdots x_n^2)$ are immediately R -connected, then all the points $(x_1^{\alpha_1}, x_2^{\alpha_2} \cdots x_n^{\alpha_n})$ are in G , for all $\alpha_i = 1, 2$ (independently of each other).*

Proof: If $n = 2$ there is nothing to prove. To clarify notation suppose $x_1^1 = x_1^2$. Then $(x_1^1, x_2^1 \cdots x_n^1) \in G$ and $(x_1^1, x_2^2 \cdots x_n^2) \in G$. Hence $\{x_1^1, x_1^\alpha\}$, $\{x_i^\alpha, x_j^\alpha\}$ are true for all $i \neq j$, $\alpha = 1, 2$.

If our system satisfies 3, then for all $i \neq j \neq 1 \neq i$ and $\alpha, \beta = 1, 2$,

$$\{x_i^\alpha, x_1^1\} \ \& \ \{x_1^1, x_j^\beta\} \rightarrow \{x_i^\alpha, x_j^\beta\}.$$

The relations on the left are always true; from those on the right, the relations already mentioned, and definition of G , the lemma follows.

If it satisfies $\mathcal{R}$, then for all $1 \neq i \neq j \neq 1$, and all $\alpha, \beta = 1, 2$,

$$\{x_1^1, x_i^\alpha\} \ \& \ \{x_i^\alpha, x_j^\alpha\} \ \& \ \{x_1^1, x_i^\beta\} \rightarrow \{x_j^\alpha, x_i^\beta\}.$$

Lemma follows as before.

Lemma 2. *If $(x_1^1, x_2^1 \cdots x_n^1)$ and $(x_1^2 \cdots x_n^2)$ are connected, then all the points $(x_1^{\alpha_1}, x_2^{\alpha_2}, \cdots x_n^{\alpha_n})$ with $\alpha_i = 1, 2$ independently for $i = 1 \cdots n$, are in G .*

Proof: Use induction on the number of points in the chain. Assume to clarify notation that $r = (x_1^2, x_2^3, x_3^3 \cdots x_n^3)$ is the next to last element of the chain. By induction assumption, $(x_1^2, x_2^1, x_3^1 \cdots x_n^1) \in G$. If $n \geq 3$, then Lemma 1, together with $(x_1^2, x_2^2, x_3^2 \cdots x_n^2) \in G$, gives $(x_1^2, x_2^{\alpha_2} \cdots x_n^{\alpha_n}) \in G$ for all $\alpha_i = 1, 2$. Applying Lemma 1 again gives $(x_1^1, x_2^{\alpha_2}, \cdots x_n^{\alpha_n}) \in G$ whenever at least one of the $\alpha_i = 1$; finally, if $n > 2$, $(x_1^1, x_2^2, x_3^1, x_4^2 \cdots x_n^2) \in G$ gives

$$(x_1^1, x_2^2, x_3^2 \cdots x_n^2) \in G.$$

If $n = 2$ we must go back to (5), which gives directly $(x_1^1, x_2^2) \in G$.

This relation of R -connectedness is reflexive, symmetric and transitive; hence

it splits G into equivalence classes, which it is convenient to call R -components. Lemma 2 shows that for each i , and every $x_i \in X_i$, all the points with x_i as their i -coördinate lie in the same R -component of G .

So we can, for each i , define a function f_i as follows: $f_i(x_i)$ = the component which contains all points of G with x_i as their i -coördinate, if there are any such points; $f_i(x_i)$ undefined if there are none. If we then count an equation $f(x) = g(y)$ as false whenever either side is undefined, we have

Theorem 1. *The graph of every R -system is also the graph of the equations*

$$(6) \quad f_1(x_1) = f_2(x_2) \cdots = f_n(x_n).$$

Proof: Every point on G obviously satisfies the equations. Suppose that (6) is satisfied by $(x_1, x_2 \cdots x_n)$. Then G contains points $(x_1, x_2^1 \cdots x_n^1)$, $(x_2^2, x_2, x_3^2 \cdots x_n^2) \cdots (x_1^n, x_2^n \cdots x_n)$, and all these points are connected. Applying Lemma 2 repeatedly we see that $(x_1, x_2 \cdots x_n) \in G$.

2. Relation between $\mathfrak{R}$ and $\mathfrak{J}$. Call a system of relations *normalized* if it has the property: *for all $x_i \in X_i$, $x_j \in X_j$, the graph contains a point with x_i and x_j as its i, j -coördinates whenever $\{x_i, x_j\}$ is true.* Since $\{x_i, x_j\}$ must be true whenever G contains such a point, we see that for a normalized set of relations, $\{x_i, x_j\}$ is just the proposition, " x_i and x_j are the i, j -coördinates of at least one point on the graph."

Now suppose a subset of P is the graph of any system of equations of form (6). Without changing the graph we can assume the functions $f_i(x_i)$ are undefined unless x_i is the i -coördinate of some point of G ; then the relations $R_{i,j}$ defined by

$$R_{i,j}(x_i, x_j) \leftrightarrow f_i(x_i) = f_j(x_j)$$

are a normalized set of relations, and a trivial check shows that they satisfy both $\mathfrak{J}$ and $\mathfrak{R}$.

This proves:

Theorem 2. *Any graph of a system (6) is the graph of a normalized R -system.*

Corollary: *Any normalized system of relations with $n \geq 3$ satisfies $\mathfrak{J}$ if and only if it satisfies $\mathfrak{R}$.*

The assumption that our set of equations is a normalized system is highly reasonable from the point of view either of physics or of implicit function theory.

To prove that for non-normalized systems neither $\mathfrak{R}$ nor $\mathfrak{J}$ implies the other, let $n = 3$, let X_i be non-null, let $R_{1,2}$ be a binary relation for which (5) does not hold, and let $R_{1,3}$ and $R_{2,3}$ be identically false. Then $\mathfrak{J}$ is true but $\mathfrak{R}$ false.

To get an example with $\mathfrak{R}$ true and $\mathfrak{J}$ false, take $n = 3$ again, and note first that the only cases of $\mathfrak{R}$ in which i, j, k, l contain among them exactly three indices p, q, r are those with i, j, k, l of form

$$p, q, p, r \quad \text{or} \quad p, q, r, q,$$

and that for both of these the pair of indices which appears on the right side also appears on the left. Let X_i be non-null, $R_{1,2}$ identically true, $R_{2,3}$ identically true, $R_{1,3}$ identically false. Then $\mathfrak{J}$ is false, but $\mathfrak{R}$ is true. For one sees easily that whenever the right side of (3) is false, the left side is also.

This may have the slight physical interest of emphasizing the unreasonableness of non-normalized systems.

3. Analytic part. Now suppose that the sets X_i are m -dimensional real euclidean spaces, eventually with the usual topology. The axiom of choice sets up a correspondence, necessarily one-one, which maps each R -component of G into one of its points; hence by cardinal number theory, the set of components is in one-one correspondence with some subset of the real numbers, so Theorem 1 is true also for certain functions f_i on X_i to the real numbers.

However, this approach lacks interest because it gives no control over continuity. Before attacking this question we shall show by examples that everywhere continuous f_i cannot be expected even when the $R_{i,j}$ are defined by equations involving extremely simple functions.

Example 1. $m = 1, n = 2$, system consisting of the equation $\sin(\pi(x_1 - x_2)) = 0$. G is the set of all straight lines $x_1 - x_2 = \text{integer}$. $f_1(x_1) = f_2(x_2) \leftrightarrow x_1 \equiv x_2 \pmod{1}$; so $f_1(x_1) = f_1(x_1') \leftrightarrow x_1 \equiv x_1' \pmod{1}$. Hence for any f_1 satisfying Theorem 1, we would have $f_1(0) = f_1(1)$ and $f_1(\xi) \neq f_1(0)$ for any ξ between 0 and 1; but if f_1 were a continuous function on the reals to the reals we would have for any such ξ and any c between $f_1(0)$ and $f_1(\xi)$ two numbers α and β with

$$f_1(\alpha) = f_1(\beta) = c$$

and $0 < \alpha < \xi < \beta < 1$, contradicting fact that $f_1(\alpha) = f_1(\beta) \leftrightarrow \alpha \equiv \beta \pmod{1}$.

Example 2. $m = 2, n = 2$, $X_1 = \text{set of } x_1 = [\xi_1, \eta_1]$, $X_2 = \text{set of } x_2 = [\xi_2, \eta_2]$, ξ_i, η_i real, system consisting of the equation, in complex numbers,

$$\xi_1 + \eta_1 i = \xi_2 + \eta_2 i,$$

which could also be written $(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 = 0$. In terms of complex numbers this is already in the form $f_1(x_1) = f_2(x_2)$; if we could replace these f_i by continuous functions g_i on X_i to the reals then we could find a continuous one-one map of the real plane into the real line—which is impossible because a circle would have to map into an interval (parametric representation and standard calculus) hence two intersecting nontangent circles into two intervals with exactly two points in common.

The first example shows that we cannot hope to do more than find a local parametrization of the components by functions f_i continuous in some small enough neighborhood of a point of G . The second example, which applies as well to any open region of the plane as to the whole plane, shows that we can't do this without some assumptions on the $R_{i,j}$ which make the dimension behave

“properly”. The most convenient and familiar forms of such assumptions involve implicit functions.

The word “local” has a very simple interpretation for R -systems. Let G be the graph of an R -system with X_i and P as described at the beginning of §1. Let $X_1^*, \dots, X_n^*$ be any subsets of the X_i and P^* the set of n -tuplets

$$(x_1, x_2, \dots, x_n)$$

with the x_i restricted to X_i^* , and let $G^* = G \cap P^*$. Then G^* is also an R -system and indeed two points of G^* are R -connected in G^* if and only if they are R -connected in G ; this is an easy consequence of Lemma 2. The components of G^* are the intersections of the components of G with P^* . Thus local theorems about R -systems can be stated as theorems about little R -systems.

Theorem 3. *Assume that an R -system satisfies the following conditions:*

1. *It is normalized, and for each i , every $x_i \in X_i$ is the i -coördinate of some point of G .*
2. *The X_i are subsets of finite dimensional euclidean spaces,² X_1 being a set of points³ $[\xi_1, \xi_2, \dots, \xi_m]$, ξ_μ real, $m \geq 1$.*
3. *There exists a set (empty, if $m = 1$) $\alpha_2, \dots, \alpha_m$ of real constants α_μ such that*
 - a. *For every $x_2 \in X_2$ there is a unique real number $\lambda^{(2)}$ such that $\{x_1, x_2\}$ is true with an $x_1 = [\lambda^{(2)}, \alpha_2, \dots, \alpha_m]$.*
 - b. *For each $j = 3 \dots n$ and every $x_j \in X_j$ there is at least one real number $\lambda^{(j)}$ such that $\{x_1, x_j\}$ is true with an $x_1 = [\lambda^{(j)}, \alpha_2, \dots, \alpha_m]$.*

Then the real numbers $\lambda^{(j)}$ of 3b are in fact uniquely determined for each $j = 3 \dots n$; also, for each $x_1 \in X_1$ there is a unique real number $\lambda^{(1)}$ such that $[\lambda^{(1)}, \alpha_2, \dots, \alpha_m]$ and x_1 are 1-coördinates of points in the same R -component of G . And if, for each $i = 1 \dots n$, we define $f_i(x_i)$ to be the real number $\lambda^{(i)}$ as just described, then G is the graph of the system

$$f_1(x_1) = f_2(x_2) = \dots = f_n(x_n).$$

The assumptions of this theorem are reasonable both from the viewpoint of physical reality and that of implicit function theory; and the latter theory can now be applied to settle questions about continuity and differentiability.

Proof of Theorem 3: Let $(x_1, x_2, \dots, x_n)$ be any point in G . For each $i = 1 \dots n$ let $\lambda^{(i)}$ be any real number satisfying 3a, 3b, or the above condition for $\lambda^{(i)}$, and define $x_1^{(i)}$ to be $[\lambda^{(i)}, \alpha_2, \dots, \alpha_m]$. Then for $j = 2 \dots n$, G contains a point with 1-coördinate $x_1^{(j)}$ and j -coördinate x_j ; hence, by Lemma 2,

$$(7) \quad (x_1^{(j)}, x_2, \dots, x_n) \in G.$$

² Following a suggestion of M. Zorn, we observe that Theorem 3 could easily be altered so that it, too, would be a theorem of pure set theory.

³ We don't need symbols for the “subcoördinates” of the elements of any X_i except X_1 .

And since $x_1^{(1)}$ is the 1-coördinate of some point in the same component as our originally given point, Lemma 2 shows that (7) holds also for $j = 1$. But (7), together with 3a, shows that all the $\lambda^{(i)}$ must be equal to $\lambda^{(2)}$. Thus all the statements about uniqueness in the theorem, as well as the equality of the $f_i(x_i)$, are verified.

Suppose on the other hand that the $f_i(x_i)$ are all equal. Let α be their common value and let $a_1 = [\alpha, \alpha_2, \dots, \alpha_m]$. Then $\{a_1, x_i\}$ is true for all $i \neq 1$. Then by definition of $G(n = 2)$, or by 3, which is allowed by our assumption of normality and the corollary in §2, when $n \geq 3$, we see that $(a_1, x_2, \dots, x_n) \in G$. By definition of f_1 , and methods now familiar, $(x_1, x_2, \dots, x_n) \in G$.

4. Application. The existence of a numerical temperature now follows from Theorem 3 if one takes the equilibrium conditions to be equations $F_{i,j}(x_i, x_j) = 0$ and makes assumptions of a sort very common in physics. But one can avoid all mention of the form in which the relations $R_{i,j}$ are given if one is willing to assume that for a certain fixed set of values of all but one of the variables describing the state of the first system there is for each state of any of the other systems a unique value of the remaining variable for which the two systems are in equilibrium, and that this variable is a continuous and sufficiently differentiable function of the others. Physically, this is very clear and plausible: in the case that the systems are gases and the variables their volume and pressure, the variable singled out could be the volume of the first gas at a fixed pressure, namely exactly the reading of a gas thermometer.

BIBLIOGRAPHY

- [1] M. BORN, *Kritische Betrachtungen zur traditionellen Darstellung der Thermodynamik*, Phys. Zeitsch. **22** (1921) pp. 218–224, 249–254, and 282–286.
- [2] C. CARATHÉODORY, *Untersuchungen über die Grundlagen der Thermodynamik*, Math. Ann. **67** (1909) pp. 355–386.
- [3] C. CARATHÉODORY, *Über die Bestimmung der Energie und der absoluten Temperatur mit Hilfe von reversiblen Prozessen*, Berl. Ber. 1925 pp. 39–47.
- [4] T. EHRENFEST-AFANASSJEW, *Zur Axiomatisierung des zweiten Hauptsatzes der Thermodynamik*, Zeitsch. für Phys. **33** (1925) pp. 933–945.
- [5] A. LANDÉ, *Axiomatische Begründung der Thermodynamik durch Carathéodory*, Handbuch der Phys., Berlin 1926, vol. 9 pp. 281–300.