On the Geometry of Two-dimensional Elastic Stress Fields

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I. Introduction. In 1933 one of the present authors¹ showed that any plane isometric net of curves can be considered as the net of principal stress trajectories of a five-parameter family of two-dimensional elastic stress systems without extraneous forces. In the following discussion, stress systems whose principal stress trajectories form an isometric net will be called isometric.

The importance of isometric nets in plane elasticity was brought out in 1934 by Wegner,² who demonstrated that an arbitrary two-dimensional elastic stress field with no body forces can be considered as the superposition of two isometric stress systems, one of which is a harmonic stress system in the sense of Föppl, that is, one whose first invariant $2S \equiv \sigma_x + \sigma_y = \sigma_1 + \sigma_2$ is everywhere zero.

A brief and elegant proof of Neményi's theorem previously cited was given by Prager³ in 1938, by employing complex variable methods first used by Kolosoff⁴ and Muschelišvili.⁵

In the present paper we will be concerned with a certain class of elastic stress systems. It will be characterized first of all by the existence of a body force potential $M(x, y)$ such that $\mathbf{f} = \nabla M$; in addition, we will assume that one of the following three sets of conditions is satisfied:

(a) The stress is plane. $\mathbf{f}$ is solenoidal; that is,

$$\nabla^2 M = 0.$$

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¹ P. F. NEMÉNYI, *Zeitschr. für Angew. Math. u. Mech.*, **13**, 364-366 (1933).

² U. WEGNER, *Ingenieurarchiv*, **5**, 449-469 (1934).

³ W. PRAGER, *Rev. math. de l'union interbalkanique*, **3**, 1-3 (1941).

⁴ G. KOLOSOFF, Doctoral Dissertation, Dorpat (1909); *Zeitschr. für Math. u. Phys.*, **62** (1914).

⁵ N. MUSCHELIŠVILI, *Bull. de l'Acad. de Sci. de l'U.S.S.R.*, 663-668 (1919); *Math. Ann.* **107**, 282-312 (1932).

(b) The strain is plane. $\mathbf{f}$ is solenoidal; that is,

$$\nabla^2 M = 0.$$

(c) The strain is plane. The material is incompressible; hence Poisson's ratio $\nu = \frac{1}{2} \left(\frac{1}{\nu} = m = 2 \right)$.

As we will see in section II, the stress function is biharmonic in x, y and hence the principal stress trajectories in a plane parallel to the x - y plane have the same geometric properties, whichever of the sets of conditions (a), (b), (c) is satisfied. Hence we will call, for the sake of brevity, any two-dimensional elastic stress field whose body forces are derivable from a potential $M(x, y)$ and which satisfies one of the sets of conditions (a), (b), (c) and M -system.

When we discuss stress trajectories in what follows, we shall mean the trajectories corresponding to the stresses $\sigma_x, \sigma_y, \tau_{xy}$ in the x - y plane. Similarly, the direction angle Θ occurring in this paper refers to the above trajectory net.

By employing the complex variable methods mentioned previously, a singularly simple expression for the direction angle Θ of a plane elastic field follows immediately and from it the generalization of Wegner's theorem to M -systems and a second theorem of similar structure related to the work of Pólya⁶ are obtained as corollaries. Further it is proved that if $\Theta(x, y)$ specifies the direction field of an elastic field of this type, $-\Theta(x, y)$ will not in general correspond to another elastic field of the same type. The exceptional cases include all isometric systems.

Finally, a brief synopsis of the deduction of the following theorem is given: any non-isometric net of stress trajectories consistent with a given plane elastic stress system whose body forces are irrotational and solenoidal, determines this stress system to within 2 to 5 arbitrary parameters. The complete proof will be given by one of the authors in a separate publication. An example which is included in the present paper shows that a certain family of non-isometric trajectory patterns actually determines the corresponding set of stress systems to within exactly 2 parameters, while in the limiting cases for which the pattern becomes isometric three additional parameters emerge.

Because of the limitations of our complex variable approach, our proofs are confined to simply-connected regions, and need to be re-examined if their application to multiply-connected regions is needed.

II. Analytical Preliminaries. We will base all our work on the analytic properties of the Airy stress function $\varphi(x, y)$ of plane elasticity, which, for the case of irrotational body forces, is related to the stresses $\sigma_x, \sigma_y, \tau_{xy}$ by the following well-known equations:

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⁶ G. PÓLYA, *Z. A. M. M.*, **10**, 353-360 (1930).

$$\left. \begin{aligned} \sigma_x &= \frac{\partial^2 \varphi}{\partial y^2} - M, \\ \sigma_y &= \frac{\partial^2 \varphi}{\partial x^2} - M, \\ \tau_{xy} &= \frac{\partial^2 \varphi}{\partial x \partial y}, \end{aligned} \right\} \quad (2.1)$$

for both plane strain and plane stress.

In the case of plane stress⁷

$$\phi = \phi_0 + \frac{1}{2}z^2 \left\{ \frac{A}{m-1} (x^2 + y^2) + M - \frac{\Theta_0}{1+m} \right\},$$

where

$$\nabla_{xy}^2 M = -2A, \quad \nabla_{xy}^4 \phi_0 = -\frac{2(m-1)}{m} A,$$

and Θ_0 is a harmonic function. Hence in case (a), A being $\neq 0$, for any given value of z , φ is a biharmonic function of x and y .

In the case of plane strain

$$\nabla_{xy}^4 \phi = \frac{m-2}{m-1} \nabla^2 M,$$

so that it is evident that φ is biharmonic both in case (a) and (c).

If, furthermore, we restrict ourselves to M -systems for which φ has no singularities in a simply-connected region R , i.e., if we demand that φ be biharmonic in x and y for any given value of z in R , it follows from the work of Muschelišvili⁸ that

$$\varphi = \Re \{ \bar{z} F(z) + G(z) \}, \quad (2.2)$$

where the symbol $\Re$ denotes the real part of the quantity in curly brackets, where $z = x + iy$, $\bar{z} = x - iy$, and $F(z)$ and $G(z)$ are regular functions of $z = x + iy$ in R .⁹

It follows from (2.1) that the quantity

$$\begin{aligned} \Omega &\equiv \frac{\sigma_y - \sigma_x}{2} - i\tau_{xy} \\ &= \left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} \right) + i \frac{\partial^2 \varphi}{\partial x \partial y}, \end{aligned} \quad (2.3)$$

⁷ Compare R. V. SOUTHWELL, *Theory of Elasticity*, 2nd ed., pp. 367-371, 1941.

⁸ N. I. MUSCHELIŠVILI, *loc. cit.*

⁹ It is hoped that no ambiguity will arise from the use of the letter z to designate $z + iy$ and the Cartesian coordinate perpendicular to the x - y plane.

whether the extraneous force field $\mathbf{f}$ is everywhere zero or whether $\mathbf{f} = -\text{grad } M \neq 0$.

Since the Kolossoff¹⁰ equation

$$\bar{\Omega} = \bar{z}F''(z) + G''(z) \quad (2.4)$$

depends only on the biharmonic character of φ and on equation (2.3), we have the following theorem: *The Kolossoff representation (2.4) of the conjugate stress deviator $\bar{\Omega}$ holds for any M -system.*

The first stress invariant, $\frac{1}{2}(\sigma_x + \sigma_y) \equiv S$, can be expressed as follows in terms of M and $F'(z)$:

$$S = -M + 2\Re\{F'(z)\}, \quad (2.5)$$

where z is a point of the region R mentioned above. This result is a direct consequence of (2.1) and the identity

$$\nabla^2 \Re\{\bar{z}F(z) + G(z)\} = 4\Re\{F'(z)\}.$$

If Θ is the direction angle of our plane elastic field, it follows from (2.3), (2.4) and the well-known formula

$$\tan 2\Theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

that

$$\tan 2\Theta = -\frac{\Im\{\bar{z}f(z) + g(z)\}}{\Re\{\bar{z}f(z) + g(z)\}},$$

where the symbol $\Im$ means the imaginary part of $\bar{z}f(z) + g(z)$ and

$$f(z) \equiv F''(z), \quad g(z) \equiv G''(z).$$

Therefore,

$$\Theta = -\frac{1}{2}\Im\{\log(\bar{z}f(z) + g(z))\}. \quad (2.6)$$

This last formula will be of basic importance in what follows.

III. Generalization of the Theorems of Wegner and Neményi. The theorem of U. Wegner,¹¹ stated in the introduction to the present paper, was deduced by him in a roundabout way. In the present paper, we will obtain the theorem immediately, as a corollary of equation (2.6), so that it will be proved not only when no extraneous forces are present, but also for the case of irrotational and solenoidal force fields.

¹⁰ G. KOLOSOFF, *loc. cit.*

¹¹ U. WEGNER, *loc. cit.*

To prove this result, we first write the conjugate stress deviator of our plane field as follows:

$$\bar{\Omega} = r^2 \frac{f(z)}{z} + g(z), \quad (3.1)$$

according to (2.4).

This stress field can be decomposed into the systems

$$\begin{cases} \bar{\Omega}_1 = g(z) \\ S_1 = 0 \end{cases} \quad (3.2)$$

and

$$\begin{cases} \bar{\Omega}_2 = r^2 \frac{f(z)}{z} \\ S_2 = M - 2\Re\{F'(z)\}, \end{cases} \quad (3.3)$$

where we have employed (2.5).

That (3.2) is harmonic is clear by definition. As to (3.3), let us compute its direction field $\Theta_2(x, y)$ by the general formula (2.6):

$$\Theta_2 = -\frac{1}{2}\Im\left\{\log r^2 \frac{f(z)}{z}\right\} = -\frac{1}{2}\Im\left\{\log \frac{f(z)}{z}\right\},$$

which is a harmonic function. But the direction field can be harmonic if and only if the corresponding net is isometric. This completes the proof of Wegner's theorem. Thus we have:

Any two-dimensional elastic field of M-type can be obtained by superposition of two isometric stress systems, one of which is harmonic in the sense of Föppl.

We will now prove another theorem of similar nature, which is related to harmonic stress systems in the sense of Pólya,¹² that is, systems for which each component of the stress tensor is harmonic.

If we restrict ourselves to plane strain, it can be shown that the conjugate stress deviator assumes the following form for systems of Pólya type:

$$\bar{\Omega} = C\bar{z} + h(z), \quad (3.4)$$

where C is a complex constant and $h(z)$ is a monogenic function of z .

From (2.6) it follows that we may express the direction field of a plane elastic system as follows:

$$\Theta = -\frac{1}{2}\Im\left\{\log\left[\bar{z} + \frac{g(z)}{f(z)}\right]\right\} - \frac{1}{2}\Im\{\log f\}, \quad (3.5)$$

where the first term, according to (3.4), corresponds to a harmonic system in

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¹² G. PÓLYA, *loc. cit.*

the sense of Pólya and the second one to a harmonic system in the sense of Föppl. Thus we have:

Any two-dimensional elastic field of M-type can be obtained through superposition of two systems, one of which is harmonic in the sense of Föppl, and the other harmonic in that of Pólya.

By employing the complex representation (2.4) of the conjugate stress deviator, Prager obtained Neményi's theorem concerning isometric nets as a corollary. Now, since it was shown in section II of the present paper that the structure of the stress deviator is identical for the cases of no body forces and of *M*-systems, we see that Prager's argument can be immediately applied to these more general elastic fields, for which we obtain the following theorem:

To any isometric net of plane curves, characterized by a complex velocity function $\omega(\bar{z}) = v_x + iv_y$ of an incompressible irrotational flow, belongs a family of two-dimensional stress systems with these curves as principal stress trajectories, whose conjugate stress deviator is

$$\bar{\Omega} = \bar{\omega}^2[\alpha(z + p)(\bar{z} + \bar{p}) + \beta] \quad (3.6)$$

and whose first stress invariant is

$$S = M - 2\Re \left\{ \alpha \int_{z_0}^z (\zeta + p) \bar{\omega}^2(\zeta) d\zeta \right\} + \gamma, \quad (3.7)$$

M being the force potential, α, β, γ real, and p a complex constant.

As a corollary we obtain a particularly simple characterization of the conjugate stress deviator of an isometric stress system:

An M-system is isometric if either

$$g = 0 \quad (3.8)$$

or

$$\frac{f}{g} = \frac{\alpha(z + p)}{\alpha\bar{p}(z + p) + \beta}. \quad (3.9)$$

IV. A General Property of *M*-Systems. In 1941 Saffet Süray¹³ discovered an interesting property of plane elastic systems. He proved that, in the absence of an extraneous force field, the only conformal transformations which transform *any* two-dimensional net of principal stress trajectories into a second net of the same nature belong to the class of linear fractional transformations.

In the light of our present analysis we see that this result of Süray is valid also for *M*-systems.

We conjecture that for an arbitrary conformal transformation only for exceptional nets will the transformed orthogonal net correspond to a set of stress

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¹³ S. SÜRAY, *Rev. de la Fac. de Sci. de l'Université d'Istanbul*, **6**, 40-43 (1946).

trajectories of a plane elastic field. The exceptional cases include, of course, all isometric systems.

In this section we will prove a companion theorem to this conjectured relationship.

Let Ω and Ω^* , defined by the equations

$$\begin{aligned}\Omega &= \bar{z}f(z) + g(z) = We^{-2i\Theta}, \\ \Omega^* &= \bar{z}f^*(z) + g^*(z) = W^*e^{-2i\Theta^*},\end{aligned}\quad (4.1)$$

where W and W^* are real, be the conjugate stress deviators corresponding to a pair of M -systems, and let $f(z)$, $g(z)$, $f^*(z)$, $g^*(z)$ be all identically different from zero. Then the stress trajectories corresponding to Ω and Ω^* are one another's reflection across the x -axis, that is

$$\Theta^* = -\Theta, \quad (4.2)$$

only if both f/g and f^*/g^* , if substituted for u , satisfy identically the equation

$$(qz^2 + \bar{p}z + \gamma)u^2 - (\bar{r}z^2 + \beta z + p)u + (\alpha z^2 + rz + \bar{q}) = 0, \quad (4.3)$$

where (α, β, γ) are real and (p, q, r) complex numbers, and where at least one of the quantities (α, q, r) and one of (γ, p, q) is different from zero.

Proof: From (4.1) and (4.2) it follows that

$$\bar{\Omega}\Omega^* = (\bar{z}g + f)(\bar{z}f^* + g^*) = \text{real}, \quad (4.4)$$

from which it can be immediately concluded that

$$\Im\{\bar{z}^2ff^* + \bar{z}(fg^* + f^*g) + gg^*\} = 0. \quad (4.5)$$

Since $\bar{z}^2ff^* + \bar{z}(fg^* + f^*g) + gg^*$ is an analytic function of order 3, in the sense of Burgatti,¹⁴ one may deduce from (4.5) that

$$\left\{ \begin{aligned} &\bar{z}^2ff^* + \bar{z}(fg^* + f^*g) + gg^* = \\ &(\alpha z^2 + \bar{r}z + q)\bar{z}^2 + (\bar{r}z^2 + \beta z + p)\bar{z} \\ &+ (qz^2 + \bar{p}z + \gamma). \end{aligned} \right\} \quad (4.6)$$

Therefore, equating coefficients of $\bar{z}$ in (4.6), one obtains:

$$\left\{ \begin{aligned} ff^* &= \alpha z^2 + rz + q, \\ fg^* + f^*g &= \bar{r}z^2 + \beta z + p, \\ gg^* &= qz^2 + \bar{p}z + \gamma. \end{aligned} \right\} \quad (4.7)$$

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¹⁴ P. BURGATTI, *Boll. della Unione Mat. Italiana*, **1**, 8-12 (1922). Equation (4.6) may also be deduced by more elementary methods.

Since, by hypothesis, none of the functions $f(z)$, $g(z)$, $f^*(z)$, $g^*(z)$ is identically equal to zero, it follows from (4.7) that at least one member of each of the sets (α, q, r) and (γ, p, g) is different from zero.

We will now show that f/g and f^*/g^* satisfy (4.3) when substituted for u in this equation.

From the first and third of equations (4.7) one sees that

$$f^* = \frac{\alpha z^2 + rz + \bar{q}}{f},$$

$$g^* = \frac{qz^2 + \bar{p}z + \gamma}{g}.$$

When these results are substituted in the second equation (4.7) one obtains:

$$(qz^2 + \bar{p}z + \gamma) \left(\frac{f}{g}\right)^2 - (\bar{r}z + \beta z + p) \left(\frac{f}{g}\right) + (\alpha z^2 + rz + \bar{q}) = 0.$$

It can be shown in an identical way that f^*/g^* obeys

$$(qz^2 + \bar{p}z + \gamma) \left(\frac{f^*}{g^*}\right)^2 - (\bar{r}z + \beta z + p) \left(\frac{f^*}{g^*}\right) + (\alpha z^2 + rz + \bar{q}) = 0,$$

and thus our proof is complete.

Equation (4.3) can be satisfied for both isometric and non-isometric pairs of M -systems. That isometric cases are included is obvious, because we know that for every isometric net there exist infinitely many solutions of (4.3) with non-vanishing f , g , f^* , g^* . This is shown by (3.6) and by the further analysis on the reflexion of isometric nets given below.

Nets consistent with (4.3) are thus a natural generalization of isometric ones. In order to show that there are non-isometric systems satisfying (4.3), we consider the following pair of four-parameter families of conjugate stress deviators:

$$\bar{\Omega} = \bar{z}f(z) + g(z) = \{\bar{z}(i\alpha z + \beta) + i\gamma z + \delta\}\varphi(z), \quad (4.8)$$

$$\bar{\Omega}^* = \bar{z}f^*(z) + g^*(z) = \{\bar{z}(-i\alpha z - i\gamma) + \beta z + \delta\} \frac{1}{\varphi(z)}, \quad (4.9)$$

where $\alpha, \beta, \gamma, \delta$ are real.¹⁵

It can be readily verified that $\bar{\Omega}\bar{\Omega}^* = \text{real}$, and thus that $\bar{\Omega}^*$ corresponds to the reflected net of $\bar{\Omega}$.

It is instructive to repeat the analysis specifically for the isometric case. Here we can start from a harmonic stress system with the conjugate deviator $g(z)$

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¹⁵ The constants in (4.3) and in (4.8), (4.9) are not intended to be the same. A similar remark applies to the equations following (4.9).

having the direction field Θ . The most general stress deviator $zf^* + g^*$ having a direction field $-\Theta$, has, then, to satisfy the equation

$$\Im\{g(\bar{z}f^* + g^*)\} = \Im\{\bar{z}gf^*\} + \Im\{gg^*\} = 0,$$

from which, by reasoning quite similar to the general case, we obtain:

$$\bar{z}f^* + g^* = \frac{\alpha z\bar{z} + p\bar{z} + \bar{p}z + \beta}{g},$$

where α and β are real and p is a complex constant. Hence for any choice of $g(z)$ a four-parameter family of solutions exists with a direction field (-1) -times the given ones.

Incidentally, we observe that a particular conjugate stress deviator obtained through reflection of the direction field of g is $1/g$; reflecting again we come back to the original direction field and obtain for this

$$\bar{z}f^{**} + g^{**} = g(z)\{\alpha'\bar{z}z + p'\bar{z} + \bar{p}'z + \beta'\} \quad (4.10)$$

where (α', β') are real, p is complex. This is Neményi's result of 1933, extended to isometric M -systems.

V. The Specification of Plane Elastic Systems by a Knowledge of their Stress Trajectories. We have seen that if we have a plane elastic field in a simply connected domain R , whose Airy stress function $\varphi(x, y)$ is biharmonic and single-valued at all points, then $\varphi(x, y)$ is given by (2.2), where the functions $F(z)$ and $G(z)$ are regular functions of z in R .

The knowledge of W and the stress trajectories in R implies that $\bar{\Omega} = We^{-2i\Theta}$ is given in this region, and that we therefore know the right-hand side of the equation

$$\bar{z}F''(z) + G''(z) = \bar{\Omega}. \quad (5.1)$$

It is readily seen by differentiating the foregoing equation with respect to $\bar{z}$ that

$$F''(z) = \frac{\partial \bar{\Omega}}{\partial \bar{z}},$$

where

$$\frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

From (5.1), due to the fact that $F''(z)$ is regular in R , it follows that

$$F'(z) = \int_{z_0}^z \frac{\partial \bar{\Omega}}{\partial \bar{\zeta}} d\zeta + \alpha,$$

where z, z_0 are arbitrary points of R and α is a complex constant. Therefore, from (2.5),

$$\left\{ \begin{aligned} S &= -M + 2\Re \left\{ \int_{z_0}^z \frac{\partial \bar{\Omega}}{\partial \bar{\zeta}} d\bar{\zeta} \right\} \\ &\quad + 2\Re \{ \alpha \} \end{aligned} \right\} \quad (5.2)$$

i.e., S is fixed to within a constant, so that we have shown:

The knowledge of $W(x, y)$ and of the stress trajectories of an M -system in a simply-connected domain R determines the corresponding stress components $\sigma_x, \sigma_y, \tau_{xy}$ ¹⁶ to within a single arbitrary constant.

This theorem is of crucial importance in the proof of our final result, which we present here in schematic form, reserving the complete proof for a later paper:

The family of plane elastic stress fields corresponding to a given set of stress trajectories is a family having between two and five parameters.

Since we are dealing with a system of stress trajectories, by hypothesis, there exists at least one stress system, say $\sigma_{0x}, \sigma_{0y}, \tau_{0xy}$, for which

$$\bar{\Omega}_0 = W_0 e^{-2i\Theta} = \bar{z}f_0(z) + g_0(z), \quad (5.3)$$

where W_0 is the maximum shearing stress of this particular system and $\bar{\Omega}_0$ its conjugate stress deviator.

For any other stress field $\sigma_x, \sigma_y, \tau_{xy}$, having the same set of stress trajectories as $\sigma_{0x}, \sigma_{0y}, \tau_{0xy}$, we have:

$$\bar{\Omega} = W e^{-2i\Theta} = \bar{z}f(z) + g(z), \quad (5.4)$$

where W and $\bar{\Omega}$ have the same significance as before, and Θ in (5.3) and (5.4) is the same function.

From (5.3) and (5.4) it follows that

$$\Re\{\bar{\Omega}_0\}\Im\{\bar{\Omega}\} = \Re\{\bar{\Omega}\}\Im\{\bar{\Omega}_0\}, \quad (5.5)$$

and thus our problem consists in determining all functions $f(z)$ and $g(z)$, analytic in R , which satisfy (5.5) for fixed $\bar{\Omega}_0$ and to show that they generate a family of stress fields $\sigma_x, \sigma_y, \tau_{xy}$, containing a number of parameters between two and five.

We first eliminate the trivial case of uniform hydrostatic pressure, $W_0 \equiv 0$. Once this case is eliminated, we can always find a point in R for which $W_0 \neq 0$. Such a point will be chosen as origin ($z = 0$) of the Cartesian co-ordinate system to which the stress components $\sigma_{0x}, \sigma_{0y}, \tau_{0xy}$, are referred. Furthermore, the Cartesian axes will be oriented parallel to the principal stress directions of this field at $z = 0$, so that $\Theta = 0$ at the latter point.

We now introduce the four functions of the two complex variables z and z' ,

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¹⁶ In what follows, we shall be interested solely in the stress components $\sigma_x, \sigma_y, \tau_{xy}$.

given by

$$\begin{cases} D_0(z, z') = z'f_0(z) + g_0(z) + z\bar{g}_0(z') + \bar{h}_0(z') \\ N_0(z, z') = z'g_0(z) + h_0(z) - z\bar{g}_0(z') - \bar{h}_0(z') \\ D(z, z') = z'g(z) + h(z) + z\bar{g}(z') + \bar{h}(z') \\ N(z, z') = z'g(z) + h(z) - z\bar{g}(z') - \bar{h}(z') \end{cases} \quad (5.6)$$

The notation $\bar{p}(z')$ means that if $p(z)$ is analytic in R , and thus can be represented by a power series

$$p(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k \quad (5.7)$$

about every interior point of R , $\bar{p}(z')$ is defined as follows:

$$\bar{p}(z') = \sum_{k=0}^{\infty} \bar{a}_k(z' - \bar{z}_0)^k. \quad (5.8)$$

From (5.6), one sees that

$$\begin{cases} D_0(z, \bar{z}) = 2\Re\{\bar{z}f_0(z) + g_0(z)\}, \\ N_0(z, \bar{z}) = 2i\Im\{\bar{z}f_0(z) + g_0(z)\}, \\ D(z, \bar{z}) = 2\Re\{\bar{z}f(z) + g(z)\}, \\ N(z, \bar{z}) = 2i\Im\{\bar{z}f(z) + g(z)\}. \end{cases} \quad (5.9)$$

Equations (5.5) and (5.9) imply that

$$D_0(z, \bar{z})N(z, \bar{z}) = D(z, \bar{z})N_0(z, \bar{z}). \quad (5.10)$$

This result, in conjunction with the analyticity of $D_0(z, z')N(z, z')$ and $D(z, z')N_0(z, z')$ for $z \in R$, $z' \in \bar{R}$, where $\bar{R}$ is the reflection of R across the axis of reals, implies, according to a known result of the theory of functions of several complex variables,¹⁷ that

$$D_0(z, z')N(z, z') = D(z, z')N_0(z, z') \quad (5.11)$$

for $z \in R$, $z' \in \bar{R}$.

The significance of (5.11) lies in the fact that the functions occurring therein are functions of *two independent* complex variables z and z' , and permit us to construct from them relations by differentiation with respect to z' and then to evaluate the results at $z' = 0$. This would not have been possible with the

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¹⁷ S. BOCHNER & W. T. MARTIN, *Functions of Several Complex Variables*, Princeton, 1948, Ch. 2, p. 36, Theorem 7.

sole aid of equation (5.5) or its equivalent, equation (5.10). A careful analytical investigation based on (5.11) shows that $\bar{\Omega} = \bar{z}f(z) + g(z)$ can be determined to within at most four real constants in the interior of the circle $|z| = \rho > 0$. The assumed analyticity of $g(z)$ and $h(z)$ then allows us to extend this result to the whole region R by the principle of analytic continuation.

The previous theorem then tells us that the family of elastic systems corresponding to the given net is at most a $4 + 1 = 5$ parameter family.

On the other hand, the number of parameters is never less than two, because one can always add an arbitrary plane hydrostatic stress system and can multiply the resulting stress system by an arbitrary constant without altering the system of stress trajectories.

Examples. For the isometric case we have seen that five arbitrary constants actually occur. But in the non-isometric case it can happen that only two, that is, the minimum number of arbitrary parameters, is consistent with a possible direction field, one arbitrary parameter being a real multiplier in the conjugate stress deviator (that is, an intensity factor of the deviator), the other a superimposed homogeneous hydrostatic pressure.

For example, the most general conjugate stress deviator $\bar{\Omega}$ corresponding to the direction field Θ defined by

$$\tan 2\Theta = r^2$$

is as follows:¹⁸

$$\bar{\Omega} = -iC\bar{z}z + C = C(1 - i\bar{z}z)$$

and the stress invariant is, in the absence of body forces,

$$S = -2\Re\left\{\int -iCz \, dz\right\} = -4Cxy + C_1,$$

where both C and C_1 are real constants.

This result and others of the same nature can be obtained by procedures similar to those in section IV.

A more instructive example is furnished by the assumption

$$\tan 2\Theta = n\frac{y}{x}$$

(where n is a real number). It is easy to see that the net will be isometric if $n = -1, 1, 0, \infty$. For these cases the conjugate stress deviators are in turn:

$$\bar{\Omega} = z[\alpha z\bar{z} + p\bar{z} + \bar{p}z + \beta],$$

$$\bar{\Omega} = \frac{1}{z}[\alpha z\bar{z} + p\bar{z} + \bar{p}z + \beta],$$

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¹⁸ This is a special case of (4.8).

$$\bar{\Omega} = [\alpha z\bar{z} + p\bar{z} + \bar{p}z + \beta],$$

$$\bar{\Omega} = i[\alpha z\bar{z} + p\bar{z} + \bar{p}z + \beta],$$

where α and β are real and p is complex. The first invariants can be also written explicitly in each of these cases, and they contain a fifth arbitrary real parameter.

As to the non-isometric case, analysis shows that only the following solution exists:

$$\bar{\Omega} = c\{(1+n)\bar{z} + (1-n)z\} \quad (5.12)$$

and the first invariant

$$S = -2c(1+n)x + c_1, \quad (5.13)$$

c and c_1 being real parameters.

Obviously the four limiting cases of (5.12) in which the net becomes isometric are included in the above list of isometric conjugate deviators; but the latter contains three additional arbitrary real parameters to which nothing corresponds in the non-isometric case. Thus we see that a non-isometric trajectory pattern may constitute a much more restrictive condition upon the stress distribution than a closely related isometric one.

VI. Summary and Conclusions. The investigation reported in this paper shows that, as far as the geometry of the stress trajectory pattern is concerned, plane elastic systems of M -type behave exactly in the same way as systems on which no extraneous forces are acting.

It was found that the simple properties of isometric stress fields, such as the possibility of mapping one system on another conformally and of obtaining new solutions by reflection of the direction field $\Theta(x, y)$, could not be directly generalized to arbitrary non-isometric M -systems. The key to the understanding of this is the fact that $\Theta(x, y)$ is proportional to the imaginary part of the logarithm of an analytic function of order 2, in the sense of Burgatti.

From this result followed the theorems in section III, concerning the possibility of obtaining a non-isometric stress system from systems of simpler type in two distinct ways.

The main result of this paper is that if the trajectory net corresponding to the stresses σ_x , σ_y , τ_{xy} is known for every plane $z = \text{constant}$, the stress system itself is determined to within 2 to 5 arbitrary constants.

A rather simple discussion of the constants entering in the above analysis shows that, *given a stress trajectory net, the knowledge of the magnitude of the two principal stresses and their gradients at a single non-isotropic point is sufficient to individualize a stress system consistent with the given net.*

These conclusions are surprising, as it has been traditionally assumed, especially by those working in experimental stress analysis, that if graphic integrations of a rather elaborate kind are to be avoided, three sets of curves have to

be obtained experimentally: isoclines, lines of constant shear (isochromatics), and lines of constant first invariant (isopachics). Neuber has shown that this is not correct because the direction field of the isopachics follows without integration from the trajectories.

It remains to be seen whether the new insight concerning the trajectories will lead practical stress analysts to improve upon current methods in a significant manner.

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