Statistical Hydromechanics and Functional Calculus

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1. Introduction. The differential law which governs the motion of a deterministic mechanical system has the symbolic form

$$\frac{du}{dt} = \mathfrak{F}(u)$$

where u is an instantaneous phase of the system and where the right hand side is completely determined by the phase. We consider only the case in which $\mathfrak{F}$ does not contain t explicitly. In general the law completely determines the future history u' of the system if its initial phase u^0 is exactly known at some moment of time t^0 . There are mechanical systems the phases of which are characterized by a very large number of independent parameters and the phase motions of which are tremendously complicated. Two examples are the classical model of a gas with its very large number of degrees of freedom and the flow of a viscous incompressible fluid at a very large value of the overall Reynolds number. In both cases the important task is not the determination of the exact phase motion with an exactly given initial phase but the determination of the statistical properties of the "typical" phase motion. In order to achieve this goal statistical mechanics studies probability distributions of simultaneous phases and their evolution in time resulting from the individual phase motions. It is particularly interested in stationary phase distributions because only these can describe statistical equilibrium of the system. But not all of these stationary distributions are of importance. Statistical mechanics constructs certain "relevant" phase distributions which characterize the "typical" phase motions and which must be used for all statistical predictions. Classical statistics of a gas has reached this goal to a large extent. The relevant distributions are the canonical ones if one confines oneself to a scale in gas space on which the interactions of the

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molecules are negligible, and it is just this scale on which the important macroscopic properties of a gas are based. On the other hand, in statistical hydro-mechanics—the theory of highly turbulent fluid flow—the small scale on which the “fluid elements” interact seems to be of decisive importance.* The relevant phase distributions based on this scale—the hypothetical Kolmogoroff distributions—must be mathematically very different from canonical distributions. So far all attempts to determine the relevant hydrodynamical phase distributions, at least to a sufficient degree of approximation, have met with considerable mathematical difficulties.† In 1940 I made a hitherto unpublished attempt to overcome these difficulties or, at least, to formulate the problem mathematically by introducing the characteristic functional or Fourier transform Φ of an arbitrary hydrodynamical phase distribution. I derived from the Navier-Stokes equations a functional differential equation which expresses $\partial\Phi/\partial t$ directly in terms of the first and second functional derivatives of Φ in the sense of Volterra. The correlations of simultaneous velocities at n space points are n -th functional derivatives of Φ . Formally, the Φ -equation is completely equivalent to the infinite system of partial differential equations for these correlation functions, but it describes the differential evolution of a phase distribution in a more concise way. Of course, the mathematical level of the problem is raised into the domain of functional analysis, but this is natural and even unavoidable if one views the problem *in toto*. My attempts at finding the “relevant” solutions of the Φ -equation have been unsuccessful even in the simplest case of boundary-free flow, but I believe that the mathematical difficulties of the problem arise from the fact that it is unprecedented and not from any intrinsic complexity. I publish the mere beginnings of the theory with some hesitation but also with the hope that they will contribute to the ultimate solution of the problem. In order to deal with correlations of velocities at different times one needs a simple extension of the theory. This and other matters will be taken up in a second paper.

2. The phase space of a flow problem. We consider an incompressible fluid of constant density $\rho = 1$ and of viscosity μ which moves between given material walls. The boundary condition is that the fluid adheres to the walls or, in mathematical terms, that the velocity vector

$$u = (u_1, u_2, u_3)$$

* The analogy between fluid elements and molecules is a merely formal one. It is known that in many applications the size of the smallest eddies for highly turbulent flow is still very large in comparison to the molecular scale. In other words, for most purposes of statistical hydromechanics a fluid may be regarded as a strictly continuous medium. Statistical hydrodynamics is the theory of the “typical” solutions of the Navier-Stokes equations.

† We refer the reader to the following monograph in which the important recent advances in the statistical theory of turbulence—the theories of Kolmogoroff, Onsager, von Weizsäcker, Heisenberg, Batchelor and others—are described. *Les théories de la turbulence*, par L. Agostini et J. Bass. Publ. scientif. et techn. du ministère de l'air. Paris 1950.

of the fluid coincides on a wall with the velocity of that wall. In order to allow for strictly stationary or statistically stationary flows we assume that the relations which express the boundary conditions do not explicitly contain the time. This means that the boundary configuration remains the same geometrical whole at all times although the bounding walls may move, and that the velocity vector is at all times the same in every geometrical boundary point. Stationariness of the boundary data restricts, of course, the shape of the walls and also their type of motion if they move rigidly. At any rate, we find it convenient to confine ourselves to the following type of boundary condition. Time-independent values of u are prescribed on the boundary* of a region which does not change with time. We speak of a definite flow problem if its boundary data are definitely fixed. Such a definite flow problem has an infinity of possible solutions, and a solution will not be determined unless initial data are prescribed at a given moment of time.

Let R denote the region of x -space, $x = (x_1, x_2, x_3)$, which is occupied by the fluid at all times and let B denote its boundary. Any sufficiently smooth velocity field $u = u(x)$ defined in $R + B$ may be considered as a possible instantaneous state or phrase of the fluid provided that it is solenoidal in R ,

$$(1) \quad \operatorname{div} u \equiv u_{\alpha,\alpha} = 0,$$

and that it assumes the boundary values given on B . The basic equations of flow are the Navier-Stokes equations

$$(2) \quad u_{\alpha,t} + u_{\beta}u_{\alpha,\beta} = -p_{,\alpha} + \mu u_{\alpha,\beta\beta} \quad (\alpha = 1, 2, 3)$$

together with (1). The notation

$$f_{\alpha,\beta} = \frac{\partial f_{\alpha}}{\partial x_{\beta}}, \quad f_{\alpha,\beta\gamma} = \frac{\partial^2 f_{\alpha}}{\partial x_{\beta}\partial x_{\gamma}}, \quad f_{\alpha,t} = \frac{\partial f_{\alpha}}{\partial t},$$

is employed and also the usual summation convention is made use of. The differential system (2) + (1) if taken in conjunction with the boundary conditions (u given on B) uniquely determines the rate of change of a phase

$$\frac{\partial u(x, t)}{\partial t} = u_{,t} = (u_{1,t}, u_{2,t}, u_{3,t})$$

in terms of the instantaneous phase $u = u(x, t)$ at the same moment of time t . The idea is that one can write (2) in the form

$$u_{\alpha,t} + p_{,\alpha} = -u_{\beta}u_{\alpha,\beta} + \mu u_{\alpha,\beta\beta}$$

or vectorially,

$$u_{,t} + \operatorname{grad} p = -u_{\beta}u_{,\beta} + \mu u_{,\beta\beta},$$

* Of course, boundary values must be compatible with the condition of incompressibility.

where the right hand vector field is completely determined by the instantaneous phase u , and the problem is to represent a given vector field in R as a sum of a gradient field and a solenoidal field which assumes given values on B (we must have $\operatorname{div} (u, \iota) = (\operatorname{div} u), \iota = 0$). This decomposition is well known to be unique.* In other words, the laws of flow plus the boundary conditions uniquely determine the pressure distribution as well as the rate of change of the phase in terms of the instantaneous phase $u = u(x, t)$. Thus they furnish the differential description of the time-succession of phases $u^t = u(x, t)$ in $R + B$. We presume that they determine this succession of phases uniquely† for all $t > 0$ once the initial phase $u^0 = u(x, 0)$ is given.

The possible phases $u = u(x)$ in $R + B$ may be regarded as points of a function space, the u -space or phase space associated with the flow problem. We denote it by Ω . The flow of the fluid in R carries a phase u , given at $t = 0$, into another well determined phase u^t at time t . This transition from a given phase to another one may be regarded as a functional operation T^t which is defined and single-valued in Ω ,

$$u^t = T^t u, \quad u^t = u(x, t), \quad u = u(x) = u(x, 0).$$

For u fixed and t variable, $t \geq 0$, T^t describes the flow of the fluid in R which starts from the initial velocity field $u = u(x)$. For t fixed and u variable, u in Ω , T^t defines a transformation of the phase space into itself or part of itself. Under our assumption that the boundary data of the flow problem be stationary the operation which carries a phase at time t_0 into another one at time $t_0 + t$ is independent of t_0 and, therefore, identical with T^t . Hence the group T^t has the property that

$$T^t T^s = T^{t+s}, \quad T^0 = \text{identity}.$$

Needless to say, for a definite flow problem the phase motions depend also upon the parameter μ , the viscosity of the fluid.

3. Phase distributions, averages and characteristic functionals. An instantaneous phase distribution (or ensemble) is completely characterized by its

* Actually the decomposition is already determined if only the normal component of the solenoidal field is given on B . This apparent overdeterminacy is but an instance of a phenomenon generally observed in mixed boundary and initial value problems for parabolic differential systems. The solution of this problem does exist in the hydrodynamical case for all times $t > 0$ and it does satisfy those apparently superfluous boundary conditions. Other methods have been used to prove the existence. See, for instance, my paper: *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachrichten **4** (1950), p. 213–231.

† Uniqueness has not yet been proved. The customary methods furnish this proof in the case of plane flow but, curiously enough, they fail to do so in the general three-dimensional case even with viscosity present. I find it hard to see something serious behind these mathematical difficulties. Moreover, it seems to me that the final conclusions of this paper—the Φ -equation—are not affected if uniqueness fails to hold.

probability distribution (or relative frequency distribution) in Ω . Let $P(A)$ denote the probability that the point of the phase distribution falls into (the relative frequency with which the ensemble occurs in) the part A of the phase space Ω . $P(A)$ is, mathematically speaking, a completely additive set function which is defined for practically all subsets $A, B, \dots$ of Ω such that

$$P(A) \geq 0, \quad P(\Omega) = 1.$$

Consider such a phase distribution at time $t = 0$. After elapse of time t all its individual phases u have moved into their new positions T_u^t . If $P(A)$ denotes the distribution at $t = 0$ then the distribution $P^t(B)$ at time $t > 0$ is evidently determined by the formula

$$(3) \quad P^t(B) = P^0(T^{-t}B), \quad P^0(A) = P(A)$$

where $T^{-t}B$ denotes the subset of precisely those phases u for which T_u^t comes to lie within the part B of Ω . From now on we mean by a "phase distribution" a whole time succession P^t of probability distributions in Ω provided that they are interconnected by the laws of flow, i.e. by (3). Thus phase distributions are to be distinguished from instantaneous phase distributions.

A phase distribution is called stationary (ergodic theory calls this an invariant measure) if

$$P^t(A) = P(A)$$

holds for all t and for all parts A of Ω .

The average of a phase function $F(u)$ with respect to a given phase distribution is defined by the Lebesgue-Stieltjes-integral

$$(4) \quad \bar{F} = \int_{\Omega} F(u) P^t(du)$$

where $P^t(du)$ denotes the "differential" of the probability distribution P^t . Note and remember later on that the average is only defined with respect to a definite phase distribution. In general it changes with time t . If, however, the underlying phase distribution is stationary then the average of any phase function stays constant in time. Note also that the following identity holds,

$$(5) \quad \bar{F} = \int_{\Omega} F(u) P^t(du) = \int_{\Omega} F(T_u^t) P(du).$$

It is easily proved if one observes that, by virtue of (3), the Lebesgue-sums of approximation are the same for both integrals. It is important to grasp fully the scope of the concept of phase function. It is a law which attaches a (real or complex) value to every (or practically every) phase u . Remember that u is a symbol for an admissible vector field $u(x)$ defined in the part $R + B$ of x -space occupied by the fluid. A phase function is a functional whose argument is a whole vector

field $u(x)$ in $R + B$ which satisfies (1) and the given boundary conditions. Examples of such phase functionals are

$$F(u) = \frac{1}{2} \int_R u^2 dx = \frac{1}{2} \int_R u_\alpha u_\alpha dx,$$

which is the kinetic energy of the velocity field $u = u(x)$;

$$F(u) = \mu \int_R u_{\alpha,\beta} u_{\alpha,\beta} dx,$$

which is the energy dissipation associated with the solenoidal field $u(x)$;

$$F(u) = \int_R p_{,\alpha} p_{,\alpha} dx,$$

which is the integral of the square of the pressure gradient associated with the field. Remember here that the laws of flow uniquely determine the instantaneous pressure gradient in terms of the instantaneous phase $u = u(x)$. Simpler examples are

$$F(u) = u_\alpha(x^1).$$

which is the value of the α -th component of the field $u(x)$ in a given geometric point x^1 of R ;

$$F(u) = u_\alpha(x^1) u_\beta(x^2) \cdots u_\omega(x^n);$$

and

$$F(u) = p_{,\alpha}(x^1).$$

Let

$$(y, u) = \int_R y \cdot u dx = \int_R y_\alpha u_\alpha dx$$

denote the scalar product of two real vector fields $y = y(x)$ and $u = u(x)$ and consider the expression

$$(6) \quad e^{i(y, u)} = \exp (i(y, u))$$

where $y(x)$ is an arbitrary continuous field in $R + B$ while $u(x)$ is an arbitrary fluid phase in $R + B$ (admissible vector field). The average of this functional of with respect to a phase distribution $P^t(A)$ is denoted by

$$(7) \quad \Phi(y, t) = \overline{e^{i(y, u)}} = \int_\Omega e^{i(y, u)} P^t(du).$$

We call Φ which is a functional of an arbitrary vector field $y = y(x)$ in $R + B$

and a function of t the “characteristic functional” of the phase distribution P^t . For t fixed, say $t = 0$, it is the functional analogue of what probability theory calls the characteristic function of a finite-dimensional probability distribution $P(A)$. If P is a probability distribution in an n -dimensional u -space,

$$u = (u_1, u_2, \dots, u_n),$$

then its characteristic function

$$\int e^{i(y,u)} P(du), \quad (y, u) = y_\nu u_\nu,$$

is a function of an arbitrary set y of n numbers $y_1, y_2, \dots, y_n$.

Since y is to be a real field

$$\Phi^*(y) = \Phi(-y)$$

must hold for the conjugate value.

(3) implies that

$$(8) \quad \Phi(y, t) = \int_{\Omega} \exp(i(y, T^t u)) P(du).$$

The incompressibility (1) of the fluid mirrors itself in the following property of the characteristic functional:

$$(9) \quad \Phi(y + \text{grad } \varphi) = \Phi(y)$$

holds for every single-valued scalar function $\varphi(X)$ in $R + B$ which vanishes on B .* This follows, upon using (1), from

$$\begin{aligned} (\text{grad } \varphi, u) &= \int_R \varphi_{,\alpha} u_\alpha dx \\ &= \int_R (\varphi u_\alpha)_{,\alpha} dx = - \int_B \varphi u_n dS = 0. \end{aligned}$$

4. The functional derivatives of Φ . A function $\Phi(y)$ of a finite set of variables $y_1, \dots, y_n$ is said to be differentiable for a particular value of the argument y

* We admitted mathematically very general values of u on the boundary B subject only to the condition

$$\int_B u_n dS = 0$$

(incompressibility). Hence (9) holds for all functions φ with a constant value on B . But if we restrict the boundary values of u to physically realizable ones — B consists of impenetrable walls, $u_n = 0$ — then (9) holds for all smooth functions in $R + B$ without restriction of φ on B .

if the change $\Delta\Phi$ resulting from a slight change $\delta y = (\delta y_1, \dots, \delta y_n)$ of its argument is predominantly linear in δy ,

$$\Delta\Phi = \delta\Phi + \eta, \quad \delta\Phi = A_\nu \delta y_\nu,$$

where $\eta/\sqrt{\delta y_\nu \delta y_\nu} \rightarrow 0$ as $\delta y \rightarrow 0$. The linear form $\delta\Phi$ is the differential or variation, and its coefficients A_ν are the partial derivatives of Φ ,

$$A_\nu = \frac{\partial\Phi}{\partial y_\nu}, \quad \delta\Phi = \frac{\partial\Phi}{\partial y_\nu} \delta y_\nu.$$

Suppose now that Φ is a functional whose argument is an arbitrary number-valued function of a single real variable x , $0 \leq x \leq 1$, $y = y(x)$. Φ is said to be differentiable (for a particular "value" of the argument, i.e. for a particular function $y = y(x)$) if its change $\Delta\Phi$ resulting from a slight change $\delta y = \delta y(x)$ of its argument $y(x)$ is predominantly linear in $\delta y(x)$, in other words, if

$$\Delta\Phi = \delta\Phi + \eta, \quad \delta\Phi = \int_0^1 A(x) \delta y(x) dx$$

where

$$\eta / \sqrt{\int \delta^2 y(x) dx} \rightarrow 0 \text{ as the denominator} \rightarrow 0.*$$

If we would strictly define the partial derivative by the coefficient of $\delta y(x)$ in the linear form $\delta\Phi$ this partial derivative would have to be not $A(x)$ but $A(x)dx$. It is customary, however, to call $A(x)$ itself the functional or Volterra derivative of Φ with respect to $y(x)$ for the value $x^\dagger$. Note that the point x is the analogue of the discrete index ν in the finite case. We choose the somewhat cumbersome but very suggestive notation

$$A(x) = \frac{\partial\Phi}{\partial y(x)dx}$$

* In the functional case there are also other types of linear forms and distances which may be used to define differentiability differently. The types chosen above are, however, those which are of import in our context.

† The rule to define the partial derivative as the coefficient of the variation of the argument in $\delta\Phi$ would no longer be violated by this choice if we look at the situation in the following way. The argument of Φ is a point function $y(x)$ but we may consider this as the density of a mass distribution (set function) on $0 \leq x \leq 1$. Φ then becomes a functional of an arbitrary mass distribution. If we denote by $Y(S)$ the mass contained in the subset S of the interval $(0,1)$ the differential $\delta\Phi$ has the form

$$\int_0^1 A(x) \delta Y(dx).$$

Since now Y is the argument and δY its variation, $A(x)$ as defined by the rule mentioned above is actually the partial derivative with respect to its argument.

for the functional derivative. The relation

$$\delta\Phi = \int_0^1 \frac{\partial\Phi}{\partial y(x)dx} \delta y(x)dx$$

becomes then the complete analogue of the relation

$$\delta\Phi = \sum \frac{\partial\Phi}{\partial y_\nu} \delta y_\nu$$

of the discrete case.

The hydrodynamical functional $\Phi(y)$ introduced above is a functional of a vector field $y = y(x) = (y_1(x), y_2(x), y_3(x))$ defined in some region R of the three dimensional x -space. In this case the differential has the form

$$(10) \quad \delta\Phi = \int_R A_\alpha(x) \delta y_\alpha(x) dx$$

where

$$dx = dx_1 dx_2 dx_3$$

is the element of volume in x -space. $A_\alpha(x)$ is again called the functional derivative of Φ and we denote it by

$$(11) \quad A_\alpha(x) = \frac{\partial\Phi}{\partial y_\alpha(x)dx}.$$

Note that the present analogue of the discrete index ν is a pair (x, α) of a point x in R and a component-subscript $\alpha = 1, 2, 3$. Again we have

$$\delta\Phi = \int_R \frac{\partial\Phi}{\partial y_\alpha(x)dx} \delta y_\alpha(x) dx.$$

The first of the two functionals

$$\Phi = \int_R a_\alpha(x) y_\alpha(x) dx, \quad \Phi = y_\alpha(x^1)$$

is differentiable with the derivative

$$\frac{\partial\Phi}{\partial y_\alpha(x)dx} = a_\alpha(x).$$

The second one is not differentiable because the differential has not the required integral form (10).

Since the first order derivative is again a functional of the field $y = y(x)$ one may consider second order derivatives

$$\frac{\partial}{\partial y_\beta(x') dx'} \left(\frac{\partial \Phi}{\partial y_\alpha(x) dx} \right) = \frac{\partial^2 \Phi}{\partial y_\beta(x') dx' \partial y_\alpha(x) dx}.$$

Thus the second derivative of the first example above is zero. The quadratic functional

$$\Phi = \int_R \int_R K_{\alpha\beta}(x, x') y_\alpha(x) y_\beta(x') dx dx', \quad K_{\alpha\beta}(x, x') = K_{\beta\alpha}(x', x),$$

has the derivatives

$$\frac{\partial \Phi}{\partial y_\alpha(x) dx} = 2 \int_R K_{\alpha\beta}(x, x') y_\beta(x') dx'$$

and

$$\frac{\partial^2 \Phi}{\partial y_\beta(x') dx' \partial y_\alpha(x) dx} = 2 K_{\alpha\beta}(x, x')$$

while the third derivative is zero. The quadratic functional

$$\Phi = \int_R K_{\alpha\beta}(x) y_\alpha(x) y_\beta(x) dx, \quad K_{\alpha\beta}(x) = K_{\beta\alpha}(x),$$

however, behaves differently. It has a first derivative

$$\frac{\partial \Phi}{\partial y_\alpha(x') dx'} = 2 K_{\alpha\beta}(x') y_\beta(x')$$

but none of the second order.

The successive derivatives of the characteristic function $\Phi(y)$ of a probability distribution, evaluated at $y = 0$, are well known to furnish the successive moments or correlations of the distribution. The same is true in our hydrodynamical case where $\Phi = \Phi(y, t)$ is defined by (7). From

$$\begin{aligned} \delta \Phi &= \overline{\delta e^{i(y,u)}} = \overline{i(\delta y, u) e^{i(y,u)}} \\ &= i \overline{\int_R u_\alpha(x) \delta y_\alpha(x) dx} e^{i(y,u)} \\ &= i \int_R \overline{u_\alpha(x) e^{i(y,u)}} \delta y_\alpha(x) dx \end{aligned}$$

we infer that

$$(12) \quad \frac{\partial \Phi}{\partial y_\alpha(x) dx} = \overline{i u_\alpha(x) e^{i(y,u)}}$$

and that, generally,

$$(13) \quad \frac{\partial^n \Phi}{\partial y_\alpha(x^1) dx^1 \cdots \partial y_\omega(x^n) dx^n} = i^n \overline{u_\alpha(x^1) \cdots u_\omega(x^n) e^{i(y,u)}}.$$

If these derivatives are taken for the vanishing vector field $y = 0$ they evidently furnish the moments of the distribution or, in hydrodynamical language, the spatial correlation functions of the velocities

$$\overline{u_\alpha(x^1) u_\beta(x^2) \cdots u_\omega(x^n)}$$

which depend upon n "index pairs" $(x^1, \alpha), (x^2, \beta), \cdots, (x^n, \omega)$. Needless to say that they are defined only with respect to a definite phase distribution P^t and that they vary in general with t .

These correlation functions must satisfy a number of obvious relations in consequence of incompressibility. However, we are not interested in the technique of the correlation functions as something *per se*. Our primary interest is in the functional Φ which embodies the statistical properties of fluid flow in a much more concise form than the complicated infinite set of those functions. Incompressibility is expressed by the property (9) of Φ . It can also be expressed by the differential relation

$$(14) \quad \frac{\partial}{\partial x_\beta} \left(\frac{\partial \Phi}{\partial y_\beta(x) dx} \right) \equiv 0.$$

This is an immediate consequence of (1) and (12). One might, by the way, forget the original definition (7) of Φ and one could then show that a functional Φ of an arbitrary field $y = y(x)$ in $R + B$ which has the property (9) must also have the property (14) and vice versa.

I want to add a remark which will have to be used again and again. The characteristic functional

$$\Phi(y) = \int_{\Omega} e^{i(y,u)} P(du)$$

of a probability distribution actually characterizes this distribution; in other words, there cannot be two essentially different probability distributions with the same characteristic functional Φ .*

5. The functional differential equation for Φ . To make this subject more clearly understood we begin with a simpler problem of the same sort. The hydrodynamical u -space is replaced by an n -dimensional u -space, $u = (u_1, \cdots, u_n)$, denoted also by Ω , and the differential law of the phase motion becomes an

* We cannot bother here with the precise formulation and proof of this uniqueness theorem which is well known in the finite-dimensional case.

ordinary differential system

$$(15) \quad \frac{du_\nu}{dt} = Q_\nu(u) = Q_\nu(u_1, \dots, u_n), \quad \nu = 1, \dots, n.$$

In the hydrodynamical case the right hand sides of the differential equations of phase motion are functionals of second degree in $u = u(x)$. Correspondingly we assume the functions Q to be polynomials though not necessarily of second degree. It is assumed, too, that the solutions u^t of (15) starting from an arbitrary initial phase u^0 exist for all $t > 0$. Concepts such as $T^t u$, A , P^t and

$$(16) \quad \Phi(y, t) = e^{\overline{i(y, u)}} = \int_{\Omega} e^{i(y, u)} P^t(du), \quad (y, u) = y_\nu u_\nu,$$

are now to be used in the corresponding sense. We assert that the characteristic function Φ of any phase distribution $P^t(A)$ satisfies the linear partial differential equation

$$(17) \quad \frac{\partial \Phi}{\partial t} = i \sum_{\nu} y_\nu Q_\nu \left(\frac{\partial}{i \partial y_1}, \dots, \frac{\partial}{i \partial y_n} \right) \Phi$$

in which the expressions Q_ν on the right are to be interpreted as symbolic differential operators. For example, if

$$Q(u_1, u_2) = 2 + 3u_1 - u_2^2$$

then

$$Q \left(\frac{\partial}{i \partial y_1}, \frac{\partial}{i \partial y_2} \right) \Phi = 2\Phi - 3i \frac{\partial \Phi}{\partial y_1} + \frac{\partial^2 \Phi}{\partial y_2^2}.$$

The summation sign is not omitted in (17) because it makes the analogy to the integral sign obtained later in the hydrodynamical case more conspicuous.

The proof of (17) begins with the formula

$$\Phi(y, t) = \int_{\Omega} \exp (i(y, T^t u)) P(du).$$

By differentiation with respect to t ,

$$(18) \quad \frac{\partial \Phi}{\partial t} = i \int_{\Omega} \left(y, \frac{dT^t u}{dt} \right) \exp (i(y, T^t u)) P(du).$$

$T^t u$ describes a solution of (17). Hence

$$\left(y, \frac{dT^t u}{dt} \right) = y_\nu Q_\nu(T^t u)$$

and

$$\frac{\partial \Phi}{\partial t} = iy_\nu \int_{\Omega} Q_\nu(T^t u) \exp(i(y, T^t u)) P(du).$$

By virtue of (5) this can be written

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= iy_\nu \int_{\Omega} Q_\nu(u) \exp(i(y, u)) P^t(du) \\ &= iy_\nu \overline{Q_\nu(u) e^{i(y, u)}}. \end{aligned}$$

Since the $Q(u)$ are polynomials the averages must satisfy the identities

$$\overline{Q_\nu(u) e^{i(y, u)}} \equiv Q_\nu \left(\frac{\partial}{i \partial y_1}, \dots, \frac{\partial}{i \partial y_n} \right) \overline{e^{i(y, u)}}.$$

This concludes the proof of (17).

(17) is a linear differential equation of an order equal to the maximum degree of the polynomials Q . Its coefficients are linear functions of the independent variables y .

One can apply precisely the same argument to hydrodynamical phase motion $T^t u$. Naturally, the Φ -equation becomes in this case a linear functional differential equation of second order. We do not intend, however, to go on with the theory on this level of generality and so we will merely mention and describe the general Φ -equation. In the following sections we confine ourselves to the important and frequently considered limit case of boundary-free flow. The Φ -equation assumes its simplest form in this case and it will there be fully derived from the Navier-Stokes equations.

The characteristic functional of any phase distribution which moves in accordance with the laws of flow in the x -region R satisfies the equation

$$(19) \quad \frac{\partial \Phi}{\partial t} = \int_R y_\alpha(x) \left[i \frac{\partial}{\partial x_\beta} \frac{\partial^2 \Phi}{\partial y_\beta(x) \partial x \partial y_\alpha(x) \partial x} + \mu \Delta_x \frac{\partial \Phi}{\partial y_\alpha(x) \partial x} - \frac{\partial \Pi}{\partial x_\alpha} \right] dx,$$

$$\Delta_x = \partial^2 / \partial x_\beta \partial x_\beta,$$

for every "value" of its argument, i.e. for every vector field $y = y(x)$ in $R + B$ and for every value of $t \geq 0$. The space-integration on the right is the analogue of the summation in (17). The three terms between the square brackets clearly arise from the inertia— and friction terms and from the pressure gradient. The essential terms are the functional derivatives of Φ of first and second order, the x -derivatives being merely operations on the "index-point" x . $\Pi = \Pi(y; x, t)$ is an auxiliary quantity which is a functional of the vector field $y = y(x)$ and a function of a point x and of t . It must be determined by the simultaneous functional Φ just as the pressure p is to be determined by the simultaneous velocity field $u(x)$. The determination of Π is this. The expression between the square

brackets is the α -th component of a certain vector field, defined in the points x of R . Π must be chosen such that this field is solenoidal in R and that it assumes certain given boundary values on B .*

One can write the Φ -equation in several different forms just as one can do this with the Navier-Stokes equations. One way to eliminate the pressure-functional Π is to use the unique decomposition

$$(20) \quad y(x) = \tilde{y}(x) + \text{grad } \varphi, \quad \text{div } \tilde{y} = 0 \text{ in } R, \quad \tilde{y}_n = 0 \text{ on } B,$$

of an arbitrary field $y(x)$ in $R + B$. One must think here of $\tilde{y}(x)$ as a field which is functionally well determined by the field $y(x)$.† It can then be shown that $\Phi(y, t)$ satisfies the differential equation

$$(21) \quad \frac{\partial \Phi}{\partial t} = \int_R \tilde{y}_\alpha(x) \left[i \frac{\partial}{\partial x_\beta} \frac{\partial^2 \Phi}{\partial y_\beta(x) \partial x} \frac{\partial^2 \Phi}{\partial y_\alpha(x) \partial x} + \mu \Delta_x \frac{\partial \Phi}{\partial y_\alpha(x) \partial x} \right] dx$$

for all fields $y(x)$ and for all $t \geq 0$. The functional relation (9) and the footnote attached to it show that Φ depends actually only on the field $\tilde{y}(x)$. The relation $\Phi = \Phi(\tilde{y}, t)$ is another expression for the condition of incompressibility. One could use it to bring the Φ -equation into a form where only $\tilde{y}$ appears.

The Φ -equation (19) or (21) furnishes a direct differential description of the evolution of a phase distribution which evolves in accordance with the laws of flow. Naturally, it is better adapted to the needs of statistical hydrodynamics than the equations of flow which describe only individual phase motions.

For a flow problem of regular type (entirely finite region R and sufficiently smooth boundary data) Φ must possess a regular Taylor expansion

$$\Phi = \Phi^0 + \Phi^1 + \Phi^2 + \dots, \quad \Phi^0 = 1,$$

where Φ^n is a homogeneous polynomial functional of degree n in $y = y(x)$,

$$\Phi^n = \int_R \dots \int_R K_{\alpha \dots \omega}(x^1, \dots, x^n, t) y_\alpha(x^1) \dots y_\omega(x^n) dx^1 \dots dx^n$$

and where the kernel function K is given by

$$K_{\alpha \dots \omega}(x^1, \dots, x^n) = \frac{1}{n!} \frac{\partial^n \Phi}{\partial y_\alpha(x^1) \partial x^1 \dots \partial y_\omega(x^n) \partial x^n} \Big|_{y=0}.$$

* Under the boundary conditions admitted by us, u given on B independent of t , those boundary values are zero.

† The relation between y and $\tilde{y}$ has a simple geometric meaning if one interprets the totality of all vector fields y in $R + B$ as a Hilbert space H with the scalar product (y, y') . The gradient fields of single-valued functions in $R + B$ form a linear subspace G of H . The linear subspace S formed by the solenoidal fields with vanishing normal component on B is the orthogonal complement of G . $\tilde{y}$ is the projection of y upon S .

We know already that this function is just the correlation function

$$\frac{i^n}{n!} \overline{u_\alpha(x^1) \cdots u_\omega(x^n)}.$$

The Taylor expansion could have been derived equally well by using the power series for the exponential function in (7).

Upon inserting the power series of Φ into the Φ -equation (21) and upon equating terms of equal degree on both sides one obtains the infinite sequence of differential equations for the correlation functions,

$$\frac{\partial \Phi^n}{\partial t} = \int_R \tilde{y}_\alpha(x) \left[i \frac{\partial}{\partial x_\beta} \left(\frac{\partial^2 \Phi^{n+1}}{\partial y_\beta(x) dx \partial y_\alpha(x) dx} \right) + \mu \frac{\partial^2}{\partial x_\beta \partial x_\beta} \left(\frac{\partial \Phi^n}{\partial y_\alpha(x) dx} \right) \right] dx.$$

Formally, there is of course a one-to-one correspondence between the solutions of the Φ -equation and the solutions $\Phi^1, \Phi^2, \dots$ of the infinitely many correlation equations but, obviously, the Φ -equation describes the evolution of place distributions more concisely.*

Let us now briefly examine what uses we can expect of the Φ -equation. For this purpose we envisage a definite flow problem with definitely fixed stationary boundary data and where only the coefficient μ of viscosity is permitted to vary. If μ is sufficiently large there exists a single stationary solution (single point in u -space)—the laminar flow— $u = a(x, \mu)$ which is approached by any other solution as $t \rightarrow \infty$. At least, it is entirely within our present mathematical means to prove this rigorously.† If μ is very large the frictional term dominates on the right of the Φ -equation which then becomes of the first order

$$\frac{\partial \Phi}{\partial t} = \mu \int_R \tilde{y}_\alpha(x) \frac{\partial^2}{\partial x_\beta \partial x_\beta} \frac{\partial \Phi}{\partial y_\alpha(x) dx} dx.$$

It could be solved by the classical method of characteristics. We shall do this later in the extreme limit use of boundary-free flow. However, the well-explored case of very small Reynolds numbers is analytically simple enough and certainly does not call for a new mathematical technique. What happens if μ takes on smaller values? In exceptional cases of flow problems—for instance, if the bounding walls are all at rest, $u = 0$ on B —the character of general phase motion, namely convergence (as $t \rightarrow \infty$) towards a single stationary point in phase space, is preserved down to the smallest values of μ . For a “general” flow problem, how-

* Any attempt to use the correlation equations for purposes of approximation must overcome the difficulty that Φ^{n+1} occurs in the equation for $\partial \Phi^n / \partial t$. For small values of μ or, in other words, for highly turbulent flow, this term is of decisive importance. Besides, only the cases $n = 1, 2$ and at most, perhaps, $n = 3$ appear to be manageable. Therefore one is forced to make a good guess about Φ^{n+1} . This is, of course a perfectly natural procedure and recent progress in this direction is quite remarkable but the question remains how good an approximation one can obtain in this way.

† The stability proof has been known to me for about twelve years.

ever, the picture of phase motion becomes different as soon as μ decreases below a certain critical value.* The extreme case of a very small μ presents in general the following picture. The phase motions or, at least, a majority of them still converge towards something as $t \rightarrow \infty$. But this something is far from being a single stationary point in u -space. It must be pictured as a high-dimensional manifold of "typical" phase motions everyone of which comes in its course arbitrarily close to every point of this manifold and which all have the same statistical properties. The number of dimensions of this (ergodic) manifold of motions increases beyond limit as $\mu \rightarrow 0$ or, physically speaking, the infinitely many degrees of freedom of typical inviscid flow come more and more into play.† The manifold is the bearer of a certain stationary phase distribution $P' \equiv P$. Its characteristic functional Φ is a stationary solution of the Φ -equation. For intermediate values of μ the determination of Φ is an exceedingly complicated task. It is only when $\mu \rightarrow 0$ that Φ should have a simpler asymptotic form. But even the solution of this problem, the problem of highly turbulent flow, is perhaps beyond our present reach. This problem is presumably governed by the case $\mu = 0$ of the Φ -equation (21),

$$\int_{\mathbf{x}} \tilde{y}_{\alpha}(x) \frac{\partial}{\partial x_{\beta}} \left[\frac{\partial^2 \Phi}{\partial y_{\beta}(x) dx \partial y_{\alpha}(x) dx} \right] dx = 0.$$

As to simply setting $\mu = 0$ in the basic equations I may suggest that the problem of the limit passage in the solutions which is so involved in the case of individual flow should become easier if attention is shifted to stationary phase distributions. Note also that the functional differential equation (21) is of second order but that the frictional term contains only functional derivatives of first order which is the opposite of what happens in the individual case.

The presence of bounding walls is a highly complicating factor in all these problems and the statistical problem must become easiest in the extreme case of boundary-free flow. It is this simplest and purest form of the problem which looms in back of the present theories of turbulence. This problem will be considered in the following pages from the point of view of the Φ -equation.

6. Spatially periodic flow. Our subject is from now on the flow of a fluid which occupies the entire x -space. We first consider a very restricted type of such flow. Throughout this section it is supposed that

* I am appealing here to known observed facts and not to proven mathematical theorems on the stability of the laminar flow.

† The picture is described in more detail in the introduction of my paper: *A mathematical example displaying features of turbulence*, Communications on Pure and Applied Math. 1 (1948), pp. 303–322. I designed this example mainly to exhibit an actual mathematical case of such a continuously changing manifold of central phase motions with more and more dimensions. The terms of second degree in this example are, however, very different in character from the hydrodynamical inertia terms, and so the analogy to the hydrodynamical case does not extend much farther.

$$u(x + X) \equiv u(x)$$

where X is an arbitrary lattice vector

$$X = (n_1 L, n_2 L, n_3 L)$$

L being a fixed number. This flow problem is, of course, equivalent to the problem in which the basic region R is a cube C of width L with the boundary condition that the velocities and their first derivatives have the same respective values in equivalent points on opposite faces.

The momentum theorem takes the form

$$\frac{d}{dt} \int_C u \, dx = P$$

where P is the resultant (vectorial) pressure force exerted upon the fluid within C . The initial value problem—to find $u(x, t)$ if $u(x, 0)$ is given—is not a determinate one unless P or, equivalently, the momentum $\int_C u \, dx$ is prescribed for all times $t \geq 0$. We confine ourselves to the case where

$$(22) \quad \int_C u \, dx = 0,$$

in other words, we assume that the fluid is at rest in the mean. The Fourier series for u is

$$(23) \quad u(x) = \sum_k v(k) e^{ik \cdot x} *$$

where the vector k runs through all the lattice vectors

$$(24) \quad k = \left(n_1 \frac{2\pi}{L}, n_2 \frac{2\pi}{L}, n_3 \frac{2\pi}{L} \right)$$

of the Fourier space or k -space. $v(k)$ is a vector with complex-valued components v_α . It may be regarded as a vector field which is defined not in the entire k -space but only in the lattice (24) of this space. On denoting by z^* the conjugate of a complex number z (or of a complex vector z) we must have

$$(25) \quad v^*(k) = v(-k).$$

The condition of incompressibility takes the form

$$(26) \quad k \cdot v(k) = k_\alpha v_\alpha(k) = 0.$$

* It is frequently used in turbulence theory. The advantage is that x -differentiations go over into simple algebraic operations.

(22) means that

$$v(0) = 0.$$

Also, $v(k) \rightarrow 0$ sufficiently rapidly as $|k| \rightarrow \infty$. Upon applying (23) and on eliminating the pressure p by means of (26) the Navier-Stokes-equations are readily seen to go over into $(v(k) = v(k, t))$

$$(27) \quad \frac{\partial v_\alpha(k)}{\partial t} = -i \sum_{k' + k'' = k} [k_\beta'' \cdot v_\beta(k')] \left[v_\alpha(k'') - \frac{k_\gamma v_\gamma(k'')}{|k|^2} k_\alpha \right] - \mu |k|^2 v_\alpha(k)$$

or, vectorially,

$$(28) \quad \frac{\partial v(k)}{\partial t} = -i \sum_{k' + k'' = k} [k'' \cdot v(k')] \left[v(k'') - \frac{k \cdot v(k'')}{|k|^2} k \right] - \mu |k|^2 v(k).$$

For later use we note here another form of (27) obtained by introducing an arbitrary vector $z(k)$. We define

$$(29) \quad \bar{z}(k) = z(k) - \frac{k \cdot z(k)}{|k|^2} k$$

which is the unique vector of the form $z(k) - \lambda(k)k$, λ complex, that is perpendicular to the vector k .^{*} On using the identity

$$z(k) \cdot \left[v(k'') - \frac{k \cdot v(k'')}{|k|^2} k \right] = \bar{z}(k) \cdot v(k'')$$

one readily obtains the following form of (27)

$$(29) \quad \bar{z}_\alpha(k) \frac{\partial v_\alpha(\cdot k)}{\partial t} = i \sum_{k' + k'' = k} [\bar{z}_\alpha(k) v_\alpha(-k'')] [k_\beta'' v_\beta(-k')] - \mu |k|^2 z_\alpha(k) v_\alpha(-k).$$

A phase of the fluid is now characterized by the infinitely many variables $v_\alpha(k)$ or by a discrete vector field $v = v(k)$ which satisfies the conditions named above, in particular (26). The phase space or v -space is again denoted by Ω . General notation such as $T^t v$ = phase motion, Δ = part of Ω , P^t = moving phase distribution, $f(v)$ = phase function is used as defined above.

$$(30) \quad \bar{f} = \int_{\Omega} f(v) P^t(dv)$$

^{*} The relation between the fields z and $\bar{z}$ in k -space is the precise Fourier-analogue of the relation (20) between the fields y and $\bar{y}$ in x -space.

denotes the average of a phase functional $f(v)$ with respect to the distribution P^t . Again we have

$$(31) \quad \bar{f} = \int_{\Omega} f(T^t v) P(dv).$$

In analogy to the former arbitrary real vector field $y = y(x)$ we now introduce an arbitrary complex-valued vector field $z = z(k)$ which is defined only in the lattice points (24) of k -space. Two restrictions only are placed on this field, first, that

$$(32) \quad z^*(k) = z(-k)$$

and, secondly, that $z(k)$ vanishes sufficiently rapidly at $k = \infty$. The scalar product of two such fields is defined by

$$(33) \quad \begin{aligned} [z, v] &= \sum_{k'} z(k') \cdot v^*(k') = \sum_{k'} z_{\alpha}(k') v_{\alpha}^*(k') \\ &= \sum_{k'} z(k') \cdot v(-k') = \sum_{k'} z_{\alpha}(k') v_{\alpha}(-k'). \end{aligned}$$

The characteristic function of a phase distribution P^t is then given by

$$(34) \quad \Phi(z(k), t) = \overline{e^{i[z, v]}} = \int_{\Omega} e^{i[z, v]} P^t(dv).$$

Obviously, $[z, v]^* = [z, v]$ and hence

$$\Phi^*(z) = \Phi(-z).$$

Since in the present case Φ is actually a function of the discrete infinity of variables $z_{\alpha}(k)$ there is no need yet to introduce functional derivatives, and derivatives are understood in the ordinary sense. The property of Φ which expresses incompressibility is now that

$$(35) \quad \Phi(z(k) + \varphi(k)k) = \Phi(z(k))$$

holds for every field z and for every complex-valued scalar $\varphi(k)$, $\varphi^*(k) = \varphi(-k)$. This follows immediately from (34), (33) and (26). Its differential equivalent is now that

$$(35') \quad k_{\alpha} \frac{\partial \Phi}{\partial z_{\alpha}(k)} = 0$$

holds for every field z and for every vector k .^{*} The z -derivatives of Φ are obtained from (34) and (33).

^{*} The equivalence of (35) and (35') is readily proved by using the formula

$$\left. \frac{d}{d\tau} \Phi(z(k) + \tau \varphi(k)k) \right|_{\tau=0} = \sum_k k_{\alpha} \frac{\partial \Phi}{\partial z_{\alpha}(k)} \varphi(k).$$

$$(36) \quad \frac{\partial \Phi}{\partial z_\alpha(k)} = i \int_{\Omega} v_\alpha(-k) e^{i[z, v]} P^t(dv) = \overline{i v_\alpha(-k) e^{i[z, v]}}$$

and

$$(37) \quad \begin{aligned} \frac{\partial^2 \Phi}{\partial z_\beta(k') \partial z_\alpha(k'')} &= - \int_{\Omega} v_\alpha(-k'') v_\beta(-k') e^{i[z, v]} P^t(dv) \\ &= - \overline{v_\alpha(-k'') v_\beta(-k') e^{i[z, v]}}. \end{aligned}$$

On the other hand, (31) implies that

$$(38) \quad \Phi = \int_{\Omega} \exp(i[z, T^t v]) P(dv).$$

Hence

$$(39) \quad \frac{\partial \Phi}{\partial t} = i \int_{\Omega} \left[z, \frac{dT^t v}{dt} \right] \exp(i[z, T^t v]) P(dv).$$

Since T^t describes a phase motion $T^t v = v(k, t)$ and by virtue of (33),

$$(40) \quad \left[z, \frac{dT^t v}{dt} \right] = \sum_k z_\alpha(k) \frac{\partial v_\alpha(-k, t)}{\partial t}.$$

Hence and from the equations (29') of phase motion $v(k) = v(k, t)$

$$(41) \quad \begin{aligned} \frac{\partial \Phi}{\partial t} &= - \sum_{k' + k'' = k} \sum_{\alpha} \tilde{z}_\alpha(k) k_\beta'' \int_{\Omega} v_\alpha(-k'') v_\beta(-k', t) \exp(i[z, T^t v]) P(dv) \\ &\quad - i\mu \sum_k |k|^2 z_\alpha(k) \int_{\Omega} v_\alpha(-k, t) \exp(i[z, T^t v]) P(dv). \end{aligned}$$

The double sum can just as well be taken over all pairs (k', k'') of lattice points if in each term k is replaced by $k' + k''$. All the phase integrals have the form (31) and may therefore be written in the form (30). By virtue of (36) and (37), (41) now becomes

$$(42) \quad \frac{\partial \Phi}{\partial t} = \sum_{k'} \sum_{k''} z_\alpha(k' + k'') k_\beta'' \frac{\partial^2 \Phi}{\partial z_\alpha(k'') \partial z_\beta(k')} - \mu \sum_k |k|^2 z_\alpha(k) \frac{\partial \Phi}{\partial z_\alpha(k)}$$

where

$$(42') \quad \tilde{z}(k) = z(k) - \frac{k \cdot z(k)}{|k|^2} k.$$

(42) can be written

$$(43) \quad \frac{\partial \Phi}{\partial t} = \sum_{k'} \frac{\partial}{\partial z_\beta(k')} \left[\sum_{k''} \tilde{z}_\alpha(k' + k'') k_\beta'' \frac{\partial \Phi}{\partial z_\alpha(k'')} \right] - \mu \sum_k |k|^2 z_\alpha(k) \frac{\partial \Phi}{\partial z_\alpha(k)}.$$

After completion of the limit passage $L \rightarrow \infty$ it will turn out that (43) stays meaningful but not (42). The proof of (43) rests upon the fact that

$$\frac{\partial \tilde{z}_\alpha(k + k')}{\partial z_\beta(k')}$$

can be different from zero only if $k'' = 0$ in which case, however, no contribution is made by this term.

(42) is, of course, the special instance of the general Φ -equation (21) and (20) for the case of spatially periodic flow.*

The connection between the two forms $y(x)$ and $z(k)$ of the arbitrary vector field is given by the Fourier-relation

$$(44) \quad y(x) = \frac{1}{L^3} \sum_k z(k) e^{ik \cdot x}.$$

The same relation obtains between the fields $\tilde{y}$ and $\tilde{z}$. (23) and (44) imply that

$$(y, u) = [z, v].$$

7. General boundary-free flow. We must envisage a much broader class of flows in the entire x -space. Some sort of condition at infinity must, however, be imposed in order to insure reasonable behaviour of the flow. This is shown by the solutions

$$u_i = a_{i\nu} x_\nu, \quad p = -(\dot{a}_{ik} + a_{i\alpha} a_{\alpha k}) x_i x_k$$

of the equations of flow where the a_{ik} are entirely arbitrary functions of t subject to the conditions

$$a_{ik} = a_{ki}, \quad a_{\alpha\alpha} = 0.$$

Such a solution can become infinite at a finite moment of time. It can also start from the state of rest with $u = p = 0$ without staying at rest. Presumably no such things happen if a condition of the following sort is imposed: u is of smaller order of magnitude than $|x|$ at infinity, spatial averages of certain flow quantities taken over larger and larger spheres have limits. We assume that the average velocity (in the sense of that spatial average) is zero. A clear mathematical understanding of this important flow problem—it is the basis of the present theories of free turbulence—is a matter of great interest. In these theories, boundary-free flow is conceived of as the limit case of spatially periodic flow if the period L becomes infinite. We will proceed along these lines in order to find the correct form of the Φ -equation for general boundary-free flow. The viscosity is kept constant during the limit passage $L \rightarrow \infty$. As $L \rightarrow \infty$ the cubic

* The decomposition (20) is in this case determined by the following conditions: $\text{grad } \varphi$ is to be periodic but φ need not be periodic. $\tilde{y}(x)$ is to be periodic.

lattice (24) becomes increasingly fine in k -space and the discrete vector field $z(k)$ which is defined in the lattice points only is made to go over into a continuous field $z(k)$ in k -space.

If one attempts the limit passage in the Fourier-form of the equations (27) of individual flow one encounters the difficulty that the limit form of the sum of the right is not an ordinary integral.* This does not concern us, however, since we are not interested in problems of individual flow.

The situation is more fortunate in the statistical theory and the limit form of the Φ -equation (46) as $L \rightarrow \infty$ is found in the following way. Φ is a function of an arbitrary discrete vector field $z(k)$ which is defined only in the lattice points (24). One may interpret this field as a stepwise constant vector field in the whole of k -space by defining the value of $z(k')$ constant in each lattice cube

$$n_\nu \frac{2\pi}{L} \leq k'_\nu < (n_\nu + 1) \frac{2\pi}{L}, \quad \nu = 1, 2, 3$$

and equal to the vector $z(k)$ in the corner point

$$k = \left(n_1 \frac{2\pi}{L}, n_2 \frac{2\pi}{L}, n_3 \frac{2\pi}{L} \right).$$

With this interpretation Φ becomes a functional of a stepwise constant field $z(k)$ and one infers that

$$\begin{aligned} \delta\Phi &= \int_k \frac{\partial\Phi}{\partial z_\alpha(k') dk'} \delta z_\alpha(k') dk' \\ &= \sum_k \left[\int_{\text{cube}} \frac{\partial\Phi}{\partial z_\alpha(k') dk'} dk' \right] \delta z_\alpha(k) \quad (dk' = dk'_1 dk'_2 dk'_3) \end{aligned}$$

* If the vector

$$w(k) = L^3 v(k) = \int_G u(x) e^{ik \cdot x} dx$$

is introduced instead of $v(k)$ then the right hand sum in the equation for $\partial w(k)/\partial t$ has just the right factor $L^{-3} \sim$ volume of the lattice cube in k -space to make this sum go over into

$$\int_{k'+k''=k} (w(k') \cdot k'') \left(w(k'') - \frac{k \cdot w(k'')}{k^2} k \right) dk'.$$

This procedure is justified if (and, in general, only if) the limit field $u(x)$ vanishes sufficiently rapidly at infinity to make the limit integral

$$w(k) = \int_\infty u(x) e^{ik \cdot x} dx$$

convergent. Flows with $u(\infty) = 0$, however, are much too restricted to be of interest for statistical hydrodynamics.

where the sum is taken over all lattice points k and where the cube is the one with this lattice point as corner point. The cube-integral is approximately equal to

$$\frac{\partial \Phi}{\partial z_\alpha(k) dk} \Delta k$$

where $\Delta k = (2\pi/L)^3$ is the volume of the cube. We must expect, therefore, that for $L \rightarrow \infty$

$$(45) \quad \frac{\partial \Phi}{\partial z_\alpha(k)} \sim \frac{\partial \Phi}{\partial z_\alpha(k) dk} \Delta k$$

holds where the left represents the partial derivative in the discrete case whereas the right is Δk times the functional derivative of the limit functional Φ . Upon application of (45) to the differential equation (43) the right hand sums become at once approximating sums of integrals and the limit form of (43) is plainly the functional differential equation of second order

$$(46) \quad \frac{\partial \Phi}{\partial t} = \int_k \frac{\partial}{\partial z_\beta(k') dk'} \left[\int_{k''} \tilde{z}_\alpha(k' + k'') k''_\beta \frac{\partial \Phi}{\partial z_\alpha(k'') dk''} dk'' \right] dk' \\ - \mu \int |k|^2 z_\alpha(k) \frac{\partial \Phi}{\partial z_\alpha(k) dk} dk$$

where $\tilde{z}$ is given by (42') or (ϵ_{ik} = Kronecker symbol)

$$(47) \quad z_\alpha(k) = \left(\epsilon_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) z_\beta(k).$$

Φ must satisfy (46) for every complex-valued vector field $z(k)$ in k -space for which

$$z^*(k) \equiv z(-k)$$

and which vanishes sufficiently rapidly at $k = \infty$. Incompressibility means that

$$\Phi = \Phi(\tilde{z}(k), t).$$

Solutions Φ are admitted only if they have the following property. At any given moment of time t Φ is the characteristic functional of a probability distribution in v -space. This property implies, for instance, that

$$\Phi^*(z) = \Phi(-z)$$

and that

$$\Phi(0) = 1, \quad |\Phi(z)| \leq 1.$$

8. Φ and the double correlations. Homogeneous phase distributions. For spatially periodic flow Φ is a function of a discrete vector field $z(k)$. The terms of its Taylor series

$$(48) \quad \Phi = 1 + \Phi^1 + \Phi^2 + \dots$$

have the form

$$(49) \quad \Phi^n = \sum \dots \sum G_{\alpha \dots \omega}(k^1, \dots, k^n; t) z_\alpha(k^1) \dots z_\omega(k^n)$$

where the sum is taken over all systems $(k^1, \dots, k^n)$ of n independent lattice points and where the coefficients G have the values

$$(50) \quad \frac{1}{n!} \frac{\partial^n \Phi}{\partial z_\alpha(k^1) \dots \partial z_\omega(k^n)} \Big|_{z=0} = \frac{i^n}{n!} \overline{v_\alpha^*(k^1) \dots v_\omega^*(k^n)}.$$

In the present limit case $L = \infty$ one might expect Φ^n to have the form

$$(51) \quad \int \dots \int g_{\alpha \dots \omega}(k^1, \dots, k^n) z_\alpha(k^1) \dots z_\omega(k^n) dk^1 \dots dk^n$$

where the integral is an n -fold k -integral and where the coefficient-function g has the value

$$(52) \quad \frac{1}{n!} \frac{\partial^n \Phi}{\partial z_\alpha(k^1) dk^1 \dots \partial z_\omega(k^n) dk^n} \Big|_{z=0}$$

It turns out, however, that (51) is too restricted a form of the n -th degree polynomial Φ^n . Sufficient generality is attained if the differential element of (51)

$$g_{\alpha \dots \omega} dk^1 \dots dk^n$$

is replaced by the element of a mass distribution in the $3n$ -dimensional $(k^1, \dots, k^n)$ -space which need not have a density g in this space. In the case of homogeneous turbulence (see farther below) this mass distribution is actually concentrated on the $3(n-1)$ -dimensional subspace

$$k^1 + \dots + k^n = 0$$

and the functional derivatives (52) are Dirac functions of the n points k with their singularity on that subspace. Therefore it is important that we first establish a sufficiently general formula which expresses Φ^n in terms of the derivatives of Φ and which stays valid in that case. Only the case $n = 2$ will be dealt with. The case $n = 1$ will turn out to be trivial.

Let $H_{\alpha\beta}(k, k')$ denote a function of two independent and continuously varying points k, k' in k -space, $\alpha, \beta = 1, 2, 3$. We assume that these functions behave, particularly at infinity, in such a way that the following sums and integrals converge. We wish to establish the limit form of the sum

$$(53) \quad \sum_k \sum_{k'} \overline{v_\alpha^*(k) v_\beta^*(k')} H_{\alpha\beta}(k, k')$$

as $L \rightarrow \infty$. By virtue of (34), (32) and (33), this sum equals

$$(53') \quad - \sum_k \sum_{k'} \frac{\partial^2 \Phi}{\partial z_\alpha(k) \partial z_\beta(k')} \Big|_{z=0} H_{\alpha\beta}(k, k').$$

It can be written

$$- \sum_k \frac{\partial}{\partial z_\alpha(k)} \left[\sum_{k'} \frac{\partial \Phi}{\partial z_\beta(k')} H_{\alpha\beta}(k, k') \right] \Big|_{z=0}.$$

On application of (45) to both derivatives this double sum evidently becomes an approximation sum for the integral expression

$$(54) \quad - \int_k \frac{\partial}{\partial z_\alpha(k) dk} \left[\int_{k'} \frac{\partial \Phi}{\partial z_\beta(k') dk'} H_{\alpha\beta}(k, k') dk' \right] dk \Big|_{z=0}.$$

Therefore

$$(55) \quad (53) = (53') \rightarrow (54) \quad \text{as } L \rightarrow \infty.$$

Two applications of (55) are made. First we identify (53) with

$$\Phi^2 = - \frac{1}{2!} \sum_k \sum_{k'} \overline{v_\alpha^*(k) v_\beta^*(k')} z_\alpha(k) z_\beta(k').$$

On writing $w(k)$ for the argument-field in Φ we obtain the limit form

$$(56) \quad \Phi^2(z) = \frac{1}{2!} \int_k \frac{\partial}{\partial w_\alpha(k) dk} \left[\int_{k'} \frac{\partial \Phi}{\partial w_\beta(k') dk'} z_\alpha(k) z_\beta(k') dk' \right] dk \Big|_{w=0}$$

of Φ^2 as $L \rightarrow \infty$. Secondly we identify (53) with

$$\overline{u_\rho(x) u_\sigma(x')} = \sum_k \sum_{k'} \overline{v_\alpha^*(k) v_\beta^*(k')} \epsilon_{\alpha\rho} \epsilon_{\beta\sigma} e^{-i(k \cdot x + k' \cdot x')}$$

and obtain

$$(57) \quad \overline{u_\rho(x) u_\sigma(x')} = - \int_k \frac{\partial}{\partial z_\rho(k) dk} \left[\int_{k'} \frac{\partial \Phi}{\partial z_\sigma(k') dk'} e^{-i(k \cdot x + k' \cdot x')} dk' \right] dk \Big|_{z=0},$$

which relates the double correlations of the velocities to the functional $\Phi = \Phi(z, t)$.

An instantaneous phase distribution $P(A)$ in u -space is called homogeneous if it is invariant under arbitrary translations of the velocity fields. The precise definition is the following. We interpreted the solenoidal vector fields $u = u(x)$ as points of a u -space. The operation which consists in replacing a field $u(x)$ by the translated one, $u_a = u(x + a)$, a being a fixed vector, may be interpreted

as a one-to-one transformation of u -space into itself. A probability distribution $P(A)$ in u -space,

$$P(A) = \text{Prob } [u \text{ in } A],$$

A being an arbitrary part of this space, is called homogeneous if

$$\text{Prob } \{u_a \text{ in } A\} = \text{Prob } [u \text{ in } A] = P(A)$$

holds for every fixed vector a and for every part A of u -space. Homogeneity of a phase distribution is equivalently expressed by the invariance of its characteristic functional $\Phi = \Phi(y(x))$ under arbitrary translations of the argument-field $y(x)$,

$$(58) \quad \Phi(y_a(x)) \equiv \Phi(y(x)).$$

Indeed, (6) and (7) imply that

$$\begin{aligned} \Phi(y_a) &= \int \exp(i(y_a, u)) P(du) = \int \exp(i(y, u_{-a})) P du \\ &= \int \exp(i(y, u)) P(du) = \Phi(y), \end{aligned}$$

since the probability differential $P(du)$ remains invariant under the transformation $u \rightarrow u_a$ of the u -space. The converse implication (58) $\rightarrow$ invariance of the phase distribution follows by application of the uniqueness theorem mentioned at the end of section 5.

If Φ is expressed in terms of the argument-field $z(k)$ in k -space, $\Phi = \Phi(z) = \Phi(z(k))$, homogeneity of the phase distribution means that

$$(59) \quad \Phi(e^{ia \cdot k} z(k)) \equiv \Phi(z(k))$$

holds for arbitrary real vectors a and arbitrary fields z . This follows from (44).

If a phase distribution is homogeneous each term Φ^n of the Taylor series for Φ must have the same property of invariance (59). Concerning the integral representation (Stieltjes-integral!) (51) of Φ^n this means that the mass distribution is concentrated upon the subspace

$$k^1 + \dots + k^n = 0.$$

For instance, Φ^1 must have the form $-iU_a z_a(0)$, where U is the average velocity of the fluid, which we supposed to be zero. Hence

$$(60) \quad \Phi^1 = 0.$$

Φ^2 must be of the form

$$(61) \quad \Phi^2 = -\frac{1}{2} \int \Gamma_{\alpha\beta}(k) z_\alpha(k) z_\beta(-k) dk, \quad \Gamma_{\alpha\beta}(k) = \Gamma_{\beta\alpha}(-k),$$

or, in virtue of the incompressibility, $\Phi = \Phi(\bar{z})$,

$$(62) \quad \Phi^2 = -\frac{1}{2} \int f_{\nu\mu}(k) \bar{z}_\nu(k) \bar{z}_\mu(-k) dk, \quad f_{\nu\mu}(k) = f_{\mu\nu}(-k).$$

The relation between Γ and f is obtained from (47),

$$(63) \quad \Gamma_{\alpha\beta}(k) = \left(\epsilon_{\alpha\nu} - \frac{k_\alpha k_\nu}{|k|^2} \right) \left(\epsilon_{\beta\mu} - \frac{k_\beta k_\mu}{|k|^2} \right) f_{\nu\mu}(k).$$

From the original definition of Φ , for instance from (7), it is obvious that

$$\Phi^2 = -\frac{1}{2} \overline{(y, u)^2}$$

can have only real and non-positive values. Reality means that

$$\Gamma_{\alpha\beta}^*(k) = \Gamma_{\alpha\beta}(-k) = \Gamma_{\beta\alpha}(k).$$

Non-positivity of (61) implies that the Hermitean form

$$\Gamma_{\alpha\beta}(k) \xi_\alpha \xi_\beta^*$$

is ≥ 0 for every complex vector and at every point k . The connection between Γ or f and the double velocity correlations can readily be established by inserting $\Phi = 1 + \Phi_2 + \dots$ into (57). Only Φ^2 makes a contribution, and the result is the relation

$$(64) \quad \overline{u_\rho(x) u_\sigma(x')} = \int \Gamma_{\rho\sigma}(k) e^{ik \cdot (x' - x)} dk.$$

We repeat the main result which we obtained for homogeneous phase distributions with $\bar{u} = 0$. If the characteristic functional $\Phi = \Phi(z, t)$ is known then the second degree term Φ^2 of the Taylor series

$$\Phi = 1 + \Phi^2 + \Phi^3 + \dots$$

can be computed from it by means of the formula (56). Φ must have the form (61) or (62). After the functions f and Γ are determined the double velocity correlations are obtained from (64).*

9. Isotropic phase distributions. An instantaneous phase distribution P —again we interpret it first in u -space—is called isotropic if it is invariant not only under any simultaneous translation but also under any simultaneous rotation or reflexion of all the velocity fields $u(x)$. The precise definition is quite analogous to the one for homogeneity.

A functional $\Phi = \Phi(y)$ will be called isotropic if it is invariant under arbitrary

* The functions Γ are the components of the spectral tensor of the double velocity correlation well known in the theory of homogeneous turbulence.

translations, rotations and reflexions of the argument-field. We can now assert that a phase distribution is isotropic if and only if its characteristic functional $\Phi(y)$ is isotropic.

To prove this consider a fixed operation of that kind. Let $u' = u'(x)$ be the field obtained by applying this operation to an arbitrary field $u = u(x)$ and denote by $y'(x)$ the field obtained by applying the inverse operation to an arbitrary field $y(x)$. The transition $u \rightarrow u'$ represents again a one-to-one transformation of the u -space into itself which leaves the probability distribution $P(A)$ invariant. From the identity

$$(y', u) = (y, u')$$

and from (7) it follows exactly as in the homogeneous case that $\Phi(y') = \Phi(y)$. Since $y \rightarrow y'$ can represent any operation of that kind Φ must be isotropic. The converse implication is proved by following the preceding proof backwards and by using the fact that the characteristic functional uniquely characterizes the probability distribution P .

We expressed Φ in terms of vector fields $z(k)$ in k -space. The correct relation between the fields $z(k)$ and $y(x)$ is obtained from (44) by proceeding to the limit $L \rightarrow \infty$,

$$y(x) = (2\pi)^{-3} \int z(k) e^{ik \cdot x} dk.$$

From this one readily concludes: The necessary and sufficient condition that Φ be isotropic is that $\Phi = \Phi(z)$ satisfies (59) and that $\Phi(z)$ be invariant under arbitrary rotations of $z(k)$ about $k = 0$ and under reflexions at arbitrary planes through $k = 0$.

The restrictions which isotropy imposes upon the form of double velocity correlations and of their spectral tensor are well known. Isotropy of Φ implies isotropy of every term of its Taylor series $1 + \Phi^2 + \dots$. One can readily show that the most general Φ^2 which is both incompressible and isotropic is given by (62) where

$$f_{\nu\mu}(k) = \epsilon_{\nu\mu} f(|k|).$$

In other words,

$$(65) \quad \Phi^2 = -\frac{1}{2} \int f(|k|) \tilde{z}(k) \tilde{z}(-k) dk = -\frac{1}{2} \int f(|k|) |\tilde{z}(k)|^2 dk.$$

If this expressed in terms of z ,

$$\tilde{z}(k) = z(k) - \frac{k \cdot z(k)}{k^2} k,$$

Φ^2 is obtained in the form (61). Hence

$$(66) \quad \Gamma_{\alpha\beta}(k) = \left(\epsilon_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) f(|k|).$$

Since the Hermitean form

$$\Gamma_{\alpha\beta}(k) \xi_\alpha \xi_\beta^*$$

is positively definite,

$$\Gamma_{\alpha\alpha}(k) = 2f(|k|)$$

is ≥ 0 for every k . (64) and (66) imply that

$$(67) \quad \frac{1}{2} \overline{u(x) \cdot u(x')} = \int f(|k|) e^{ik \cdot (x-x')} dk$$

and, in particular, that the average kinetic energy per unit mass equals

$$(67') \quad \frac{1}{2} \overline{u^2(x)} = \int f(|k|) dk = 4\pi \int_0^\infty f(k) k^2 dk$$

The indefinite integral

$$\int f(|k|) dk$$

taken over an arbitrary part of k -space is called the amount of average energy contained in that part. $f(|k|)$ therefore represents the density of average energy in k -space. (64) and (66) also imply that

$$(68) \quad \frac{1}{2} \overline{(u(x') - u(x))^2} = 2 \int f(|k|) (1 - e^{-ik \cdot (x'-x)}) dk$$

and that the average dissipation of energy per unit mass ϵ equals

$$(69) \quad \epsilon \equiv \overline{\mu u_{\alpha,\beta} u_{\alpha,\beta}} = 2\mu \int f(|k|) k^2 dk = 8\pi\mu \int_0^\infty f(k) k^4 dk$$

The integrals (67) and (68) can, of course, be transformed into one-dimensional integrals.

10. Some exact stationary solutions of the Φ -equation in the case $\mu = 0$. We first transform the Φ -equation (48) by using $\bar{z}(k)$ instead of $z(k)$ as the argument-field. Remember that Φ must be a functional of $\bar{z}$ only. $\bar{z}(k)$ is however not an arbitrary field with the property $\bar{z}^*(k) \equiv \bar{z}(-k)$ since it must satisfy the orthogonality relation

$$(70) \quad k \cdot \bar{z}(k) = 0.$$

But this relation is characteristic for fields of the form

$$\tilde{z}(k) = z(k) - \frac{k \cdot z(k)}{k^2} k$$

because every field $w(k)$ which satisfies the relation $k \cdot w(k) = 0$ has the property that $w(k) = \tilde{w}(k)$ holds. In order to perform the transformation we must first imagine Φ as a functional of an entirely arbitrary field $\tilde{z}(k)$ which not necessarily satisfies (70). From

$$\delta\Phi = \int \frac{\partial\Phi}{\partial\tilde{z}_\nu(k)dk} \delta\tilde{z}_\nu(k)dk$$

and from

$$\delta\tilde{z}_\nu(k) = \left(\epsilon_{\nu\alpha} - \frac{k_\nu k_\alpha}{k^2} \right) \delta z_\alpha(k)$$

we infer that

$$\frac{\partial}{\partial z_\alpha(k)dk} = \left(\epsilon_{\nu\alpha} - \frac{k_\nu k_\alpha}{k^2} \right) \frac{\partial}{\partial\tilde{z}_\nu(k)dk}.$$

If this is applied to the Φ -equation (46) the viscosity term becomes

$$(71) \quad -\mu \int k^2 \tilde{z}_\alpha(k) \frac{\partial\Phi}{\partial\tilde{z}_\alpha(k)dk} dk$$

and the inertia term is found to be equal to

$$(72) \quad \int_{k'} \frac{\partial}{\partial\tilde{z}_\alpha(k')dk'} \left[\int_{k''} \tilde{z}_\lambda(k' + k'') K_{\alpha\beta\lambda} \frac{\partial\Phi}{\partial\tilde{z}_\beta(k'')dk''} dk'' \right] dk',$$

where

$$(72') \quad K_{\alpha\beta\lambda} = \left(k''_\alpha - \frac{k' \cdot k''}{k'^2} k'_\alpha \right) \left(\epsilon_{\beta\lambda} - \frac{k''_\beta k''_\lambda}{k''^2} \right).$$

Hence the Φ -equation becomes

$$(73) \quad \frac{\partial\Phi}{\partial t} = (72) + (71),$$

which equation is to be satisfied not for all fields $\tilde{z}(k)$ but only for those which satisfy (70).

We use this form of the Φ -equation to find some stationary solutions in the case $\mu = 0$ or, in other words, to find solutions Φ of the differential equation

$$(72) = 0.$$

First it is shown that an arbitrary quadratic functional

$$(74) \quad \Phi = \int F_{\alpha\beta}(k) \bar{z}_\alpha(k) \bar{z}_\beta(-k) dk, \quad F_{\alpha\beta}(k) = F_{\beta\alpha}(-k),$$

is such a solution.

Proof. By virtue of

$$\frac{\partial \Phi}{\partial \bar{z}_\beta(k'') dk''} = 2F_{\nu\beta}(-k'') \bar{z}_\nu(-k'')$$

the square bracket in (72) becomes

$$\int_{k'',} \left(k''_\alpha - \frac{k' \cdot k''}{k'^2} k'_\alpha \right) \bar{z}_\lambda(k' + k'') \bar{z}_\nu(-k'') \cdots dk''.$$

The derivative

$$\frac{\partial}{\partial \bar{z}_\alpha(k') dk'}$$

of this quadratic functional consists of two terms which are obtained by differentiating the two $\bar{z}$ -factors separately. The first factor contributes only if $k'' = 0$ whereas a contribution of the second one is possible only if $k'' = -k'$. But in both cases

$$k''_\alpha - \frac{k' \cdot k''}{k'^2} k'_\alpha$$

vanishes. Consequently, the expression after the first integral sign in (72) vanishes. Incidentally, it does this for entirely arbitrary fields $\bar{z}$ and not only for those that satisfy (70).

This solution is however without interest to us because it cannot be a characteristic functional of a phase distribution. A polynomial functional is unbounded whereas a characteristic functional must satisfy the inequality $|\Phi| \leq 1$.

It is shown next that

$$(75) \quad \Phi = W \left(\int \bar{z}(k) \cdot \bar{z}(-k) dk \right) = W \left(\int \bar{z}_\alpha(k) \bar{z}_\alpha(-k) dk \right)$$

is a solution of (72) = 0 if $W(\xi)$, $\xi \geq 0$, is an arbitrary function of a single variable which possesses a continuous second derivative.

Proof. By virtue of

$$\frac{\partial \Phi}{\partial \bar{z}_\beta(k'') dk''} = 2W' \bar{z}_\beta(-k'')$$

and of (70) the square bracket of (72) becomes

$$(76) \quad 2W' \int_{k''} z_{\lambda}(k' + k'') \bar{z}_{\lambda}(-k'') \left(k''_{\alpha} - \frac{k' \cdot k''}{k'^2} k'_{\alpha} \right) dk''.$$

Its functional derivative

$$(76') \quad \frac{\partial}{\partial \bar{z}_{\alpha}(k') dk'}$$

is the derivative of the product of two functionals, namely $2W'$ and the integral of (76). The derivative of the integral-factor vanishes. In fact, this derivative is but a special case of a similar derivative dealt with in the preceding example. The derivative (76') of the first factor $2W'$ is obtained just as the one of $\Phi = W$. Hence (76') of (76) equals $4W'' z_{\alpha}(-k')$ times the integral of (76), and (72) equals $4W''$ times

$$\iint \bar{z}_{\alpha}(-k') \bar{z}_{\lambda}(-k'') \bar{z}_{\lambda}(k' + k'') \left(k''_{\alpha} - \frac{k' \cdot k''}{k'^2} k'_{\alpha} \right) dk' dk''.$$

The term

$$-\frac{k' \cdot k''}{k'^2} k'_{\alpha}$$

may be omitted by virtue of (70). After changing the signs of k' and k'' the integral in question becomes

$$(77) \quad - \iint_{k+k'+k''=0} (\bar{z}(k') \cdot k'') (\bar{z}(k) \cdot \bar{z}(k'')) dV,$$

where

$$dV = dk dk' = dk dk'' = dk' dk''$$

is the volume element of the six-dimensional subspace $k + k' + k'' = 0$ of nine-dimensional (k, k', k'') -space. The value of (77) is invariant under interchange of k and k'' . If these two equal integrals are added the integrand is found to contain the factor

$$\bar{z}(k') \cdot k'' + \bar{z}(k') \cdot k = -\bar{z}(k') \cdot k' = 0.$$

This concludes the proof.

The isotropic solutions (75) are not without interest. For instance,

$$(78) \quad \Phi = \exp \left(-\frac{\kappa^2}{2} \int \bar{z}(k) \cdot \bar{z}(-k) dk \right)$$

can be shown to be the characteristic functional of an isotropic Gaussian dis-

tribution of solenoidal vector fields. (78) is the particular distribution of this sort in which flows in disjoint regions of x -space are as independent as is possible under the condition of incompressibility. Comparing the second form of the Taylor series for (78),

$$-\frac{\kappa^2}{2} \int \mathbf{z}(k) \cdot \mathbf{z}(-k) dk,$$

with (65) we see that the spectral energy density $f(k)$ has the constant value κ^2 . By the way, the same is true for the distributions corresponding to the general functionals (75). Hence (75) leads to equipartition of average energy in k -space. The Gaussian case (78) is the perfect analogue to the classical canonical distribution of the phases of a gas. (75) represents an arbitrary combination of micro-canonical distributions.

The characteristic functional of a general homogeneous Gauss-distribution of solenoidal vector fields with $\bar{u} = 0$ can be shown to have the form

$$\exp \left(-\frac{1}{2} \int f_{\alpha\beta}(k) \mathbf{z}_\alpha(k) \mathbf{z}_\beta(-k) dk \right)^*$$

where

$$f_{\alpha\beta}(k) = f_{\beta\alpha}(-k) = f_{\beta\alpha}^*(k).$$

We have not determined all functionals of this type which are stationary solutions of the case $\mu = 0$ of the Φ -equation.† There must be axisymmetric solutions which correspond to two-dimensional flow. However, one can easily prove that (78) is the only three-dimensional isotropic solution of Gaussian type. This remark will be used later.

11. The basic problem of isotropic turbulence. Current theories of isotropic turbulence presuppose and suggest that stationary isotropic phase distributions actually exist even with viscosity present, $\mu > 0$. Their characteristic functionals $\Phi(z)$ are stationary solutions of the Φ -equation (46), $\mu > 0$ ‡. Stationariness is not incompatible with the loss of kinetic energy due to viscosity. It obviously implies that the average energy per unit mass (67') be infinite. There is one exceptional stationary solution, namely $\Phi = 1$. The corresponding phase distribution consists of the single stationary phase $u = v = 0$ (state of rest). The probability $P'(A) = P(A)$ equals one if the phase region contains this phase $u = v = 0$ and zero if it does not do so. Presumably there exist other stationary isotropic solutions of (46), $\mu > 0$. Among them there should be certain "relevant"

* More generally, a Stieltjes integral should be used. However, this generality is hardly needed in statistical hydromechanics.

† A pupil of mine is working on questions of this nature.

‡ Solutions of (46) which represent typical decaying turbulence are not considered in this paper.

ones which represent the statistical pattern of observed turbulent flow, and the problem of isotropic turbulence amounts to just finding them. Remember that the averages of the physically important quantities can be computed once Φ is known. At the present time it is not possible completely to clarify the problem of "relevance." The points involved are brought out best by a reexamination of a basic idea of the current theory.

We start with the familiar two-parameter-group of simple transformations—coupled changes of scales in x , t and change of μ —which preserve the form of the Navier-Stokes equations (2). Remember that we supposed the density ρ of the fluid to be always equal to one. This group evidently induces a two-parameter-group which leaves the form of the Φ -equation (46) invariant and which we will write down farther below. Naturally, this group carries solutions Φ again into solutions, in general with a different value of μ , and it singly preserves properties of solutions such as homogeneity, axisymmetry, isotropy, stationarity. It is expected that the "relevant" solutions of the Φ -equation, $\mu > 0$ too form a two-parameter family of solutions—more precisely, that the relevant solutions are completely exhausted by applying the group transformations to just one of them. This is clearly a sort of ergodic hypothesis.* The original physical form of this idea is this. One may choose for group parameters two physically basic quantities, μ and the average dissipation ϵ of energy. In statistical equilibrium, and with no preferred direction in space, these two quantities must completely determine the statistical behavior of typical boundary-free flow (Kolmogoroff's first hypothesis). We write

$$(79) \quad \Phi(z(k); \mu, \epsilon)$$

for the corresponding characteristic functional. The group relation between the different functionals (79) is found to be

$$(80) \quad \Phi(z(k); \mu, \epsilon) \equiv \Phi(\alpha z(\beta k); \mu', \epsilon'),$$

where

$$(80') \quad \alpha = \left(\frac{\epsilon}{\epsilon'}\right)^{\frac{1}{3}} \left(\frac{\mu}{\mu'}\right)^{\frac{1}{3}}, \quad \beta = \left(\frac{\epsilon}{\epsilon'}\right)^{\frac{1}{3}} \left(\frac{\mu}{\mu'}\right)^{\frac{1}{3}}.$$

Hence the remaining basic problem is to find the single functional

$$\Phi(z(k); 1, 1).$$

The functional (79) must, as μ varies from 0 to ∞ , represent the whole continuous scale of possible degrees of turbulent disorder. As $\mu \rightarrow \infty$ it should reduce to the trivial case $\Phi = 1$ mentioned above. This is the turbulence-free

* To be more explicit, this hypothesis is made for three-dimensional flow. It is intuitively clear that general spatial flow can and will produce enough disorder to make this hypothesis valid. However, it cannot be expected to hold for strictly plane flow.

limit case with its exclusive preference for the single "laminar flow" which, in our boundary-free case, is simply the state of rest $u = 0$. What happens if $\mu \rightarrow 0$? This is the inviscid limit case in which turbulent disorder reaches its maximum degree. It must be expected that the limit functional

$$(81) \quad \Phi_0(z(k); \epsilon) = \lim_{\mu \rightarrow 0} \Phi(z(k); \mu, \epsilon)$$

exists (Kolmogoroff's second hypothesis) and that it is a stationary solution of the case $\mu = 0$ of the Φ -equation (46). (81), (81') imply that Φ_0 satisfies the following functional equation:

$$\Phi_0(z(k); \epsilon) = \Phi_0(\alpha z(\beta k); \epsilon'),$$

where

$$\alpha = \left(\frac{\epsilon}{\epsilon'}\right)^{\frac{1}{3}} \gamma, \quad \beta = \left(\frac{\epsilon}{\epsilon'}\right)^{\frac{1}{3}} \gamma^{-3},$$

with γ being arbitrary. This relation does not determine the full functional, but Kolmogoroff was the first to notice that dimensional arguments do determine the spectral density $f(|k|)$ in the limit case $\mu \rightarrow 0$. In other words, the functional equation does determine the important term Φ_0^2 of the Taylor series for Φ_0 . It is worthwhile to consider this matter in more detail.

Denoting the spectral density of average energy by

$$f(|k|; \epsilon, \mu)$$

we infer from (80), (80')—we apply this functional equation to Φ^2 —that

$$\int f(|k|; \mu, \epsilon) |z(k)|^2 dk = \alpha^2 \int f(|k|; \mu', \epsilon') |z(\beta k)|^2 dk$$

holds identically in $z = z(k)$. This implies the functional equation for f

$$(82) \quad f(|k|; \mu, \epsilon) = \alpha^2 \beta^{-3} f\left(\frac{1}{\beta} |k|; \mu', \epsilon'\right),$$

where α, β are given by (80'). Its general solution can be obtained if one chooses $\epsilon' = 1$ and μ' such that $\beta = |k|$,

$$\mu' = \mu \epsilon^{-\frac{1}{3}} |k|^{\frac{1}{3}}.$$

On letting

$$(83) \quad g(\mu') = f(1; \mu', 1)$$

the general solution of (82) is found to be

$$(84) \quad f(|k|; \mu, \epsilon) = g(\mu \epsilon^{-\frac{1}{3}} |k|^{\frac{1}{3}}) \epsilon^{\frac{1}{3}} |k|^{\frac{11}{3}},$$

where $g(\mu')$ is an arbitrary function of a single variable. Of course, the full conditions of our problem must make this function a completely determined one. At present, however, we have nothing but Kolmogoroff's second hypothesis at our disposal. It implies that the limit density

$$f_0(|k|; \epsilon) = \lim_{\mu=0} f(|k|; \mu, \epsilon)$$

exists. This means that the limit

$$\lim_{\mu'=0} g(\mu') = g(0)$$

exists. Hence we obtain Kolmogoroff's law

$$f_0(|k|; \epsilon) = g(0)\epsilon^{2/3} |k|^{-11/3}.$$

(84) also implies that, for any fixed values $\mu > 0$ and ϵ ,

$$f(|k|; \epsilon, \mu) \sim g(0)\epsilon^{2/3} |k|^{-11/3} \quad \text{as } k \rightarrow 0.*$$

As far as the quadratic term Φ^2 of Φ is concerned, these are the only conclusions one can draw from the two hypotheses of Kolmogoroff.† The determination of the full functional Φ_0 is an outstanding problem of great interest *per se*. One might consider Φ_0 as representing the statistical structure of the typical flow of an ideal incompressible fluid and one should expect that this same limit case obtains if viscosity is replaced by another agent which similarly cuts off the full turbulent disorder of the ideal case. However, "typical ideal fluid flow" must be very different from the picture one usually has of individual smooth ideal flow with perhaps only a discrete set of slip surfaces and vortex lines. In the strict ideal case phase motions in general tend completely to lose their properties of regularity and only certain statistical quantities (the functional Φ_0 !) can be expected to stay meaningful. Loss of regularity in space and time is indicated even by dimensional considerations. Kolmogoroff's law, expressed in x -space, says that the average velocity difference in two points at distance r and at the same time is $\sim r^{1/3}$ and not $\sim r$. One can also deduce that the average displacement of a "fluid particle" in a time interval t is $\sim t^{3/4}$ and not $\sim t$. All this is attenuated by an agent such as viscosity.‡

* This shows, by the way that the integral for Φ^2 converges only if the argument-field $z(k)$ vanishes sufficiently rapidly at $k = 0$. It will therefore be necessary to restrict the range of arbitrary fields $z(k)$, $z^*(k) = z(-k)$ to those which vanish sufficiently rapidly at $k = \infty$ and at $k = 0$.

† Of course, (84) can be translated into a statement about the double velocity correlations.

‡ Mathematically speaking, viscosity introduces a certain measure of compactness into the group T^t of phase motions. It was an attempt at proving this which inspired my paper: *Ein allgemeiner hydrodynamischer Endlichkeitssatz*, Math. Annalen **117** (1941), pp. 764-775. These mathematical investigations should however be carried much further, particularly

From the remark at the end of section 10 we can now draw a negative conclusion. Kolmogoroff's law is incompatible with a Gaussian phase distribution. The Gaussian solution (78) and the hypothetical non-Gaussian functional Φ_0 are two completely different solutions of the case $\mu = 0$ of the Φ -equation (46).*

in the important case of boundary-free flow. Those compactness properties are probably much weaker in this extreme case (even with $\mu > 0$) than in regular cases where the fluid is enclosed by entirely finite walls. The contrary remark I made about the boundary-free case at the end of my paper is not justified.

Ergodic theory of deterministic systems confined itself, at least at the time when I was active in this field, to transformation groups which enjoy certain compactness properties. Viscous fluid flow belongs to this category; ideal fluid flow considered as something *per se* (separate from the limit properties $\mu \rightarrow 0$ of viscous flow) does not. Here the situation is probably this. The group T_u^t of phase motions is actually defined only for certain not too discontinuous fields u , and it has no reasonable properties of compactness. "Limit fields" of fields T_u^t , $t \rightarrow \infty$ in general do not belong to those fields. The group cannot be extended to a larger u -space in which it stays a group of single-valued transformations (single-valued = uniquely determined flow) and in which it becomes compact. However there is the following question. Is an extension possible in which T^t becomes a stochastic process? In other words, is "typical ideal fluid flow" an indeterministic process?

* Boundary-free flow in the sense indicated before is approximately realized under very general boundary conditions if the Reynolds number is very large and if attention is confined to sufficiently small x -regions. Furthermore, if these small regions are properly dimensioned and if the flow in them is viewed from a properly moved system of reference the statistical structure of the flow should become a universal one, namely the Kolmogoroff structure corresponding to the functional (79). Viewed in x -regions a certain order larger but still small as compared to the extent of the boundaries, the structure is the one corresponding to Φ_0 . It seems feasible to me that the structure becomes the Gaussian one if the flow is viewed on a scale still an order larger than the former one. The difference in the order of magnitude of the latter two scales would explain why in the limit, Reynolds number $= \infty$, the two functionals, Φ_0 and the Gaussian one, appear as completely separate solutions of the case $\mu = 0$ of boundary-free flow.