*Intrinsic Deformation Theory of Subspaces in a Riemann Space**

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Synopsis. The recent papers by Prim-Truesdell (4) and Synge (5) revealed again the importance of deformation theory in some fields of applied mathematics. In this paper we are dealing with a deformation theory of subspaces. The main scope may be described as follows. In an n -dimensional Riemann space V_n a set of subspaces $V_m[V_{n-m}]$ is given with the following properties: There is only one $V_m[V_{n-m}]$ going through each point P of a certain region R of V_n and the corresponding tangential spaces of V_m and V_{n-m} in P have only P in common. This configuration is investigated with respect to a deformation (1.2) and the results thus obtained depend on the configuration and the deformation only. Because the deformation theory of the V_{n-m} 's may be dealt with along the same lines as the deformation theory of the V_m 's we confine ourselves to the latter one, starting with the given set of V_m 's and using the V_{n-m} 's merely as "pseudonormal" subspaces whose deformation theory (not dealt with in this paper) follows the same pattern as the deformation theory of the V_m 's given *a priori*. The situation is described analytically by the equations (1.1), where the variables are subject to transformations (1.8).† The resulting theory is independent of these transformations. While in a previous paper (cf. (1), where the references to other important papers of the same field may be found) we restricted the generality of the problem by (1.4a), we are basing the present considerations on the more general assumption (1.4c).

* Prepared under Army Contract No. 33-008 Ord. 224.

† The last two of these transformations preserve the V_m 's and V_{n-m} 's given by (1.1). A

I. INTRODUCTION

1. Introduction. Let V_n be a n -dimensional Riemann space ($n \geq 2$) referred to a real coördinate system ξ^r .* Consider a set of subspaces V_m in V_n ($1 \leq m < n$) such that in a certain region R of V_n through each point of V_n there is only one V_m . We call such a set a family (V_m) of V_m 's in R and we confine ourselves to the region R . The family (V_m) may be represented by parametric equations

$$(1.1) \quad \xi^r = \varphi^r(\tau^1, \dots, \tau^m; \nu^{m+1}, \dots, \nu^n) \equiv \varphi^r(\tau; \nu).$$

Each element V_m of (V_m) is referred to its coördinates τ^a and is defined in (V_m) by constant values of its parameters ν^i . In this paper we shall investigate the family (V_m) with regard to an infinitesimal transformation

$$(1.2) \quad * \xi^r = \xi^r + \epsilon V^r(\xi)$$

where $\epsilon \rightarrow 0$ is a constant and $V^r(\xi)$ is a contravariant vector field over R . This investigation may be based on two different principles:

a) Denote by Y the space (referred to the coördinate τ^a) which one obtains by identifying all points of our (V_m) with the same values of the τ 's. Then (1.1) may be symbolically written as follows

$$(1.3) \quad R = \varphi(Y)$$

and this equation represents a *mapping* of Y into R . Any transformation $\tau^{a'} \leftrightarrow \tau^a$ of coördinates of Y which by assumption does not involve the parameters ν is independent of the mapping φ . Hence if we put $A_a^{a'} \equiv \frac{\partial \tau^{a'}}{\partial \tau^a}$ the A 's depend only on the τ 's, $A_a^{a'} = A_a^{a'}(\tau)$ and does not depend on the mapping φ , so that

$$(1.4a) \quad \frac{\partial}{\partial \varphi^\mu} A_a^{a'} = 0.$$

Hence if for instance a contravariant vector $v^a \equiv v^a(\varphi)$ is given in Y then we have

$$(1.4b) \quad \frac{\partial}{\partial \varphi^\mu} v^{a'} = A_a^{a'} \frac{\partial}{\partial \varphi^\mu} v^a.$$

The deformation theory based on (1.4) was dealt with in my previous paper.† It

more general problem (not considered here) arises if we start with (1.1) and introduce the transformations which preserve the V_m 's without preserving the V_{n-m} 's.

* Greek }
 Latin } indices run from $\begin{cases} 1 \text{ to } n \\ 1 \text{ to } m (< n) \\ m+1 \text{ to } n \end{cases}$
 German }

† Cf. (1). This paper will be referred to as DI.

leads to results which are independent of the choice of the ξ 's in V_n as well as of the choice of the τ 's in Y . On the other hand some of these results depend on the mapping φ which one starts with.

b) In this paper we will introduce a method which is independent not only of the choice of the coordinates ξ , but also of the mapping we start with. For this reason we call the corresponding deformation theory an *intrinsic* one. Because (1.1) represents a family these equations may be solved for τ^a and ν^n

$$(1.5a) \quad \tau^a = \pi^a(\xi^1, \dots, \xi^n), \quad b) \quad \nu^e = \mu^e(\xi^1, \dots, \xi^n).$$

In this case we have instead of (1.4a)

$$(1.4c) \quad \frac{\partial}{\partial \xi^\mu} A_a^{a'} \neq 0.$$

If we introduce the notation

$$(1.6) \quad T_a^\nu \equiv \frac{\partial \varphi^\nu}{\partial \tau^a}, \quad P_\lambda^a = \frac{\partial \pi^a}{\partial \xi^\lambda}, \quad N_\epsilon^\nu = \frac{\partial \varphi^\nu}{\partial \nu^\epsilon}, \quad M_\lambda^\epsilon = \frac{\partial \mu^\epsilon}{\partial \xi^\lambda}$$

we have

$$(1.7a) \quad T_a^\nu P_\nu^b = \delta_a^b$$

$$(1.7b) \quad N_\epsilon^\nu M_\nu^a = \delta_\epsilon^a$$

$$(1.7c) \quad T_b^\nu M_\nu^\epsilon = N_\epsilon^\nu P_\nu^a = 0.$$

The symbols (1.6) are clearly (mixed) tensors with respect to transformations

$$(1.8) \quad \xi^{\nu'} \leftrightarrow \xi^\nu, \quad \tau^{a'} \leftrightarrow \tau^a, \quad \nu^{\epsilon'} \leftrightarrow \nu^\epsilon.$$

We assume that in the $\begin{cases} \text{first} \\ \text{second of (1.8) only the} \\ \text{third} \end{cases} \begin{cases} \xi\text{'s} \\ \tau\text{'s} \\ \nu\text{'s} \end{cases}$ are involved. The space spanned in a generator point P of V_n by $\begin{cases} T_a^\nu \\ N_\epsilon^\nu \end{cases}$ will be referred to as the $\begin{cases} \text{tangential} \\ \text{pseudonormal} \end{cases}$ space of (V_m) in P .

2. (V_m) -Connection. In this section we introduce a new type of connection needed later on. Denote by $g_{\lambda\mu}$ the metric tensor of V_n , by $g^{\lambda\nu}$ its contravariant components and by $\Gamma_{\lambda\mu}^\nu$ the corresponding Christoffel symbols. If Γ_{bc}^a is any connection whatsoever in V_m and D_a the corresponding symbolic v.d. Waerden-Bortolotti vector of the corresponding covariant derivative then among all possible connections Γ_{bc}^a there is only one which satisfies the equation

$$(2.1a) \quad T_b^\nu D_a T_b^\nu = 0,$$

where

$$T_{\nu}^c = g^{cd} g_{\lambda\mu} T_d^{\lambda},$$

and g^{cd} are the contravariant components of

$$(2.2) \quad g_{ab} = g_{\lambda\mu} T_a^{\lambda} T_b^{\mu}.$$

If D_a denotes the symbols of the covariant derivative with respect to this connection then we have from (2.1a) and (2.2)

$$(2.1b) \quad D_a g_{bc} = 0^*$$

so that Γ_{ab}^c satisfying (2.1) are Christoffel symbols built up by means of g_{ab} and g^{cd} .

Lemma (2.1). *The expressions*

$$(2.3a) \quad \Gamma_{b\mu}^a \equiv \Gamma_{bc}^a P_{\mu}^c$$

transform according to

$$(2.3b) \quad \Gamma_{b',\mu'}^{a'} = A_{\mu'}^{\mu} (A_{b'}^{a'} A_b^{\mu} \Gamma_{b\mu}^a + A_{b'}^{a'} \partial_{\mu} A_b^{a'})$$

$$\left(A_{\mu'}^{\mu} = \frac{\partial \xi^{\mu}}{\partial \xi^{\mu'}}, \quad \partial_{\mu} = \frac{\partial}{\partial \xi^{\mu}} \right).$$

Hence the expressions

$$(2.4a) \quad D_{\mu} v^a \equiv \partial_{\mu} v^a + \Gamma_{b\mu}^a v^b$$

$$D_{\mu} w_b \equiv \partial_{\mu} w_b - \Gamma_{b\mu}^a w_a$$

are components of a mixed tensor, provided that $v^a(w_a)$ is a vector of V_m

$$(2.4b) \quad D_{\mu'} v^{a'} = A_{\mu'}^{\mu} A_{a'}^{a'} D_{\mu} v^a$$

$$(2.4b) \quad D_{\mu'} w_{b'} = A_{\mu'}^{\mu} A_{b'}^b D_{\mu} w_b.$$

Proof. The equation (2.3b) is consequence of the well known transformation formula for the Γ_{bc}^a and of

$$A_c^{c'} A_{\mu'}^{\mu} P_{\mu}^c A_d^{a'} \frac{\partial}{\partial \tau^{c'}} A_b^d = \frac{\partial \tau^c}{\partial \xi^{\mu'}} \frac{\partial}{\partial \tau^c} A_b^d.$$

* Another connection could be obtained from

$$P_{\nu}^c D_a T_b^{\nu} = 0.$$

For this connection (2.1b) holds only in special cases.

Since A_b^d does not contain the parameters ν^e , we have

$$\frac{\partial \tau^e}{\partial \xi^{\mu'}} \frac{\partial}{\partial \tau^e} A_b^d = \frac{\partial \tau^e}{\partial \xi^{\mu'}} \frac{\partial}{\partial \tau^e} A_b^d + \frac{\partial \nu^e}{\partial \xi^{\mu'}} \frac{\partial}{\partial \nu^e} A_b^d = \partial_{\mu'} A_b^d,$$

and this equation yields (2.3b). The first of (2.4b) is a consequence of (2.3b), the first of (2.4a) and of

$$\partial_{\mu'} v^{a'} = [(\partial_{\mu'} A_a^{a'}) v^a + A_a^{a'} \partial_{\mu'} v^a] A_{\mu'}^{\mu}.$$

The second of (2.4b) may be proved in the same manner. The set (2.3a) will be termed a (V_m) -connection.

We denote by ∇_{μ} the symbol of the covariant derivative with respect to $\Gamma_{\lambda\mu}^{\nu}$ and by D_{μ} the symbol of covariant derivative with respect to $\Gamma_{\lambda\mu}^{\nu}$ as well as with respect to $\Gamma_{b\mu}^a$. Thus for instance

$$\nabla_{\mu} v^{\nu} = \partial_{\mu} v^{\nu} + \Gamma_{\lambda\mu}^{\nu} v^{\lambda}, \quad D_{\mu} v^{\nu a} = \partial_{\mu} v^{\nu a} + \Gamma_{\lambda\mu}^{\nu} v^{\lambda a} + \Gamma_{b\mu}^a v^{\nu b},$$

where v^{ν} is a vector defined over V_n while $v^{\nu a}$ is a mixed tensor. Moreover we shall use sometimes the symbols $\nabla_a \equiv T_a^{\mu} \nabla_{\mu}$ and $D_a (\equiv T_a^{\mu} D_{\mu})$. Let (T) be an arbitrary tensor field. We say that it consists of V_n -parallel tensors along the field V^{μ} if

$$V^{\mu} \nabla_{\mu} (T) = 0.$$

We say that (T) consists of (V_m) -parallel tensors along the field V^{ν} if

$$V^{\mu} D_{\mu} (T) = 0.$$

Both parallelisms coincide only for tensors defined over V_n (tensors with Greek indices only). The (V_m) -parallelism of a mixed tensor is independent of the choice of the ξ 's and τ 's, while the V_n -parallelism of a mixed tensor is independent of the ξ 's while in general it depends on the choice of the τ 's.

Lemma (2.2). *The set of functions*

$$(2.5a) \quad P_{\omega\mu a}^e \equiv \partial_{\mu} \Gamma_{a\omega}^e - \partial_{\omega} \Gamma_{a\mu}^e + \Gamma_{b\mu}^e \Gamma_{a\omega}^b - \Gamma_{b\omega}^e \Gamma_{a\mu}^b$$

is a mixed tensor (the curvature tensor of the (V_m) -connection). It has the following properties:

1) *For any vector $v^a(w_a)$*

$$(2.6) \quad D_{[\omega} D_{\mu]} v^a = -\frac{1}{2} P_{\omega\mu b}^a v^b,$$

$$D_{[\omega} D_{\mu]} w_b = \frac{1}{2} P_{\omega\mu b}^a w_a;$$

2) *Also*

$$(2.5b) \quad P_{\omega\mu a}^e = P_{\omega}^i \partial_{\mu} \Gamma_{a i}^e - P_{\mu}^i \partial_{\omega} \Gamma_{a i}^e + 2P_{[\mu}^i P_{\omega]}^j \Gamma_{b i}^e \Gamma_{a j}^b;$$

3) It is connected with the curvature tensor

$$P_{ija}^c = \partial_j \Gamma_{ai}^c - \partial_i \Gamma_{aj}^c + \Gamma_{bj}^c \Gamma_{ai}^b - \Gamma_{bi}^c \Gamma_{aj}^b$$

$$\left(\partial_j \equiv \frac{\partial}{\partial \tau^j} \right)$$

of Γ_{ab}^c by

$$(2.7) \quad P_{ija}^c = T_i^\omega T_j^\mu P_{\omega\mu a}^c.$$

Proof. The equations (2.6) follow by direct computation and yield the tensorial character of (2.5a). The equation (2.5b) follows from (2.3a) and

$$\partial_{[\omega} P_{\mu]}^i = \frac{\partial^2 \tau^i}{\partial \xi^{[\omega} \partial \xi^{\mu]}} = 0$$

and (2.7) is a consequence of (2.5b) and (1.7a).

Any arbitrary vector field V^ν admits a decomposition in its tangential component W^ν and its pseudonormal component N^ν :

$$(2.8a) \quad V^\nu = W^\nu + N^\nu,$$

$$(2.8b) \quad W^\nu = V^a T_a^\nu, \quad V^a = P_\lambda^a V^\lambda,$$

$$(2.8c) \quad N^\nu = V^\epsilon N_\epsilon^\nu, \quad V^\epsilon = M_\lambda^\epsilon V^\lambda.$$

We shall use this decomposition in the following

Lemma (2.3). *We have*

$$(2.9a) \quad N^\omega T_b^\mu P_{\omega\mu a}^c = -N^\omega \partial_\omega \Gamma_{ab}^c,$$

$$(2.9b) \quad V^\omega T_b^\mu P_{\omega\mu a}^c = V^i P_{ib a}^c + N^\omega T_b^\mu P_{\omega\mu a}^c = V^i P_{ib a}^c - N^\omega \partial_\omega \Gamma_{ba}^c.$$

Proof. The equation (2.9a) follows from (2.5b) and (1.7a-c). The equation (2.9b) follows from (2.8), (2.7) and (2.5b).

Note. The right hand member of (2.9a) is a tensor, as is seen from (2.9a). Its tensorial character follows also from the transformation formula for Γ_{ab}^c and from

$$N^\omega \partial_\omega A_a^\alpha = M_\lambda^\epsilon V^\lambda N_\epsilon^\omega \partial_\omega A_a^\alpha,$$

$$= V^\lambda \frac{\partial \nu^\epsilon}{\partial \xi^\lambda} \frac{\partial \xi^\omega}{\partial \nu^\epsilon} \frac{\partial}{\partial \xi^\omega} A_a^\alpha,$$

$$= V^\epsilon \frac{\partial}{\partial \nu^\epsilon} A_a^\alpha = 0,$$

for any V^ϵ .

II. DEFORMATION THEORY

3. Fundamental definitions and some known results. Denote by ξ' the coördinates in V_n of a generic point P which describes a V_m of our family. Denote by $*P$ the point with the coördinates $*\xi'$ given by (1.2). $*P$ describes a subspace $*V_m$ as P describes V_m . We shall assume throughout this paper that the set of all the $*V_m$ (obtained from all the V_m of (V_m)) constitutes also a family, which we denote by $(*V_m)$. We have in general $(V_m) \neq (*V_m)$.

In the following definition $X = X(P)$ denotes a tensor field (a set of tensor fields) defined either over (V_m) or over V_n and $F[X] \equiv F(\xi)$ is a function of X .

Definition (3.1). X' is the value of X in $*P$, $X' \equiv X(*P)$; 0X (${}_0X$) is the tensor at $*P$ which one gets by V_n -parallel displacement ((V_m) -parallel displacement) of $X(P)$ in $*P$ along V' . If X is defined over (V_m) then $*X$ is the tensor field (the set of tensor fields) defined over $(*V_m)$ in the same way as X is defined over V_m . If X is defined over V_n then $*X = X'$. Furthermore

$$(3.1a) \quad *F \equiv F[*X], \quad {}^0F \equiv F[{}^0X], \quad {}_0F \equiv F[{}_0X], \quad F' \equiv F(*\xi),$$

in the first case and

$$(3.1b) \quad *F \equiv F' \equiv F(*\xi), \quad {}^0F \equiv F[{}^0X]$$

in the second case.

We use these symbols in the following definition of the operators τ , ω and Δ :

Definition (3.2). The operators τ , ω , Δ are defined by the following equations:

$$(3.2a) \quad \tau F \equiv \lim_{\epsilon \rightarrow 0} \frac{*F - {}^0F}{\epsilon} \quad (\text{translation of } F),$$

$$(3.2b) \quad \omega F \equiv \lim_{\epsilon \rightarrow 0} \frac{F' - {}_0F}{\epsilon} \quad (\text{variation of } F),$$

$$(3.2c) \quad \Delta F \equiv (\tau - \omega)F \quad (\text{deformation of } F).$$

In this paper we shall investigate the application of these operators on the objects of (V_m) . Comparing the operator τ as defined by (3.2a) with the operator τ defined in DI, we see that they are identical, while the operator ω , Δ as defined by (3.2b) and (3.2c) are different from the operator ω , Δ defined in DI. Because the operators τ defined here and in DI are identical we may use all the results of DI where only τ is involved. These results are condensed in the following theorem where we put

$$(3.3a) \quad W_{ab} \equiv T_{(a}^{\lambda} T_{b)}^{\mu} \nabla_{\lambda} V_{\mu},$$

$$(3.3b) \quad W_{abd} \equiv D_a W_{db} + D_b W_{da} - D_d W_{ba},$$

$$(3.3c) \quad W_{ab}^c \equiv W_{abd} g^{dc},$$

and denote by

$$(3.3d) \quad K_{a_r \dots a_1}^r \equiv D_{a_r} \dots D_{a_2} T_{a_1}^r, \quad r = 2, 3, \dots, N$$

the mixed tensors which describe the embedding theory of a V_m (and N is the number of different osculating spaces of V_m).

Theorem (3.1). *We have*

$$(3.4a) \quad \tau T_a^r = \nabla_a V^r,$$

$$(3.4b) \quad \tau g_{ab} = 2W_{ab},$$

$$(3.4c) \quad \tau \Gamma_{bc}^a = W_{bc}^a$$

$$(3.4d) \quad \tau K_{ab}^r = D_b D_a V^r - V^\omega T_b^\omega T_a^\lambda R_{\omega\mu\lambda}^r - W_{bu}^c T_c^r$$

where $R_{\omega\mu\lambda}^r$ is the curvature tensor of $\Gamma_{\lambda\mu}^r$:

$$R_{\omega\mu\lambda}^r = \partial_\mu \Gamma_{\lambda\omega}^r - \partial_\omega \Gamma_{\lambda\mu}^r + \Gamma_{\alpha\mu}^r \Gamma_{\lambda\omega}^\alpha - \Gamma_{\alpha\omega}^r \Gamma_{\lambda\mu}^\alpha.$$

The proof was given in DI.

4. Auxiliary theorems. Later on we shall see that the intrinsic deformation theory is based on the mixed tensor

$$(4.1) \quad L_a^r \equiv \nabla_a V^r - V^\mu D_\mu T_a^r.$$

In this section we shall prove some theorems about L_a^r needed later on.

Theorem (4.1). *We have*

$$(4.2) \quad L_a^r = \partial_a V^r - V^\mu (\partial_\mu T_a^r - \Gamma_{a\mu}^c T_c^r),$$

and consequently L_a^r does not depend on $\Gamma_{\lambda\mu}^r$ (and $\Gamma_{a\mu}^c$) if and only if V^r is pseudo-normal, $V^r = N^r$ (cf. (2.8)).*

Proof. The equation (4.2) follows from (4.1) and

$$\nabla_a V^r = \partial_a V^r + \Gamma_{\lambda\mu}^r T_a^\mu V^\lambda$$

$$V^\mu D_\mu T_a^r = V^\mu (\partial_\mu T_a^r + \Gamma_{\lambda\mu}^c T_a^\lambda - \Gamma_{a\mu}^c T_c^r)$$

by virtue of $\Gamma_{\lambda\mu}^r = \Gamma_{\mu\lambda}^r$. The $\Gamma_{\lambda\mu}^r$ are involved in (4.2) through $\Gamma_{a\mu}^c V^\mu$. Hence the remaining statement follows at once from (2.3a) and (1.7c).

The next theorem deals with W_{ab} :

Theorem (4.2). *The tensor W_{ab} may be expressed as follows:*

$$(4.3) \quad W_{ab} \equiv g_{\lambda\mu} T_{(b}^\lambda L_{a)}^\mu + \frac{1}{2} N^\mu D_\mu g_{ab}.$$

Proof: First of all we obtain by virtue of (2.1b) and (2.2)

* We exclude here the trivial case $\Gamma_{\lambda\mu}^r(\xi) = 0$ and $\Gamma_{b\sigma}^c(\varphi) = 0$.

$$\begin{aligned}
(4.4) \quad g_{\lambda\mu} V^\omega (D_\omega T_{(a)}^\lambda T_b^\mu) &= \frac{1}{2} g_{\lambda\mu} V^\omega D_\omega T_a^\lambda T_b^\mu = \frac{1}{2} V^\omega D_\omega T_a^\lambda T_b^\mu g_{\lambda\mu} \\
&= \frac{1}{2} V^\omega D_\omega g_{ab} = \frac{1}{2} (V^i D_i g_{ab} + N^\mu D_\mu g_{ab}) \\
&= \frac{1}{2} N^\mu D_\mu g_{ab} .
\end{aligned}$$

Consequently

$$\begin{aligned}
(4.5) \quad W_{ab} &= T_{(a)}^\lambda T_{(b)}^\mu \nabla_\lambda V_\mu = g_{\lambda\mu} T_{(b)}^\mu \nabla_a V^\lambda = g_{\lambda\mu} T_{(b)}^\mu (L_a^\lambda + V^\omega D_{[\omega} T_{a]}^\lambda) \\
&= g_{\lambda\mu} T_{(b)}^\mu L_a^\lambda + g_{\lambda\mu} (V^\omega D_\omega T_{(a)}^\lambda) T_{(b)}^\mu \\
&= g_{\lambda\mu} T_{(b)}^\mu L_a^\lambda + \frac{1}{2} N^\mu D_\mu g_{ab} .
\end{aligned}$$

Later on we shall need to express W_{ab}^c as a function of L_b^ν and the pseudonormal component N^ν of V^ν .

Theorem (4.3). *We have*

$$\begin{aligned}
(4.6) \quad W_{ab}^c &\equiv W_{abd} g^{dc} = g^{dc} g_{\lambda\mu} \{ K_{ab}^\lambda L_d^\mu + T_a^\mu D_{[b} L_{d]}^\lambda + T_b^\mu D_{[a} L_{d]}^\lambda + T_d^\mu D_{(a} L_{b)}^\lambda \} \\
&\quad + \frac{1}{2} g^{cd} \{ L_b^\mu D_\mu g_{ad} + L_a^\mu D_\mu g_{bd} - L_d^\mu D_\mu g_{ab} \} + T_a^\mu N^\omega P_{\mu\omega b}^c .
\end{aligned}$$

Proof. First of all we obtain by virtue of (2.1b) and (2.8)

$$(4.7) \quad (L_a^\mu - D_a W^\mu + W^\omega D_\omega T_a^\mu) D_\mu g_{ab} = (L_a^\mu - (D_a V^c) T_c^\mu) D_\mu g_{ab} = L_a^\mu D_\mu g_{ab} .$$

On the other hand we have by virtue of (4.7)

$$\begin{aligned}
(4.8) \quad -\frac{1}{2} N^\omega (D_\omega T_a^\mu) D_\mu g_{ab} &= -\frac{1}{2} (V^\omega - W^\omega) (D_\omega T_a^\mu) D_\mu g_{ab} \\
&= \frac{1}{2} (L_a^\mu - \nabla_a V^\mu) D_\mu g_{ab} + \frac{1}{2} W^\omega (D_\omega T_a^\mu) D_\mu g_{ab} \\
&= \frac{1}{2} (L_a^\mu - \nabla_a W^\mu + W^\omega D_\omega T_a^\mu) D_\mu g_{ab} - \frac{1}{2} (\nabla_a N^\mu) D_\mu g_{ab} \\
&= \frac{1}{2} L_a^\mu D_\mu g_{ab} - \frac{1}{2} (\nabla_a N^\mu) D_\mu g_{ab} .
\end{aligned}$$

Remembering (2.1b) we may write by virtue of (4.8)

$$\begin{aligned}
(4.9) \quad \frac{1}{2} N^\omega D_\omega D_\mu g_{ab} &= \frac{1}{2} N^\omega (D_a D_\omega - D_\omega D_a) g_{ab} \\
&= \frac{1}{2} N^\omega T_a^\mu (P_{\mu\omega a}^c g_{cb} + P_{\mu\omega b}^c g_{ac}) + \frac{1}{2} L_a^\mu D_\mu g_{ab} - \frac{1}{2} (\nabla_a N^\mu) D_\mu g_{ab} .
\end{aligned}$$

On the other hand

$$(4.10) \quad g_{\lambda\mu} D_a T_{(b)}^\mu L_a^\lambda = g_{\lambda\mu} (K_{d(b}^\mu L_a^\lambda + (D_d L_{(a)}^\lambda) T_{b)}^\mu) .$$

Substituting from (4.9) and (4.10) in the equation

$$(4.11) \quad D_a W_{ab} = g_{\lambda\mu} D_a T_{(b)}^\mu L_a^\lambda + \frac{1}{2} (\nabla_a N^\omega) D_\omega g_{ab} + \frac{1}{2} N^\omega D_a D_\omega g_{ab}$$

(obtained from (4.3)) we get

$$(4.12) \quad D_d W_{ab} = g_{\lambda\mu} (K_{d(b} L_a^\lambda) + (D_d L_{(a}^\lambda) T_{b)}^\mu) + \frac{1}{2} L_d^\mu D_\mu g_{ab} \\ + \frac{1}{2} N^\omega T_d^\mu (P_{\mu\omega a}^c g_{cb} + P_{\mu\omega b}^c g_{ac}) ,$$

and this equation together with (3.3b, c) leads to (4.6).

5. The fields T_a^ν , g_{ab} , and Γ_{ab}^c .

Theorem (5.1). *We have*

$$(5.1a) \quad \omega T_a^\nu = V^\mu D_\mu T_a^\nu$$

$$(5.1b) \quad \Delta T_a^\nu = L_a^\nu$$

where L_a^ν is defined by (4.1).

Proof. Let $X \equiv T_a^\nu$. Then

$$X' \equiv T_a^\nu(\star\xi) = T_a^\nu + \epsilon V^\mu \partial_\mu T_a^\nu + \dots \star \\ {}_0X \equiv {}_0T_a^\nu = T_a^\nu - \epsilon V^\mu (\Gamma_{\lambda\mu}^\nu T_a^\lambda - \Gamma_{a\mu}^c T_c^\nu) + \dots$$

and consequently

$$T_a^\nu(\star\xi) - {}_0T_a^\nu = \epsilon V^\mu \{ \partial_\mu T_a^\nu + \Gamma_{\lambda\mu}^\nu T_a^\lambda - \Gamma_{a\mu}^c T_c^\nu \} + \dots$$

This equation yields (5.1a) and from it and (3.4a) we obtain (5.1b).

Using the pseudonormal component N^ν of V^ν we may prove easily the following.

Theorem (5.2). *We have*

$$(5.2) \quad G_{ab} \equiv N^\omega \partial_\omega g_{ab} = N^\omega D_\omega g_{ab} .$$

The proof follows from (2.3a), (1.7c) and

$$D_\omega g_{ab} \equiv \partial_\omega g_{ab} - \Gamma_{a\omega}^c g_{cb} - \Gamma_{b\omega}^c g_{ac} .$$

Theorem (5.3). *We have*

$$(5.3) \quad \text{a) } \omega g_{ab} = G_{ab} , \quad \text{b) } \Delta g_{ab} = 2g_{\lambda\mu} T_{(b}^\mu L_{a)}^\lambda .$$

Proof. Let X be the set of tensors $g_{\lambda\mu}$, T_a^λ and put

$$F[X] \equiv g_{\lambda\mu} T_a^\lambda T_b^\mu = g_{ab} .$$

* Throughout this paper the dots denote the coefficient of ϵ^2 .

Then

$$\begin{aligned}
F' &\equiv g'_{ab} = g_{\lambda\mu}(*\xi)T_a^{\lambda'}T_b^{\mu'} \\
&= (g_{\lambda\mu} + \epsilon V^\omega \partial_\omega g_{\lambda\mu} + \dots) (T_a^\lambda + \epsilon V^\rho \partial_\rho T_a^\lambda + \dots) (T_b^\mu + \epsilon V^\sigma \partial_\sigma T_b^\mu + \dots) \\
&= g_{ab} + \epsilon V^\omega \{T_a^\lambda T_b^\mu \partial_\omega g_{\lambda\mu} + g_{\lambda\mu} T_b^\mu \partial_\omega T_a^\lambda + g_{\lambda\mu} T_a^\lambda \partial_\omega T_b^\mu\} + \dots \\
&= g_{ab} + \epsilon V^\omega \partial_\omega g_{ab} + \dots. \\
(5.4) \quad {}_0F &\equiv {}_0g_{ab} = {}_0g_{\lambda\mu} {}_0T_a^\lambda {}_0T_b^\mu = {}_0g_{\lambda\mu} T_a^\lambda T_b^\mu \\
&= [g_{\lambda\mu} + \epsilon V^\omega (\Gamma_{\lambda\omega}^\alpha g_{\alpha\mu} + \Gamma_{\mu\omega}^\alpha g_{\lambda\alpha}) + \dots] \\
&\quad [T_a^\lambda - \epsilon V^\rho (\Gamma_{\alpha\rho}^\lambda T_a^\alpha - \Gamma_{\alpha\rho}^c T_c^\lambda) + \dots] \\
&\quad [T_b^\mu - \epsilon V^\sigma (\Gamma_{\alpha\sigma}^\mu T_b^\alpha - \Gamma_{\alpha\sigma}^c T_c^\mu) + \dots] \\
&= g_{ab} + \epsilon V^\omega [T_a^\lambda T_b^\mu (\partial_\omega g_{\lambda\mu} - \Gamma_{\lambda\omega}^\alpha g_{\alpha\mu} - \Gamma_{\mu\omega}^\alpha g_{\lambda\alpha}) \\
&\quad + (\Gamma_{\alpha\omega}^c T_c^\lambda T_b^\mu + \Gamma_{\beta\omega}^c T_c^\mu T_a^\lambda) g_{\lambda\mu}] + \dots \\
&= g_{ab} + \epsilon V^i \partial_i g_{ab} + \dots.
\end{aligned}$$

Hence

$$F' - {}_0F = \epsilon V^\omega \partial_\omega g_{ab} - V^i \partial_i g_{ab} + \dots = \epsilon N^\omega \partial_\omega g_{ab} + \dots$$

and this equation together with (5.2) leads to (5.3a). The equations (3.4b), (4.3), and (5.3a) give us at once (5.3b).

Theorem (5.4). *We have*

$$(5.5a) \quad \omega \Gamma_{ab}^c = G_{ab}^c,$$

$$(5.5b) \quad \Delta \Gamma_{ab}^c \equiv Q_{ab}^c,$$

where

$$(5.6a) \quad G_{ab}^c \equiv N^\omega \partial_\omega \Gamma_{ab}^c = N^\omega T_b^\mu P_{\mu\omega\alpha}^c$$

$$(5.6b) \quad Q_{ab}^c \equiv W_{ab}^c - G_{ab}^c$$

and W_{ab}^c is defined by (3.3b) and (3.3c).

Proof. If we put

$$*\tau^a = \tau^a + \epsilon V^a$$

we obtain from (5.4)

$${}_0g_{ab} = g_{ab}(*\tau) = g_{ab} + \epsilon V^i \partial_i g_{ab} + \dots$$

and in a similar manner

$${}_0g^{ab} = g^{ab}(*\tau) = g^{ab} + \epsilon V^i \partial_i g^{ab} + \dots$$

Hence if X denotes the set of tensors g_{ab} , g^{cd} we obtain for $F[X] \equiv \Gamma_{ab}^c$

$$(5.7) \quad {}_0F \equiv {}_0\Gamma_{ab}^c = \frac{1}{2}g^{cd}(*\tau) \left(\frac{\partial}{\partial * \tau^b} g_{da}(*\tau) + \frac{\partial}{\partial * \tau^a} g_{db}(*\tau) - \frac{\partial}{\partial * \tau^d} g_{ab}(*\tau) \right).$$

A straightforward computation of the last member of (5.7) shows that

$${}_0\Gamma_{ab}^c = \Gamma_{ab}^c + \epsilon V^i \partial_i \Gamma_{ab}^c + \dots$$

and this equation together with (2.9a) and

$$F' \equiv \Gamma_{ab}^{'c} = \Gamma_{ab}^c + \epsilon V^\omega \partial_\omega \Gamma_{ab}^c + \dots$$

leads to (5.5a) and (5.6a). The equations (5.5b) and (5.6b) follow from (5.5a) and (3.4c).

6. The fields $K_{a_1 \dots a_n}^{\nu}$ *

We shall start first with

$$K_{ab}^\nu \equiv D_a T_b^\nu = D_b T_a^\nu.$$

Theorem (6.1). *We have*

$$(6.1) \quad \omega K_{ab}^\nu = V^\omega \{ D_b D_\omega T_a^\nu - T_b^\mu T_a^\lambda R_{\omega\mu\lambda}^\nu + (D_\omega T_b^\nu)(D_\mu T_a^\mu) + T_b^\mu T_c^\nu P_{\omega\mu a}^c \}.$$

Proof. First of all we have

$$(6.1a) \quad D_b D_\omega T_a^\nu \doteq \partial_b \partial_\omega T_a^\nu + T_a^\lambda \partial_b \Gamma_{\lambda\omega}^\nu - T_c^\nu \partial_b \Gamma_{a\omega}^c$$

$$(6.1b) \quad \partial_\omega \partial_b T_a^\nu - \partial_b \partial_\omega T_a^\nu = (\partial_\omega T_b^\mu) \partial_\mu T_a^\nu \doteq (D_\omega T_b^\mu)(D_\mu T_a^\nu)$$

$$(6.1c) \quad V^\omega T_c^\nu (\partial_b \Gamma_{a\omega}^c - \partial_\omega \Gamma_{ab}^c) \doteq V^\omega T_b^\mu T_c^\nu P_{\omega\mu a}^c$$

$$(6.1d) \quad V^\omega T_b^\mu T_a^\lambda (\partial_\omega \Gamma_{\lambda\mu}^\nu - \partial_\mu \Gamma_{\lambda\omega}^\nu) \doteq -V^\omega T_b^\mu T_a^\lambda R_{\omega\mu\lambda}^\nu$$

and moreover according to (6.1)

* Throughout this section we shall sometimes make use of special coördinate systems and shall symbolize the statements: "in the special coördinate system $\left\{ \begin{smallmatrix} \xi \\ \tau \end{smallmatrix} \right\}$ we have $\left\{ \begin{smallmatrix} \Gamma_{\lambda\mu}^\nu = 0 \\ \Gamma_{ca}^b = \Gamma_{c\mu}^b = 0 \end{smallmatrix} \right.$ at P " by using the symbol $\doteq$, e.g. $\left\{ \begin{smallmatrix} \Gamma_{\lambda\mu}^\nu \doteq 0 \\ \Gamma_{ca}^b \doteq \Gamma_{c\mu}^b \doteq 0 \end{smallmatrix} \right.$. An equation with $\doteq$ has to be understood as valid in a special coördinate system at P , where all the above mentioned equations holds. On the other hand the sign $=$ does not impose any restriction whatsoever on the choice of the coördinate system.

$$\begin{aligned}
K_{ab}^{\nu'} &= K_{ab}^{\nu} + \epsilon V^{\omega} \partial_{\omega} K_{ab}^{\nu} + \dots \\
&\doteq K_{ab}^{\nu} + \epsilon V^{\omega} (\partial_{\omega} \partial_b T_a^{\nu} + (\partial_{\omega} \Gamma_{\lambda\mu}^{\nu}) T_a^{\lambda} T_b^{\mu} - (\partial_{\omega} \Gamma_{ab}^c) T_a^{\nu}) + \dots \\
&\doteq K_{ab}^{\nu} + \epsilon V^{\omega} \{ (D_{\omega} T_b^{\mu}) (D_{\mu} T_a^{\nu}) + D_b D_{\omega} T_a^{\nu} + T_c^{\nu} (\partial_b \Gamma_{\omega\omega}^c - \partial_{\omega} \Gamma_{ab}^c) \\
&\quad + T_a^{\lambda} T_b^{\mu} (\partial_{\omega} \Gamma_{\lambda\mu}^{\nu} - \partial_{\mu} \Gamma_{\lambda\omega}^{\nu}) \} + \dots \\
&\doteq K_{ab}^{\nu} + \epsilon V^{\omega} \{ D_b D_{\omega} T_a^{\nu} - T_b^{\mu} T_a^{\lambda} R_{\omega\mu\lambda}^{\nu} + (D_{\omega} T_b^{\mu}) (D_{\mu} T_a^{\nu}) + T_b^{\mu} T_c^{\nu} P_{\omega\mu a}^c \} + \dots
\end{aligned}$$

On the other hand

$${}_0 K_{ab}^{\nu} \doteq K_{ab}^{\nu} + \dots$$

and consequently the last two equations lead to (6.1) with $\doteq$. Since this equation is a tensorial one we may substitute in it $=$ for $\doteq$.

Theorem (6.2). *Put*

$$(6.2a) \quad L_{ab}^{\nu} \equiv D_b L_a^{\nu} + L_b^{\mu} D_{\mu} T_a^{\nu} - R_{ba}^c T_c^{\nu}$$

$$(6.2b) \quad R_{ba}^c \equiv V^{\omega} T_b^{\mu} P_{\omega\mu a}^c + W_{ba}^c T_c^{\nu}.$$

Then

$$(6.3) \quad \Delta K_{ab}^{\nu} = L_{ab}^{\nu}.$$

Proof. We obtain first by virtue of (4.1)

$$\begin{aligned}
D_b L_a^{\nu} &= D_b (\nabla_a V^{\nu} - V^{\mu} D_{\mu} T_a^{\nu}) \\
&= D_b \nabla_a V^{\nu} - (L_b^{\mu} + V^{\omega} D_{\omega} T_b^{\mu}) D_{\mu} T_a^{\nu} - V^{\mu} D_b D_{\mu} T_a^{\nu},
\end{aligned}$$

and consequently

$$(6.4) \quad D_b L_a^{\nu} + L_b^{\mu} D_{\mu} T_a^{\nu} = D_b \nabla_a V^{\nu} - V^{\omega} \{ (D_{\omega} T_b^{\mu}) (D_{\mu} T_a^{\nu}) + D_b D_{\omega} T_a^{\nu} \}.$$

On the other hand the equations (6.1) and (3.4d) yield

$$\begin{aligned}
(6.5) \quad (\tau - \omega) K_{ab}^{\nu} &= D_b V_a V^{\nu} - V^{\omega} \{ (D_{\omega} T_b^{\mu}) (D_{\mu} T_a^{\nu}) + D_b D_{\omega} T_a^{\nu} \} \\
&\quad - T_c^{\nu} (T_b^{\mu} P_{\omega\mu a}^c V^{\omega} + W_{ba}^c),
\end{aligned}$$

and the theorem follows from (6.4) and (6.5).

Note. We see from (6.3) that $L_{[ab]}^{\nu} = 0$. Hence we must have

$$(6.6) \quad D_{[b} L_{a]}^{\nu} + (D_{\mu} T_{[a}^{\nu}) L_{b]}^{\mu} = R_{[ba]}^c T_a^{\nu}.$$

We shall deal later on with this equation.

Theorem (6.2) may be easily generalized :

Theorem (6.3). *Put*

$$(6.7) \quad L_{a_r \dots a_1}^\nu \equiv D_{a_r} L_{a_{r-1} \dots a_1}^\nu + L_{a_r}^\mu D_\mu K_{a_{r-1} \dots a_1}^\nu - \sum_1^{r-1} R_{a_r a_s}^c K_{a_{r-1} \dots a_{s+1} \dots a_{s-1} \dots a_1}^\nu.$$

Then

$$(6.8) \quad \Delta K_{a_r \dots a_1}^\nu = L_{a_r \dots a_1}^\nu \\ (r = 2, 3 \dots, N, K_a^\nu \equiv T_a^\nu)$$

Proof. Assume that the theorem is true for some $r \geq 2$ (it was already proved for $r = 2$). Then we have according to the definition (3.2) and the equation (6.8)

$$\epsilon L_{a_r \dots a_1}^\nu + \dots = *K_{a_r \dots a_1}^\nu - K_{a_r \dots a_1}^{\prime \nu} - ({}^0 K_{a_r \dots a_1}^\nu - {}^0 K_{a_r \dots a_1}^\nu),$$

or

$$(6.9) \quad *K_{a_r \dots a_1}^\nu = K_{a_r \dots a_1}^{\prime \nu} + \epsilon [L_{a_r \dots a_1}^\nu - V^\omega \sum_1^r \Gamma_{a_s \omega}^c K_{a_r \dots a_{s+1} c a_{s-1} \dots a_1}^\nu] + \dots \\ = K_{a_r \dots a_1}^\nu + \epsilon \left\{ V^\omega \partial_\omega K_{a_r \dots a_1}^\nu + L_{a_r \dots a_1}^\nu - V^\omega \sum_1^r \Gamma_{a_s \omega}^c K_{a_r \dots a_{s+1} c a_{s-1} \dots a_1}^\nu \right\} + \dots.$$

Substituting for $*K_{a_r}^\nu \dots a_1$ from (6.9) in

$$*D_{a_{r+1}} *K_{a_r \dots a_1}^\nu \equiv *K_{a_{r+1} \dots a_1}^\nu \\ \doteq \left(\partial_{a_{r+1}} *K_{a_r \dots a_1}^\nu + * \Gamma_{\lambda \mu}^\nu *T_{a_{r+1}}^\mu *K_{a_r \dots a_1}^\lambda - \sum_1^r * \Gamma_{a_s a_{r+1}}^c *K_{a_r \dots a_{s+1} c a_{s-1} \dots a_1}^\nu \right),$$

we obtain according to (2.4)

$$*K_{a_{r+1} \dots a_1}^\nu \doteq \partial_{a_{r+1}} K_{a_r \dots a_1}^\nu + \epsilon \left\{ \partial_{a_{r+1}} L_{a_r \dots a_1}^\nu + (\partial_{a_{r+1}} V^\omega) (\partial_\omega K_{a_r \dots a_1}^\nu) \right. \\ + V^\omega \left[\partial_{a_{r+1}} \partial_\omega K_{a_r \dots a_1}^\nu - \sum_1^r (\partial_{a_{r+1}} \Gamma_{a_s \omega}^c) K_{a_r \dots a_{s+1} c a_{s-1} \dots a_1}^\nu \right. \\ \left. \left. + (\partial_\omega \Gamma_{\lambda \mu}^\nu) T_{a_{r+1}}^\mu K_{a_r \dots a_1}^\lambda \right] - \sum_1^r W_{a_s a_r}^c K_{a_r \dots a_{s+1} c a_{s-1} \dots a_1}^\nu \right\} + \dots.$$

On the other hand

$$\begin{aligned}
K_{a_{r+1}\dots a_1}^{\nu'} &= K_{a_{r+1}\dots a_1}^\nu + \epsilon V^\omega \partial_\omega D_{a_{r+1}} K_{a_r\dots a_1}^\nu + \dots \\
&\doteq K_{a_{r+1}\dots a_1}^\nu + \epsilon V^\omega \left\{ \partial_\omega \partial_{a_{r+1}} K_{a_r\dots a_1}^\nu + (\partial_\omega \Gamma_{\lambda\mu}^\nu) T_{a_{r+1}}^\mu K_{a_r\dots a_1}^\lambda \right. \\
&\quad \left. - \sum_1^r {}^s (\partial_\omega \Gamma_{a_s a_{r+1}}^c) K_{a_r\dots a_{s+1} c a_{s+1} \dots a_1}^\nu \right\} + \dots, \\
{}^0 K_{a_{r+1}\dots a_1}^\nu &\doteq {}^0 K_{a_{r+1}\dots a_1}^\nu \doteq K_{a_{r+1}\dots a_1}^\nu, \\
\partial_{a_{r+1}} K_{a_r\dots a_1}^\nu &\doteq D_{a_{r+1}} K_{a_r\dots a_1}^\nu = K_{a_{r+1}\dots a_1}^\nu.
\end{aligned}$$

Hence

$$\begin{aligned}
(6.10) \quad *K_{a_{r+1}\dots a_1} - {}^0 K_{a_{r+1}\dots a_1}^\nu - (K_{a_{r+1}\dots a_1}^{\nu'} - {}^0 K_{a_{r+1}\dots a_1}^\nu) &\doteq *K_{a_{r+1}\dots a_1}^\nu - K_{a_{r+1}\dots a_1}^{\nu'} \\
&\doteq \epsilon \left\{ \partial_{a_{r+1}} L_{a_r\dots a_1}^\nu + (\partial_{a_{r+1}} V^\mu) (D_\mu K_{a_r\dots a_1}^\nu) + V^\omega \left[(\partial_{a_{r+1}} \partial_\omega - \partial_\omega \partial_{a_{r+1}}) K_{a_r\dots a_1}^\nu \right. \right. \\
&\quad \left. \left. + \sum_1^r {}^s (\partial_\omega \Gamma_{a_s a_{r+1}}^c - \partial_{a_{r+1}} \Gamma_{a_s \omega}^c) K_{a_r\dots a_{s+1} c a_{s-1} \dots a_1}^\nu \right] \right. \\
&\quad \left. - \sum_1^r {}^s W_{a_s a_{r+1}}^c K_{a_r\dots a_{s+1} c a_{s-1} \dots a_1}^\nu \right\} + \dots.
\end{aligned}$$

Substituting in (6.10) from

$$\begin{aligned}
\partial_{a_{r+1}} L_{a_r\dots a_1}^\nu &\doteq D_{a_{r+1}} L_{a_r\dots a_1}^\nu, \\
(\partial_{a_{r+1}} V^\omega) (\partial_\omega K_{a_r\dots a_1}^\nu) &\doteq (L_{a_{r+1}}^\omega + V^\mu D_\mu T_{a_{r+1}}^\omega) D_\omega K_{a_r\dots a_1}^\nu, \\
V^\omega (\partial_{a_{r+1}} \partial_\omega - \partial_\omega \partial_{a_{r+1}}) K_{a_r\dots a_1}^\nu &= -V^\omega (\partial_\omega T_{a_{r+1}}^\mu) \partial_\mu K_{a_r\dots a_1}^\nu \\
&\doteq -V^\omega (D_\omega T_{a_{r+1}}^\mu) D_\mu K_{a_r\dots a_1}^\nu, \\
-V^\omega (\partial_{a_{r+1}} \Gamma_{a_s \omega}^c - \partial_\omega \Gamma_{a_s a_{r+1}}^c) &\doteq -V^\omega T_{a_{r+1}}^\mu P_{\omega \mu a_s}^c
\end{aligned}$$

one obtains

$$\Delta K_{a_{r+1}\dots a_1}^\nu \doteq L_{a_{r+1}\dots a_1}^\nu.$$

This equation being a tensorial one, one may write it with $=$ instead of $\doteq$ and the theorem follows easily.

7. System of partial differential equations (4.1). In this section we shall assume $m > 1$ and consider the system (4.1) which we write in the form

$$(7.1) \quad \nabla_a V^\nu = V^\mu D_\mu T_a^\nu + L_a^\nu$$

and where we shall assume L_a^ν to be given *a priori* and V^ν to be the unknown field. Hence we have an overdetermined system of $m + n$ equations for n unknowns and it is to be expected that a solution may be obtained—at least partly—algebraically. In order to do so we prove first the following:

Lemma (7.1). *The skew symmetric part of the tensor R_{ba}^c satisfies the condition.*

$$(7.2) \quad R_{[ba]}^c = V^i P_{i[ba]}^c.$$

Proof. Remembering $W_{abd} = W_{(ab)d}$ and (2.9), we obtain (7.2) at once from (6.2b).

Note. The equation (6.6) simplifies by virtue of (1.2) to

$$(7.3) \quad D_{[b} L_{a]}^\nu + (D_\mu T_{[a}^\nu) L_{b]}^\mu = V^i P_{i[b a]}^c T_{c]}^\nu.$$

Theorem (7.1). *The equation (7.3) is a consequence of the integrability condition*

$$(7.4) \quad D_{[b} \nabla_{a]} V^\nu = D_{[b} L_{a]}^\nu + D_{[b} V^\mu D_{|\mu|} T_{a]}^\nu$$

of (7.1).

Proof. First we have

$$(7.5) \quad D_{[b} \nabla_{a]} V^\nu = -\frac{1}{2} T_b^\omega T_a^\mu R_{\omega\mu\lambda}^\nu V^\lambda.$$

On the other hand,

$$(7.6a) \quad D_{[b} V^\mu D_{|\mu|} T_{a]}^\nu = (D_\mu T_{[a}^\nu) (L_{b]}^\mu + V^\omega D_{|\omega|} T_{b]}^\mu) + V^\mu D_{[b} D_{|\mu|} T_{a]}^\nu.$$

This equation may be simplified, for

$$0 = V^\mu D_\mu T_{[b}^\omega D_{|\omega|} T_{a]}^\nu = (V^\mu D_\mu T_{[b}^\omega) D_{|\omega|} T_{a]}^\nu + V^\mu T_{[b}^\omega D_{|\mu|} D_{|\omega|} T_{a]}^\nu.$$

Hence

$$(7.6b) \quad \begin{aligned} D_{[b} V^\mu D_{|\mu|} T_{a]}^\nu &= (D_\mu T_{[a}^\nu) L_{b]}^\mu + V^\mu T_{[b}^\omega (D_{|\omega|} D_{|\mu|} - D_{|\mu|} D_{|\omega|}) T_{a]}^\nu \\ &= (D_\mu T_{[a}^\nu) L_{b]}^\mu - V^\mu T_{[b}^\omega T_{a]}^\lambda R_{\omega\mu\lambda}^\nu + V^\mu P_{\omega\mu[a}^c T_{b]}^\omega T_{c]}^\nu. \end{aligned}$$

Substituting from (7.5) and (7.6b) in (7.4) and remembering

$$R_{\omega\mu\lambda}^\nu + R_{\mu\lambda\omega}^\nu + R_{\lambda\omega\mu}^\nu = 0,$$

we obtain (7.3) by virtue of (2.9b).

As long as it is not stated otherwise we shall assume that the tensor P_{ba}^i is of rank m so that there is a tensor R^{ca} satisfying

$$(7.7) \quad P_{ba}^i R^{ca} = \delta_b^c.$$

This assumption enables us to prove the following

Theorem (7.2). *A necessary and sufficient condition that (7.1) admits a solution V^r is*

$$(7.8) \quad D_{[b}L_{a]}^r + (D_\mu T_{[a]}^r)L_{b]}^\mu = -2T_c^r P_{i[ba]}^c P_\lambda^j R^{ik} [D_{[j}L_{k]}^\lambda + (D_\mu T_{[k]}^\lambda)L_{j]}^\mu].$$

If this equation is satisfied then the tangential component W^r of V^r is $V^i T_i^r$, where

$$(7.9) \quad V^i = -2P_\lambda^j R^{ik} [D_{[j}L_{k]}^\lambda + (D_\mu T_{[k]}^\lambda)L_{j]}^\mu],$$

while the pseudonormal component N^r is a solution of

$$(7.10) \quad \nabla_a N^r = H_a^r + N^\mu D_\mu T_a^r,$$

where

$$(7.11) \quad H_a^r \equiv L_a^r - (D_a V^b) T_b^r.$$

The system (7.10) is completely integrable.

Proof. Suppose that (7.1) admit a solution. Then (7.3) holds and we obtain from it

$$(7.12) \quad P_i^c [D_{[b}L_{a]}^r + (D_\mu T_{[a]}^r)L_{b]}^\mu] = V^i P_{i[ba]}^c.$$

Because $P_{ba}^a = 0$ we have from (7.12)

$$P_i^c [D_{[b}L_{c]}^r + (D_\mu T_{[c]}^r)L_{b]}^\mu] = -\frac{1}{2} V^i P_{iab}^c = \frac{1}{2} V^i P_{cib}^c,$$

and this equation yields (7.9). Substituting from (7.9) in (7.3) we obtain (7.8). Suppose now that (7.8) is satisfied. If we decompose V^r according to (2.8) and remember that

$$D_a W^r - W^\mu D_\mu T_a^r = (D_a V^c) T_c^r + V^c D_a T_c^r - V^c D_c T_a^r = (D_a V^c) T_c^r,$$

we obtain from (7.1) the equation (7.10) with two sets of unknowns, namely V^c and N^r . If we select V^c arbitrarily we can not assume *a priori* that N^r satisfying (7.10) is pseudonormal. The integrability conditions of (7.10) are based on

$$D_{[b} \nabla_{a]} N^r = D_{[b} H_{a]}^r + D_{[b} N^\mu D_{|\mu|} T_{a]}^r$$

and yield (by means of the method used in the proof of Theorem (7.1))

$$(7.13a) \quad D_{[b} H_{a]}^r + (D_\mu T_{[a]}^r) H_{b]}^\mu = N^i P_{i[ba]}^c T_c^r,$$

where

$$N^i \equiv N^\lambda P_\lambda^i$$

is not equal to zero in general (for arbitrarily fixed V^c). Equations (7.13a) may

be rewritten: first of all we have

$$-[D_{[b}(D_{a]}V^c)T_c^\nu + (D_\mu T_{[a}^\nu)(D_{b]}V^c)T_c^\mu] = \frac{1}{2}P_{ba}^c V^i T_c^\nu = -P_{i[ba]}^c V^i T_c^\nu.$$

Consequently (7.13a) reduces by virtue of (7.11) to

$$(7.13b) \quad D_{[b}L_{a]}^\nu + (D_\mu T_{[a}^\nu)L_{b]}^\mu - V^i P_{i[ba]}^c T_c^\nu = N^i P_{i[ba]}^c T_c^\nu.$$

For V^i given by (7.9) and only for such a V^i the left hand member of (7.13b) vanishes by virtue of (7.8):

$$(7.13c) \quad N^i P_{i[ba]}^c = 0,$$

and this equation yields $N^i = 0$ because the rank of $P_{c ba}^c$ is m . Hence there is only one value of V^i , namely (7.9) such that if the solution N^ν of (7.10) exists it is pseudonormal. The integrability conditions (7.13a) of (7.10), where V^i is given by (7.9), reduce in virtue of (7.13c) to

$$D_{[b}L_{a]}^\nu + (D_\mu T_{[a}^\nu)L_{b]}^\mu = V^i P_{i[ba]}^c T_c^\nu,$$

and this equation (with V^i given by (7.9)) reduces to the identity $0 \equiv 0$ by virtue of (7.8). Hence the conditions of integrability of (7.10) (where V^i is given by (7.9)) are identically satisfied, and the system is completely integrable.

We say that a tensor L_a^ν is *admissible* if it satisfies (7.8).

Theorem (7.3). *Let C be any arbitrary (real) constant, which may even be equal to zero. Then*

$$(7.14) \quad L_a^\nu = C T_a^\nu.$$

is an admissible tensor, and the solution V^ν of (7.1) is pseudonormal.

Proof. If (7.14) holds we have

$$D_{[b}L_{a]}^\nu + (D_\mu T_{[a}^\nu)L_{b]}^\mu = C\{D_{[b}T_{a]}^\nu + D_{[b}T_{a]}^\nu\} = 0,$$

and (7.8) is satisfied, while (7.9) yields $V^i = 0$.

Note: This theorem may be easily generalized: *A sufficient condition that (7.1) admits a pseudonormal solution is*

$$D_{[b}L_{a]}^\nu + (D_\mu T_{[a}^\nu)L_{b]}^\mu = 0.$$

The proof follows from (7.8) and (7.9).

III. APPLICATION

8. A preserved family (V_m). In this section we do not assume first that the rank of P_{ba}^i be m .

Lemma (8.1). *Let*

$$(8.1a) \quad \xi^\nu = \varphi^\nu(\tau; \nu)$$

be the equations of a V_m (of our (V_m)) which goes through a point ξ^ν . Let

$$(8.1b) \quad \xi^\nu = \varphi^\nu(\tau; * \nu)$$

be the equation of another V_m (of our (V_m)) which goes through a point

$$(8.2) \quad * \xi^\nu = \xi^\nu + \epsilon V^\nu \quad (V^\nu \equiv V^\nu(\xi)).$$

Then

$$(8.3) \quad * \nu^\epsilon = \nu^\epsilon + \epsilon V^\epsilon + \dots \quad (V^\epsilon \equiv V^\epsilon M_\lambda^\epsilon(\xi)).$$

Proof. We obtain from the equation (1.5b)

$$\nu^\epsilon = \mu^\epsilon(\xi),$$

and by virtue of (1.6)

$$\begin{aligned} * \nu^\epsilon &= \mu^\epsilon(* \xi) = \mu^\epsilon(\xi) + \epsilon \left(V^\lambda \frac{\partial \mu^\epsilon}{\partial \xi^\lambda} \right)_\xi + \dots \\ &= \nu^\epsilon + \epsilon (V^\lambda M_\lambda^\epsilon)_\xi + \dots = \nu^\epsilon + \epsilon V^\epsilon + \dots \end{aligned}$$

Note: Starting with a V_m (of our (V_m)) given by (8.1a) we obtain another V_m of the same family given by (8.1b) and (8.3). However, if we apply on every point ξ of the V_m given by (8.1a) the operation

$$(8.4) \quad * \xi^\nu = \xi^\nu + \epsilon V^\nu$$

we obtain a $*V_m$

$$(8.5a) \quad \xi^\nu = \psi^\nu(\tau; \nu),$$

where

$$(8.5b) \quad \psi^\nu \equiv \varphi^\nu(\tau; \nu) + \epsilon V^\nu(\varphi(\tau; \nu)).$$

We assumed that the set of all $*V_m$ obtained in this way from all V_m 's constitutes a family $(*V_m)$. We use these notions in the following

Definition (8.1). *We say that the family (V_m) is preserved under (8.4), if for each point of any individual V_m (given in (V_m)) by special constant values of the ν 's) we have*

$$(8.6a) \quad \lim_{\epsilon \rightarrow 0} \frac{\psi^\nu(\tau; \nu) - \varphi^\nu(\tau; * \nu)}{\epsilon} = 0,$$

where

$$(8.6b) \quad *v^t = v^t + \epsilon V^t + \dots$$

In the following consideration we assume again that P_{ba}^i be of rank m .

Theorem (8.1). *A necessary and sufficient condition that (V_m) be preserved under (8.4) is*

$$(8.7) \quad L_a^v = 0,$$

so that V^v is pseudonormal.

Proof. We have first by virtue of (8.6b) and (1.6)

$$(8.8a) \quad \varphi'(\tau; *v) = \varphi'(\tau; v) + \epsilon V^t N_t^v + \dots,$$

$$(8.8b) \quad \begin{aligned} \frac{\partial \varphi'(\tau; *v)}{\partial \tau^a} &= T_a^v + \epsilon \left\{ \frac{\partial V^t}{\partial \tau^a} N_t^v + V^t \frac{\partial^2 \varphi'(\tau, v)}{\partial \tau^a \partial v^t} \right\} + \dots \\ &= T_a^v + \epsilon \left\{ \frac{\partial V^t}{\partial \tau^a} N_t^v + V^t \frac{\partial}{\partial v^t} T_a^v \right\} + \dots, \end{aligned}$$

$$(8.8c) \quad \frac{\partial \psi^v}{\partial \tau^a} = T_a^v + \epsilon \partial_a V^v + \dots,$$

and consequently

$$(8.9a) \quad \psi^v(\tau; v) - \varphi^v(\tau; *v) = \epsilon(V^v - V^t N_t^v) + \dots,$$

$$(8.9b) \quad \frac{\partial}{\partial \tau^a} (\psi^v(\tau; v) - \varphi^v(\tau; *v)) = \epsilon(\partial_a V^v - N_t^v \partial_a V^t - V^t \frac{\partial}{\partial v^t} T_a^v) + \dots$$

Suppose now that our (V_m) is preserved, so that (8.6) hold. Consequently we must have from (8.9a, b)

$$(8.10a) \quad V^v = V^t N_t^v,$$

$$(8.10b) \quad \varphi^v(\tau; *v) = \psi^v(\tau; v) + \dots,$$

and

$$(8.10c) \quad \partial_a V^v - N_t^v \partial_a V^t - V^t \frac{\partial}{\partial v^t} T_a^v = 0.$$

Starting with a particular V_m whose parameters v are constant $v = c$, we obtain $a * V_m$ with the same parameters and another V_m with the *constant* parameters $*v$. (These two subspaces are related by (8.10b).) Consequently we have from (8.6b)

$$(8.11a) \quad \frac{\partial V^t}{\partial \tau^a} = 0.$$

On the other hand we have from (8.10a) $W^\nu = V^a T_a^\nu = 0$, and consequently

$$(8.11b) \quad V^\epsilon \frac{\partial}{\partial \nu^\epsilon} = V^\lambda \left(\frac{\partial \nu^\epsilon}{\partial \xi^\lambda} \frac{\partial}{\partial \nu^\epsilon} + \frac{\partial \tau^a}{\partial \xi^\lambda} \frac{\partial}{\partial \tau^a} \right) = V^\lambda \frac{\partial}{\partial \xi^\lambda}.$$

Therefore the equation (8.10c) simplifies by virtue of (8.11a) to

$$(8.12) \quad \partial_a V^\nu - V^\lambda \partial_\lambda T_a^\nu = 0.$$

If we add $\Gamma_{\lambda\mu}^\nu T_a^\mu V^\lambda$ and $\Gamma_{a\lambda}^c V^\lambda = \Gamma_{a\lambda}^c N^\lambda = 0$ and subtract $\Gamma_{\lambda\mu}^\nu T_a^\mu V^\lambda$ we see that (8.12) is equivalent to (8.7). Hence (8.7) is necessary. Its sufficiency may be proved in the following way: take a particular V_m (of our family (V_m)) given by particular values $\nu = \nu$. Comparing (8.5) and Lemma (8.1) we obtain

$$(8.13) \quad \psi^\lambda(\nu; \nu) - \varphi^\lambda(\tau; \star \nu) = \epsilon(V^\lambda - V^\epsilon N_\epsilon^\lambda)_{\nu=\nu} + \dots$$

and (8.3). If now (8.7) is satisfied, we have (according to Theorem (7.3)) the equation (8.10a) so that (8.13) reduces to

$$(8.14) \quad \varphi^\nu(\tau, \star \nu) = \psi^\nu(\tau; \nu) + \dots$$

The equations (8.14) and (8.3) being valid for an arbitrary set $\nu = \nu$ we may drop the index 0 and obtain (8.6a, b).

Theorem (8.2). *If the family (V_m) is preserved under (8.4) we have*

$$(8.15a) \quad \Delta g_{ab} = 0$$

$$(8.15b) \quad \Delta \Gamma_{ab}^c = 0$$

$$(8.15c) \quad \Delta K_{a_r \dots a_1}^\nu = 0, \quad (r = 1, \dots, N; K_a^\nu \equiv T_a^\nu).$$

Proof. The equation (8.15a) follows from (8.7) and (5.3b). On the other hand we have from (8.7) and (4.6)

$$(8.16) \quad W_{ab}^c = T_a^\mu N^\omega P_{\mu\omega b}^c,$$

so that we obtain by virtue of (5.6) and (2.9a)

$$(8.17) \quad Q_{ab}^c = T_a^\mu N^\omega P_{\mu\omega b}^c - N^\omega T_b^\mu P_{\mu\omega a}^c = 2N^\omega \partial_\omega \Gamma_{[ab]}^c = 0,$$

and (8.15b) follows from (8.17) and (5.5b). The first of (8.15c) follows at once from (8.7) and (5.1b). From (6.2b) and (8.16) we obtain for $V^\omega = N^\omega$

$$(8.18) \quad R_{ba}^c = N^\omega (T_b^\mu P_{\omega\mu a}^c - T_a^\mu P_{\omega\mu b}^c) = 0.$$

Hence if (8.7) holds we obtain from (8.18) and (6.7)

$$(8.19) \quad L_{a_r \dots a_1}^\nu = 0, \quad (r = 2, 3, \dots, N).$$

9. Totally geodesic subspaces. In this section we at first do not make any

assumption about the rank of P_{ibc}^i , and we shall be concerned with a family of totally geodesic V_m 's:

$$(9.1a) \quad D_a T_b^\nu \equiv K_{ab}^\nu = 0.$$

If $m > 1$ then the integrability conditions of (9.1a), namely

$$(9.1b) \quad P_{cba}^d T_d^\nu = T_c^\omega T_b^\mu T_a^\lambda R_{\omega\mu\lambda}^\nu,$$

impose some restriction on the structure of the large space. In the first place we prove the following

Lemma (9.1). *Put*

$$(9.2) \quad v_c \equiv V_\lambda T_c^\lambda.$$

If (9.1a) holds then

$$(9.3) \quad W_{bad} = P_{bda}^c v_c + D_a D_b v_d.$$

Proof. If (9.1a) holds then we have for the tensor W_{ab} defined by (3.3a)

$$W_{ab} = T_{(a}^\lambda \nabla_{b)} V_\lambda = D_{(b} T_{a)}^\lambda V_\lambda = D_{(b} v_{a)}.$$

Consequently we get from (3.3b)

$$(9.4) \quad W_{abd} = D_{[a} D_{d]} v_b + D_{[b} D_{d]} v_a + D_{[b} D_{a]} v_d + D_a D_b v_d.$$

Since for any vector P_a whatsoever

$$D_{[a} D_{d]} P_b + D_{[d} D_{b]} P_a + D_{[b} D_{a]} P_d = 0, \quad D_{[b} D_{d]} P_a = \frac{1}{2} P_{bda}^c P_c,$$

(9.4) reduces to (9.3).

Definition (9.1a). *If $A = 0$ is an analytic expression for a property of (V_m) then we say that this property is preserved under (8.4), if $\Delta A = 0$.*

Theorem (9.1a). *Let (V_m) be a family of totally geodesic subspaces. A necessary and sufficient condition that this property be preserved by (8.4) is: The vector V^ν is a solution of any one of the following equations:*

$$(9.5a) \quad L_{ab}^\nu = 0$$

$$(9.5b) \quad D_b D_a V^\nu - V^\omega T_b^\mu T_a^\lambda R_{\omega\mu\lambda}^\nu - W_{ba}^c T_c^\nu = 0.$$

Proof. The necessity and sufficiency of (9.5a) follows at once from (6.3). If (9.1a) is satisfied we obtain

$$K_{ab}^{\nu'} = K_{ab}^{\nu'} + \epsilon V^\omega \partial_\omega K_{ab}^\nu + \dots = 0 + \dots,$$

$${}_0 K_{ab}^\nu = K_{ab}^\nu + \epsilon V^\mu \{ \Gamma_{a\mu}^c K_{cb}^\nu + \Gamma_{b\mu}^c K_{ac}^\nu - \Gamma_{\lambda\mu}^\nu K_{ab}^\lambda \} + \dots = 0 + \dots,$$

* We have in general $v_c \neq V^a g_{ac}$, $V_\lambda^\alpha \equiv P^\alpha V_\lambda$.

and consequently

$$\omega K_{ab}^{\nu} = 0.$$

Hence

$$(9.6) \quad \Delta K_{ab}^{\nu} = \tau K_{ab}^{\nu},$$

and the statement about (9.5b) follows from (3.4d).

In the next theorem we shall express the condition (9.5b) by means of the normal component n^{ν} of V^{ν} . If we put

$$v^{\nu} \equiv V_{\mu} T_b^{\mu} T_a^{\nu} g^{ba} = v^a T_a^{\nu}$$

then the normal component n^{ν} of V^{ν} is defined by

$$(9.7a) \quad n^{\nu} = V^{\nu} - v^{\nu,*} \quad n^{\lambda} v^{\mu} g_{\lambda\mu} = 0.$$

Theorem (9.1b). *If (9.1a) holds then the equation (9.5b) is equivalent to*

$$(9.7b) \quad D_b D_a n^{\nu} - n^{\omega} T_b^{\mu} T_a^{\lambda} R_{\omega\mu\lambda}^{\nu} = 0.$$

Proof. From (9.7a) and (9.5b) we have

$$(9.8) \quad D_b D_a n^{\nu} - n^{\omega} T_b^{\mu} T_a^{\lambda} R_{\omega\mu\lambda}^{\nu} = -(D_b D_a v^{\nu} - v^{\omega} T_b^{\mu} T_a^{\lambda} R_{\omega\mu\lambda}^{\nu} - W_{ba}^c T_c^{\nu}).$$

But if (9.1a) holds we have

$$(9.9a) \quad D_b D_a v^{\nu} = (D_b D_a v^c) T_c^{\nu},$$

and by virtue of (9.1b)

$$(9.9b) \quad -v^{\omega} T_b^{\mu} T_a^{\lambda} R_{\omega\mu\lambda}^{\nu} = -v^c T_c^{\omega} T_b^{\mu} T_a^{\lambda} R_{\omega\mu\lambda}^{\nu} = -v^c P_{cba}^d T_d^{\nu}.$$

Consequently the right hand member of (9.8) reduces according to (9.3) to

$$\begin{aligned} & -[D_b D_a v^c - v^d P_{dba}^c - v_e P_{bda}^e g^{dc} - D_a D_b v^c] T_c^{\nu} \\ & = -[-P_{bad}^c - P_{dba}^c + P_{dab}^c] v^d T_c^{\nu} \\ & = +[P_{bad}^c + P_{adb}^c + P_{dba}^c] v^d T_c^{\nu} \equiv 0. \end{aligned}$$

Theorem (9.1c). *Let the rank of P_{iba}^i be m and assume that (V_m) consists of totally geodesic subspaces. A sufficient (but not necessary) condition that this property be preserved is (8.7). If this condition is satisfied then (V_m) is preserved.*

Proof. If P_{iba}^i is of rank m then Theorem (8.1) holds and consequently (8.18) holds and we have (9.5a) which is equivalent in our case to (9.5b) and (9.7b). However, we may have (9.5) without (8.7) being satisfied. Then the property (9.1a) is preserved but (V_m) is not preserved.

* The pseudonormal component N^{ν} is in general different from the normal component.

Note I. The statement of the theorem (9.1a) about the equation (9.5b) is a generalization of Levi-Civita's result for $m = 1$,[†] while the theorem (9.1b) reduces for $m = 1$, $n = 2$ to the well known result by Jacobi.

Note II. The equation (1.1) may be looked upon as defining a family (V_{n-m}) of subspaces V_{n-m} , each described by the ν 's and defined by constant values of the τ 's. The corresponding intrinsic deformation theory of (V_{n-m}) may be developed along the same lines as the intrinsic deformation theory of (V_m) , with N'_τ and M'_λ playing the role of T'_a and P'_λ and vice versa.

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[†] Cf. (2) and (3).