Existence Theorems for Some Hydrodynamical Free Boundary Problems

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PART I. INTRODUCTION

1. The problem of the wake. Suppose that a fixed obstacle (arc) AB is immersed in a uniform plane flow of an ideal incompressible fluid. We shall con-

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sider flows in which a *wake* or *cavity* forms behind AB , the flows being characterized by the following conditions:

- 1) the flow is steady and irrotational;
- 2) the flow is bounded by the arc AB , by a streamline Σ detaching from B , and by a streamline Σ^* detaching from A ;
- 3) the flow speed on Σ and Σ^* is constant.

It is assumed, of course, that Σ and Σ^* are sufficiently smooth for the complex velocity $u + iv$ to exist and be continuous on them. Σ and Σ^* are customarily referred to as *free streamlines*; the constant speed will be called the *cavity speed*.

With the addition of a fourth condition we obtain the usual set of postulates for the *Helmholtz plane wake problem* [20].[‡] Indeed this problem may be formulated as follows: given the obstacle AB , find a flow satisfying conditions 1-3), and

- 4a) Σ and Σ^* extend downstream to infinity.

FIG. 1. Flow plane, Helmholtz problem

It is convenient to denote the point at infinity by E , and to denote the stagnation point on AB by O (figure 1). Helmholtz's investigation of this problem [1] initiated the study of the free boundary problem of classical hydrodynamics. It is important to observe that intersection of the free streamlines is not excluded by our formulation, and that the cavity speed in this flow model must be equal to the speed at infinity.

It has been noticed that condition 4a) may be replaced by alternate proposals which give rise to equally interesting problems. It is the purpose of this paper to consider the existence of solutions of three problems arising in this manner, and to present, in simpler form than has appeared before, a proof that solutions of the Helmholtz problem exist.

The first of the alternate problems is obtained by replacing 4a) in the state-

[‡] Numbers in brackets refer to the bibliography at the end of the paper.

ment of the Helmholtz problem by

4b) the cavity shall be finite in extent.

In this case it is apparent that i) the free streamlines meet and end at a point E behind the obstacle, and ii) the point at infinity, denoted by I , is an interior point of the flow (figure 2). We shall call this problem the *finite cavity problem*.¹ Again the cavity speed is not arbitrary; when the flow is symmetric it can be shown to be less than the speed at infinity (§10).

We shall see in the course of the paper that, for the class of obstacles considered, there is a problem which is in many ways more natural than either of the preceding. Namely, given the obstacle AB , find *either* a flow solving the finite cavity problem, *or* a flow solving the Helmholtz problem, subject in the latter case to the additional condition that, sufficiently distant from the obstacle the free streamlines do not overlap. It has been shown elsewhere (see §§9 and 10) that there is no more than one symmetric solution of this problem for a given starlike obstacle, and that in this solution the free streamlines nowhere intersect.

FIG. 2. Flow plane, finite cavity problem

Because of the latter property this problem will be called the *schlicht cavity problem*.

Through consideration of cavitation phenomena,² Wagner was led to propose the alternate condition

4c) Σ and Σ^* turn back, the flow forming a reentrant jet.

In order to preserve a steady flow, it is assumed that the jet is not interrupted by striking the rear of the obstacle, but rather forms a second sheet of the

¹ Apparently, attention was first drawn to the alternative 4b) by Schmieden [6]. Other theoretical work on this model is due to Kolscher [13] and Lavrentieff [11]. Tiller [17] has also considered the finite cavity problem, but his work is in large part incorrect.

² If a flow model based on conditions 1)–3) is proposed to represent cavitation phenomena, it must possess the following property: if V is the cavity speed and U is the speed at infinity, it must be possible to preassign V such that the cavitation number

$$N \equiv (V^2 - U^2)/U^2,$$

may take on any value $N > 0$. None of the preceding models has this property, but we shall see that the model proposed by Wagner (his work is unpublished) is, in this respect, satisfactory. For a more adequate discussion see Gilbarg & Serrin [19].

flow surface (figure 3). The point at infinity of the main flow plane, denoted by I , is an interior point of the flow, while the point at infinity on the jet, denoted by E , is a boundary point. The problem of determining a flow satisfying conditions 1)–3) and 4c) will be called the *reentrant jet problem*.

In the reentrant jet model the cavity speed as well as the speed of the incident flow may be arbitrarily preassigned. This is, in fact, the striking advantage of this model. Although cavitation phenomena indicates that the cavity speed should be larger than the speed at infinity, we shall not retain this restriction. We shall see (§19) that the reentrant jet and schlicht cavity problems are connected in the following remarkable way: a symmetric solution of the former problem exists for a given starlike obstacle if and only if the preassigned cavity speed is greater than the cavity speed in the (unique) symmetric solution of the latter problem.

2. Methods and results. We shall consider obstacles AB which are simple arcs with continuously turning tangents (§4), and, *except in Part II, the shape of the obstacle and the flow itself are restricted to be symmetric about a line through O parallel to the direction of the uniform flow.*

FIG. 3. Flow plane, reentrant jet problem

The brilliant analytical formulation of the Helmholtz problem, due to Levi-Civita [2] and Villat [4], is briefly derived in Part II. Villat's integral equation, whose solution enables us to construct the required flow, is an immediate consequence of the formulation. With this work as a guide, a similar formulation is obtained for the other problems (§5–7). Finally we develop certain functional equations which correspond, in their respective settings, to the Villat equation.

Parts III and IV are devoted to the proof that, for any regular obstacle, there exist symmetric flows solving the Helmholtz problem (§16), the schlicht cavity problem (§17), and the reentrant jet problem (§19). Existence theorems for the two former problems have already been given by Leray [8] and Lavrentieff [11]; however, the class of obstacles which we consider neither includes the class of obstacles discussed by Leray nor is it included in that class, while our results include those of Lavrentieff as special cases. The existence of flows solving the finite cavity problem is shown under appropriate conditions (Theorem 18.2); this seems to be a new result.

Following Leray, Kravtchenko [14], and others, the existence theorems are based on the fixed point theory of Leray and Schauder [7]. The method of limit-

ing the unknown $l(s)$ is, I believe, new and simpler than that given by the earlier writers. It is based on a theorem, due to Warschawski [5], on the behavior of the derivative of the mapping function on the boundary of a partially known domain. The application of Warschawski's theorem depends on the results of §§9 and 10 where the geometry of the free streamline is discussed in detail.

The existence of the Helmholtz flow where $\omega'(0) = 0$ is shown under appropriate conditions (§18). This flow is thought to be of importance in wing theory [15], [16]. Finally it is possible to give a simple proof that there exists a solution to the problem of detachment *en proue* (§18).

There are several histories and bibliographies of the Helmholtz problem, of which I mention Weinstein [9], [10], von Kármán [12], and Kravtchenko. In addition, there is a bibliography of the reentrant jet problem in Gilbarg & Serrin [19].

Terminology. The letter V will be used consistently to denote the cavity speed in the problems we consider (except in the Helmholtz problem, where the cavity speed is the same as the uniform speed U at infinity). Occasionally, when we wish to specify to which one of several flows a cavity speed V corresponds, we use subscripts: thus V_J , V_k , etc. This is not to be confused with the symbol $V(P, R)$ which is used to denote the speed of some flow at a point P in the flow.

Functional transformations and function spaces will be denoted by script capitals: $\mathfrak{F}$, $\mathfrak{S}$, etc.

In general the same letter will be used for analogous quantities appearing in the study of different flow problems. This should not be a cause of confusion, the context indicating the particular meaning. I have kept the same notation throughout the paper, so that if a symbol is not defined explicitly, its meaning can always be determined by a reference to the basic work in Part II.

I wish to express my thanks to Professor David Gilbarg for his continuing interest in this work, and for his invaluable help and encouragement. Also my thanks are due Dr. Jerald Ericksen for the many suggestions he has made.

PART II. DERIVATION OF THE CENTRAL FUNCTIONAL EQUATIONS

3. Preliminaries. Suppose throughout this chapter, except where otherwise indicated, that the respective free boundary problems have been solved, in the sense that a flow of the required type has been exhibited.³ Let the region occupied by the fluid be denoted by R . The considerations of this preliminary section apply to all four models.

Let R be in the z -plane, where $z = x + iy$. Let the stagnation point O of the

³ The obstacle AB is not assumed to be symmetric in Part II.

obstacle be the origin and take the positive x -axis in the direction of the uniform incident flow. Let

$$w(z) = \varphi(x, y) + i\psi(x, y),$$

where $\varphi(x, y)$ is the velocity potential and $\psi(x, y)$ is the stream function of the flow; thus $w(z)$ is the well-known complex velocity potential. $w(z)$ is defined, up to an additive constant, throughout R , is analytic at all interior points of R (but possibly many valued in the second and third models), and continuous together with its first derivative throughout R . We may take $w(0) = 0$.

Now $dw/dz = qe^{-i\theta}$, where q is the magnitude and θ is the inclination of the

FIG. 4. ζ -plane, all models

velocity vector at a point z of R . Therefore we may define an analytic function in R by

$$Q(z) = \theta + i \log \frac{q}{V} = i \log \frac{1}{V} \frac{dw}{dz},$$

where V denotes the cavity speed. If U is the speed of the uniform incident flow, then for the Helmholtz problem V is equal to U ; for the jet problem V is a positive real number which is arbitrarily preassigned; and for the finite cavity problem V is a positive real number which cannot be arbitrarily preassigned but is one of the unknowns of the problem.

We consider only obstacles AB which have a continuously turning tangent. Let the region in the ζ -plane, $\zeta = \xi + i\eta$, whose points satisfy the inequalities $|\zeta| < 1$, $\eta > 0$, be denoted by Γ , and let Γ' denote that part of the boundary of Γ for which $|\zeta| = 1$. Then there exists a unique function $z(\zeta)$, defined throughout Γ , which maps Γ conformally onto the interior of R , such that the images of -1 , $+1$, and 0 are, respectively, B , A , and E ⁴ (figure 4). For ζ in Γ , define $w(\zeta) = w[z(\zeta)]$, and $\omega(\zeta) = Q[z(\zeta)]$.

⁴ Of course, E is a different point in the physical plane in each flow model (see §1). In figure 4 the points in the ζ -plane labeled A , B , E , etc. correspond under the mapping $z = z(\zeta)$ to the points A , B , E , of the flow plane; the same convention will be employed in later figures.

It is essential to note that with knowledge of $w(\zeta)$ and $\omega(\zeta)$ we can determine R and $w(z)$. For $z(\zeta)$ may be determined through the formula,

$$(3.1) \quad z(\zeta) = \frac{1}{V} \int^{\zeta} e^{i\omega(\zeta)} \frac{dw}{d\zeta} d\zeta,$$

R is the image of Γ by $z(\zeta)$, and w is given as a function of z through the parameter ζ . Hence the functions $w(\zeta)$ and $\omega(\zeta)$ serve to determine the required flow.

4. The Helmholtz problem. Here the free streamlines Σ and Σ^* extend downstream to infinity (figure 1). In this section we shall derive formulas for the functions $\omega(\zeta)$ and $w(\zeta)$, the formulas depending solely on a certain real function of a real variable, $l(s)$. We shall then derive an integral equation for $l(s)$.⁵

It follows from its definition that $w(z)$ is a single valued function which maps R conformally onto a plane region in the w -plane. The map is finite, in the sense that all interior points of R map into finite points; the point ∞ in the z -plane

FIG. 5. w -plane, Helmholtz problem

is a boundary point. Along that part of the boundary of R formed by OB and Σ , $w(z)$ is real and increases from 0 to ∞ ; along the remaining boundary arc consisting of OA and Σ^* , $w(z)$ again is real and increases from 0 to ∞ . The image of R is thus the entire w -plane bounded by the slit positive real axis (figure 5).

It is apparent now that $w(\zeta) = w[z(\zeta)]$ maps Γ conformally onto this latter region, such that the origin (E) in the ζ -plane corresponds to ∞ in the w -plane. Let the point on Γ' corresponding to O in the w -plane be $e^{i\theta_0}$. The mapping $w(\zeta)$ thus described is an elementary one and may easily be found:

$$(4.1) \quad w(\zeta) = 2M[\tfrac{1}{2}(\zeta + \zeta^{-1}) - \cos \theta_0]^2,$$

or, equivalently, in differential form,

$$(4.1a) \quad w'(\zeta) = M\zeta^{-3}(\zeta^2 - 1)(\zeta - e^{i\theta_0})(\zeta - e^{-i\theta_0}).$$

⁵ Although the work contained in this section has appeared elsewhere [4], [8], it is presented again with a twofold purpose: first, in order that the paper shall be largely self contained, and second, because our point of view, which seems most appropriate for extensions to the other models, is slightly different from that previously adopted.

A formula for $\omega(\zeta) = \theta(\zeta) + i\tau(\zeta)$ may likewise be given. It is, however, more complicated and depends on a real function rather than constants. From the definition, $\omega(\zeta)$ is analytic in Γ , real and continuous on the segment $(-1, 1)$ of the real axis, and $\omega(0) = 0$. In consequence, it may be continued analytically, by reflection, throughout the entire unit circle. Letting $\zeta = e^{is}$ on the circumference of the unit circle, the Schwarz-Poisson formula for $\omega(\zeta)$ reads

$$\omega(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \theta(e^{is}) \frac{e^{is} + \zeta}{e^{is} - \zeta} ds;$$

since $\theta(0) = 0$, we have,

$$\int_0^{2\pi} \theta(e^{is}) ds = 0.$$

Since $\theta(e^{is}) = \theta(e^{-is})$, these equations become

$$(4.2) \quad \omega(\zeta) = \frac{1}{\pi} \int_0^\pi \theta(e^{is}) \frac{1 - \zeta}{1 - 2\zeta \cos s + \zeta^2} ds,$$

$$(4.3) \quad \int_0^\pi \theta(e^{is}) ds = 0.$$

Let l , $0 \leq l \leq 2b$, denote arc length along AB measured from A . Recall that the obstacle AB has a continuous tangent, and let $\Psi(l)$ be the inclination of the tangent in terms of arc length. $\Psi(l)$ is assumed to satisfy a Hölder condition. Then the correspondence between Γ' and AB set up by the function $z = z(e^{is})$ determines a *boundary correspondence* $l = l(s)$, $0 \leq s \leq \pi$, between the argument of a point on Γ' and the arc length of the image point on AB ; it is convenient to assume that the real function $l(s)$, which is necessarily continuous, possesses a continuous derivative.⁶ Now $\theta(e^{is})$ is the inclination of the streamline at the point $z = z(e^{is})$ on the obstacle; thus

$$(4.4) \quad \theta(e^{is}) = \begin{cases} \Psi[l(s)] - \pi, & \text{for } 0 \leq s < s_0, \\ \Psi[l(s)], & \text{for } s_0 < s \leq \pi. \end{cases}$$

Eliminating $\theta(e^{is})$ from (4.2) by means of (4.4) we obtain the formula

$$(4.5) \quad \omega(\zeta) = \Omega(\zeta) + i \log \frac{\zeta - e^{is_0}}{\zeta e^{is_0} - 1},$$

where

$$(4.6) \quad \Omega(\zeta) = \Theta(\zeta) + iT(\zeta) = \frac{1}{\pi} \int_0^\pi \Psi[l(s)] \frac{1 - \zeta^2}{1 - 2\zeta \cos s + \zeta^2} ds.$$

⁶ The consequence of this assumption will be an integro-differential equation for $l(s)$. If this equation can be solved, then it is proved that $l(s)$ has a continuous derivative.

The function $\Omega(\zeta)$ is analytic for $|\zeta| < 1$. Since $\Psi[l(s)]$ satisfies a Hölder condition, a theorem of Fatou and Privaloff⁷ shows that $\Omega(\zeta)$ is continuous for $|\zeta| \leq 1$.

In order to complete the formulas for $w(\zeta)$ and $\omega(\zeta)$ it is only necessary to determine s_0 and M in terms of $l(s)$. From (4.3) and (4.4) we have

$$(4.7) \quad s_0 = \frac{1}{\pi} \int_0^\pi \Psi[l(s)] ds,$$

which gives s_0 . Now using (4.1a) and (4.5), equation (3.1) may be written in the form

$$(4.8) \quad \frac{dz}{d\zeta} = \frac{M}{U} e^{i\Omega(\zeta)} \frac{(\zeta^2 - 1)(\zeta - e^{-is_0})(\zeta e^{is_0} - 1)}{\zeta^3}.$$

If we put $\zeta = e^{is}$ in (4.8) and take absolute values, we get

$$(4.9) \quad \frac{dl}{ds} = \frac{8M}{U} \exp[-T(e^{is})] \sin^2 \frac{s + s_0}{2} \sin s. *$$

Integration of (4.9) between the limits $s = 0$ and $s = \pi$ leads to

$$(4.10) \quad b = \frac{4M}{U} \int_0^\pi \exp[-T(e^{is})] \sin^2 \frac{s + s_0}{2} \sin s ds,$$

where $2b$ is the length of the obstacle. Since $T(e^{is})$ and s_0 are given in terms of $l(s)$ by formulas (4.6) and (4.7), respectively, relation (4.10) determines M .

The Villat equation. With $T(e^{is})$, s_0 , and M defined by (4.6), (4.7), and (4.10), it is clear that (4.9) is a non-linear integro-differential equation, satisfied by the boundary correspondence $l(s)$. This is the Villat integro-differential equation.

Conversely, let $l(s)$, $0 \leq s \leq \pi$, be any function which solves the Villat equation, and for which $0 < s_0 < \pi$. On the basis of this function, let us define $w(\zeta)$ and $\omega(\zeta)$ by the formulas of this section. Then the flow which is determined by the functions $w(\zeta)$ and $\omega(\zeta)$ solves the Helmholtz problem for the obstacle defined by $\Psi(l)$. The existence theorem given in §16 is based on this result.

The Villat equation: symmetric case. When symmetry is assumed we need only consider the function $l(s)$, $0 \leq s \leq \pi/2$. For in this case, the equation

$$(4.11) \quad l(s) = 2l(\pi/2) - l(\pi - s), \quad \pi/2 \leq s \leq \pi,$$

defines $l(s)$ over the remainder of the interval $0 \leq s \leq \pi$.

Formulas (4.6), (4.7), and (4.10) become

⁷ The important theorem referred to states that if the real part of an analytic function defined in the unit circle, $|\zeta| < 1$, takes continuous values on $|\zeta| = 1$, and if these values satisfy a Hölder condition in the argument s , then the function itself satisfies a Hölder condition with the same exponent in $|\zeta| \leq 1$. The theorem is proved in Privaloff [3], and will be used frequently throughout the paper.

* For convenience in printing we shall occasionally use the symbol $\exp[x]$ in place of e^x .

$$(4.6a) \quad \Omega(\zeta) = \Theta(\zeta) + iT(\zeta) = \frac{1}{\pi} \int_0^\pi \Psi[l(s)] \frac{1 - \zeta^2}{1 - 2\zeta \cos s + \zeta^2} ds;$$

$$(4.7a) \quad s_0 = \pi/2;$$

$$(4.10a) \quad b = \frac{2M}{U} \int_0^\pi \exp[-T(e^{is})](1 + \sin s) \sin s \, ds.$$

The Villat integral equation becomes

$$(4.9a) \quad l(s) = \frac{4M}{U} \int_0^s \exp[-T(e^{is})](1 + \sin s) \sin s \, ds, \quad 0 \leq s \leq \pi/2.$$

Let $\mathcal{E}$ be the space of continuously differentiable real functions $l(s)$, $0 \leq s \leq \pi/2$, and let $\mathcal{E}^*$ be the subset of $\mathcal{E}$ whose elements satisfy the inequality $0 \leq l(s) \leq b$.

The expression on the right side of (4.6a) has meaning for an arbitrary $l(s) \in \mathcal{E}^*$. Hence, for $l(s) \in \mathcal{E}^*$, we may consider the expression on the right side of (4.9a), together with (4.6a) and (4.10a), as defining a *functional transformation* $\mathcal{V}_H$ of $l(s)$. Clearly $\mathcal{V}_H$ takes $\mathcal{E}^*$ into itself. The Villat equation becomes

$$(4.9b) \quad l(s) = \mathcal{V}_H[l(s); \Psi(l)], \quad l(s) \in \mathcal{E}^*.$$

5. The finite cavity problem. In this section we shall follow an analogous procedure in determining formulas for $\omega(\zeta)$ and $w(\zeta)$. It will turn out, however, that the resulting formulas are more complicated and depend on several constants as well as on a real function. We shall be able to give five real relations which must be satisfied by these constants. Finally an integral equation is derived for the real function.

The complex potential $w(z)$ for the finite cavity problem is analytic (but possibly many valued) and $w'(z) \neq 0$ at all interior points of R , excluding the point I_∞ . At I_∞ , $w(z)$ is of the form, $w(z) = Uz + (k/2\pi i) \log z + P(1/z)$,⁸ where k is a real number, possibly zero. For consider any circuit around the cavity in the z -plane. The value of $w(z)$ after completing such a circuit in, say, the positive direction, is seen to differ from its starting value by a real constant k , which is independent of the path and the starting point. In particular, the stream function is unchanged, while there is a change in the potential function equal to the circulation of the flow about the cavity. (If the circulation is zero, $w(z)$ is single valued.) The function $w(z) - (k/2\pi i) \log z$ is therefore single valued near I_∞ . Hence, it has a Laurent expansion about I_∞ and since

⁸ The values in the various complex planes corresponding to the points labeled $A, B, I, \dots$, in the flow plane are designated in many places by the use of appropriate subscripts; thus the value of ζ at the point in the ζ -plane corresponding to I in the flow plane is designated by ζ_I . The notation $P(z - z_P)$ will be used to represent any function regular at the point z_I , but if a specific function is meant, it will be represented differently. Finally, the subscript ∞ on a letter means that the complex value attached to the point is ∞ .

$$\lim_{z \rightarrow I_{\infty}} w'(z) = U,$$

where U is the velocity of the uniform flow, this expansion has the form $Uz + P(1/z)$.

If the z -plane is cut along the streamline EI the function $w(z)$ is made single valued, and along the streamline formed by OB and Σ it is real and increases from 0 to φ_E , while along OA and Σ^* it is real and increases from 0 to $\varphi_E + k$. The image of R is thus a Riemannian manifold bounded by two cuts of unequal length on the real axis (figure 6).

It is apparent that $w(\zeta) = w[z(\zeta)]$ maps Γ conformally onto this latter manifold. With the help of the following theorem which is due to Gilbarg [18] we will obtain a formula for $w(\zeta)$.

Mapping theorem. *Let $w(\zeta)$ be an analytic function which maps the interior of Γ on a Riemannian manifold with closed polygonal boundary. In the neighbor-*

FIG. 6. w -plane, finite cavity problem

hood of the interior points $\zeta = \sigma_j$, ($j = 1, \dots, n$), let

$$w'(\zeta) = (\zeta - \sigma_j)^{\alpha_j} P(\zeta - \sigma_j), \quad P(0) \neq 0,$$

where α_j is real and unequal to zero. At all other interior points let $w(\zeta)$ be regular and $w'(\zeta) \neq 0$. If the boundary has interior angles $\beta_k\pi$, ($k = 1, \dots, m$), at vertices corresponding to $\zeta = \tau_k$, then

$$\frac{dw}{d\zeta} = M \prod_j \left\{ \frac{(\zeta - \sigma_j)(\zeta - \bar{\sigma}_j)(\sigma_j\zeta - 1)(\bar{\sigma}_j\zeta - 1)}{\zeta^2} \right\}^{\alpha_j} \prod_k \left\{ \frac{(\zeta - \tau_k)(\tau_k\zeta - 1)}{\zeta} \right\}^{\beta_k-1} \left\{ \frac{1 - \zeta^2}{\zeta^2} \right\},$$

where M is a complex constant.

From the prescribed local behavior of $w(\zeta)$ we have, at ζ_I , $w'(\zeta) = (\zeta - \zeta_I)^{-2} P(\zeta - \zeta_I)$, $P(0) \neq 0$; at all other interior points, $w(\zeta)$ is regular and $w'(\zeta) \neq 0$. At the vertices O , E of the boundary of the w -manifold, which, respectively, correspond to e^{is_0} and the origin (E) in the ζ -plane, the interior angles are 2π . Hence

$$(5.1) \quad \frac{dw}{d\zeta} = M \frac{\zeta(\zeta^2 - 1)(\zeta - e^{is_0})(\zeta - e^{-is_0})}{(\zeta - \zeta_I)^2(\zeta - \bar{\zeta}_I)^2(\zeta\zeta_I - 1)^2(\bar{\zeta}\bar{\zeta}_I - 1)^2}.$$

M is a real constant, since $w(\zeta)$ is real for ζ real. $w(\zeta)$ itself is never explicitly needed, although it may always be determined by integration.

A formula for $\omega(\zeta)$ in terms of the boundary correspondence $l(s)$ can be derived. In fact, observing that $\omega(\zeta)$ is analytic in Γ , real and continuous on the segment $(-1, +1)$ of the real axis, and $\tau(0) = 0$, we can write immediately (see §4),

$$(5.2) \quad \omega(\zeta) = i \log \frac{\zeta - e^{is_0}}{\zeta e^{is_0} - 1} + \Omega(\zeta),$$

where

$$(5.3) \quad \Omega(\zeta) = \Theta(\zeta) + iT(\zeta) = \frac{1}{\pi} \int_0^\pi \Psi[l(s)] \frac{1 - \zeta^2}{\zeta - 2\zeta \cos s + \zeta^2} ds.$$

There are three relations among the constants M , V , ζ_I , and s_0 . Namely,

$$(5.4) \quad \omega(\zeta_I) = i \log (U/V)$$

$$(5.5) \quad 2b = \frac{M}{V} \int_0^\pi \exp [-T(e^{is})] \left| \frac{(e^{2is} - 1)(e^{i(s+s_0)} - 1)^2}{(e^{is} - \zeta_I)^2 (e^{is} - \bar{\zeta}_I)^2 (e^{is}\zeta_I - 1)^2 (e^{is}\bar{\zeta}_I - 1)^2} \right| ds$$

$$(5.6) \quad \text{Res}_{\zeta=\zeta_I} \frac{dz}{d\zeta} = 0$$

It is not difficult to verify these equations. First (5.4) expresses the fact that the velocity at I_∞ is U . Next from (3.1), (5.1), and (5.2) we obtain

$$(5.7) \quad \frac{dz}{d\zeta} = \frac{M}{V} e^{i\Omega(\zeta)} \frac{\zeta(\zeta^2 - 1)(\zeta - e^{-is_0})(\zeta e^{is_0} - 1)}{(\zeta - \zeta_I)^2 (\zeta - \bar{\zeta}_I)^2 (\zeta\zeta_I - 1)^2 (\zeta\bar{\zeta}_I - 1)^2}.$$

Putting $\zeta = e^{is}$ in (5.7) and taking absolute values we get

$$(5.8) \quad \frac{dl}{ds} = \frac{M}{V} \exp [-T(e^{is})] \left| \frac{(e^{2is} - 1)(e^{i(s+s_0)} - 1)^2}{(e^{is} - \zeta_I)^2 (e^{is} - \bar{\zeta}_I)^2 (e^{is}\zeta_I - 1)^2 (e^{is}\bar{\zeta}_I - 1)^2} \right|.$$

Integration of (5.8) yields (5.5).

To account for (5.6) we recall that the function $z(\zeta)$ mapping Γ onto R has a pole of exactly order one at ζ_I , since the map is conformal at I_∞ . This being the case, the derivative $dz/d\zeta$ has a pole of order two at ζ_I , and the residue there is zero.

Upon using (5.7) and the identity

$$\text{Res}_{\zeta=\zeta_I} \frac{dz}{d\zeta} = \lim_{\zeta \rightarrow \zeta_I} \left\{ \frac{d}{d\zeta} \left[(\zeta - \zeta_I)^2 \frac{dz}{d\zeta} \right] \right\},$$

it is clear that (5.6) is equivalent to

$$(5.9) \quad i\Omega'(\zeta_I) + \frac{2}{\zeta_I - e^{-is_0}} - \frac{2}{\zeta_I - \bar{\zeta}_I} - \frac{2\bar{\zeta}_I}{\zeta_I\bar{\zeta}_I - 1} + \frac{1}{\zeta_I} = 0.$$

The central equations of the finite cavity problem. With $T(e^{is})$ defined by (5.3) in terms of $l(s)$, it is clear that (5.8) is a nonlinear integro-differential equation, satisfied by the boundary relation $l(s)$ and the constants M , V , s_0 , and ζ_I . A system of equations satisfied by these quantities is attained by adjoining to (5.8) the equations (5.4), (5.5), and (5.9).

Conversely, suppose a set of quantities $l(s)$, M , V , s_0 , and ζ_I , satisfy this system, and $0 < s_0 < \pi$, $\zeta_I \in \Gamma$. On the basis of this set, let us define $\omega(\zeta)$ and $w(\zeta)$ by the formulas of this section. Then the flow determined by $\omega(\zeta)$ and $w(\zeta)$ solves the finite cavity problem for the obstacle defined by $\Psi(l)$. This will be the basis for the existence theorem of §17.

Since there are five real conditions contained in equations (5.4), (5.5), and (5.6), and five real constants contained in M , V , s_0 , and ζ_I , it is apparent why V cannot be arbitrarily preassigned.

The symmetric case. As in the Helmholtz problem, we consider only the function $l(s)$, $0 \leq s \leq \pi/2$, and define $l(s)$ throughout the interval $0 \leq s \leq \pi$ by (4.11).

When symmetry is assumed, we have $s_0 = \pi/2$, $\zeta_I = ih$, $\Theta(i\eta) = \pi/2$, where h is a real number, $0 < h < 1$. Then (5.4), (5.5), and (5.9) become

$$(5.4a) \quad \frac{V}{U} = \frac{1+h}{1-h} \exp[-T(ih)]$$

$$(5.5a) \quad b = \frac{2M}{V} \int_0^\pi \exp[-T(e^{is})] \frac{(1+\sin s) \sin s \, ds}{(1+2h^2 \cos 2s + h^4)^2}$$

$$(5.9a) \quad 0 = \Omega'(ih) + 2(h^2 - 1)^{-1}.$$

The integral equation (5.8) becomes

$$(5.8a) \quad l(s) = \frac{4M}{V} \int_0^\pi \exp[-T(e^{is})] \frac{(1+\sin s) \sin s \, ds}{(1+2h^2 \cos s + h^4)^2}.$$

For $l(s) \in \mathcal{E}^*$ ($\mathcal{E}^*$ is defined in §4) and real h , $-1 < h < 1$, the expressions on the right sides of (5.3) and (5.5a) have meaning. For such an $l(s)$ and h , therefore, $T(e^{is})$ and the ratio M/V may be defined by (5.3) and (5.5a), and we may consider the expression on the right side of (5.8a) as giving a functional transformation $\mathcal{U}_c$ of $l(s)$ and h . It is clear that the range of $\mathcal{U}_c$ is again $\mathcal{E}^*$.

Our system of equations reduces to

$$(5.10) \quad \begin{aligned} l(s) &= \mathcal{U}_c[l(s), h; \Psi(l)], \\ 0 &= \Omega'(ih) + 2(h^2 - 1)^{-1}, \end{aligned} \quad \text{where } \begin{cases} l(s) \in \mathcal{E}^*, \\ 0 < h < 1. \end{cases}$$

Notice that the cavity speed is then given by the side relation (5.4a).

6. The schlicht cavity problem. Symmetric case. It is well known [21] that the free streamlines in a flow solving the Helmholtz problem overlap at infinity if, and only if, $\omega'(0) > 0$. By differentiation of (4.5) this is equivalent to $\Omega'(0) > 2$. Thus the schlicht cavity problem may be formulated as follows: find, for a

given obstacle AB , either a flow solving the Helmholtz problem for which $\Omega'(0) \leq 2$, or a flow solving the finite cavity problem.

We suppose that our problem has been solved, in the sense that a symmetric flow of one of the desired types has been exhibited. Let $l = l(s)$, $0 \leq s \leq \pi/2$, be the usual boundary correspondence (§4), and let h , $0 \leq h < 1$, be taken zero if the flow solves the Helmholtz problem, and otherwise defined by $\zeta_r = ih$. The formulas of §4 and 5 show that $\omega(\zeta)$ and $w(\zeta)$ may be determined from $l(s)$ and h .

Let

$$(6.1) \quad G[\Omega'(ih), h] = \{h + \text{Max } [0, \Omega'(ih) - 2]\} \{\Omega'(ih) + 2(h^2 - 1)^{-1}\}.$$

We assert that $l(s)$ and h satisfy the system of equations,

$$(6.2) \quad \begin{aligned} l(s) &= \mathfrak{U}_c[l(s), h; \Psi(l)], \\ 0 &= G[\Omega'(ih), h]. \end{aligned}$$

The assertion is obviously true if the flow in question solves the finite cavity problem. Otherwise we have

$$l(s) = \mathfrak{U}_H[l(s); \Psi(l)],$$

and

$$(6.3) \quad \Omega'(0) \leq 2, \quad h = 0.$$

Noting that

$$\mathfrak{U}_H[l(s), \Psi(l)] = \mathfrak{U}_c[l(s), 0; \Psi(l)],$$

and that $G = 0$ in virtue of (6.3), our result is immediate.

Conversely, suppose that $l(s)$ and h satisfy (6.2). Then, if $h = 0$, we have

$$\{\text{Max } [0, \Omega'(0) - 2]\} \{\Omega'(0) - 2\} = 0,$$

from which it follows that $\Omega'(0) \leq 2$. Also

$$l(s) = \mathfrak{U}_c[l(s), 0; \Psi(l)] = \mathfrak{U}_H[l(s), \Psi(l)],$$

so that the flow defined by $l(s)$ solves the Helmholtz problem.

On the other hand, if $h > 0$,

$$h + \text{Max } [0, \Omega'(ih) - 2] \neq 0$$

so that $G = 0$ implies

$$\Omega'(ih) + 2(h^2 - 1)^{-1} = 0.$$

It follows that $l(s)$ and h solve the finite cavity problem.

The cavity speed in a flow solving the schlicht cavity problem is given in all cases by the formula

$$(6.4) \quad V = U \frac{1 + h}{1 - h} \exp [-T(ih)].$$

manifold, such that the origin (E) and e^{is_0} of the ζ -plane correspond to the vertices E and O of the w -plane. The behavior of $w(\zeta)$ at ζ_I and ζ_D is easily ascertained, and elsewhere in Γ , $w'(\zeta) \neq 0$. By Gilbarg's theorem,

$$(7.1) \quad \frac{dw}{d\zeta} = M \frac{(\zeta^2 - 1)(\zeta - e^{is_0})(\zeta - e^{-is_0})(\zeta - \zeta_D)(\zeta - \bar{\zeta}_D)(\zeta\bar{\zeta}_D - 1)(\zeta\bar{\zeta}_D - 1)}{\zeta(\zeta - \zeta_I)^2(\zeta - \bar{\zeta}_I)^2(\zeta\bar{\zeta}_I - 1)^2(\zeta\bar{\zeta}_I - 1)^2}.$$

M is a real constant, since $w(\zeta)$ is real for ζ real. $w(\zeta)$ itself is never explicitly needed, although it may always be determined by integration.

Introducing the function $\Psi(l)$ and the boundary correspondence, $l(s)$, with definitions and agreements paralleling those of §4, we shall derive a formula for $\omega(\zeta)$. From its definition, $Q(z)$ is analytic at all points in the interior of R with the exception of stagnation points. It follows that the only singularity of $\omega(\zeta) = Q[z(\zeta)]$ in Γ occurs at ζ_D where

$$\omega(\zeta) = i \log (\zeta - \zeta_D) + P(\zeta - \zeta_D).$$

Evidently $\omega(\zeta)$ is many valued; however, the function

$$(7.2) \quad f(\zeta) = \omega(\zeta) - i \log \frac{(\zeta - \zeta_D)(\zeta\bar{\zeta}_D - 1)}{(\zeta - \bar{\zeta}_D)(\zeta\bar{\zeta}_D - 1)}$$

is single valued and free from singularities in Γ .

On the circumference Γ' the real part of $f(\zeta)$ equals the real part of some branch of $\omega(\zeta)$; we can take this branch to be the one where

$$(7.3) \quad \operatorname{Re}[\omega(e^{is})] = \theta(e^{is}) = \begin{cases} \Psi[l(s)] - \pi, & \text{for } 0 \leq s < s_0, \\ \Psi[l(s)], & \text{for } s_0 < s \leq \pi. \end{cases}$$

Since any branch of $\omega(\zeta)$ is real and continuous on the segment $(-1, +1)$ of the real axis, $f(\zeta)$ is also real and continuous on this segment. In virtue of this fact, $f(\zeta)$ may be analytically continued, by reflection, throughout the unit circle. Hence, by the Schwarz-Poisson formula,

$$(7.4) \quad f(\zeta) = \frac{1}{\pi} \int_0^\pi \theta(e^{is}) \frac{1 - \zeta^2}{1 - 2\zeta \cos s + \zeta^2} ds.$$

From (7.2), (7.3), and (7.4), we obtain

$$(7.5) \quad \omega(\zeta) = i \log \frac{(\zeta - \zeta_D)(\zeta\bar{\zeta}_D - 1)(\zeta - e^{is_0})}{(\zeta - \bar{\zeta}_D)(\zeta\bar{\zeta}_D - 1)(\zeta e^{is_0} - 1)} + \Omega(\zeta),$$

where

$$(7.6) \quad \Omega(\zeta) = \Theta(\zeta) + iT(\zeta) = \frac{1}{\pi} \int_0^\pi \Psi[l(s)] \frac{1 - \zeta^2}{1 - 2\zeta \cos s + \zeta^2} ds.$$

The following three relations hold among the constants ζ_I , ζ_D , M and s_0 .

⁹ A complete derivation of this formula appears elsewhere [19].

$$(7.7) \quad \omega(\zeta_I) = i \log (U/V),$$

$$(7.8) \quad 2b = \frac{M}{V} \int_0^\pi \exp [-T(e^{is})] \left| \frac{(e^{2is} - 1)(e^{i(s+s_0)} - 1)^2(e^{is} - \bar{\zeta}_D)^2(e^{is}\zeta_D - 1)^2}{(e^{is} - \zeta_I)^2(e^{is} - \bar{\zeta}_I)^2(e^{is}\zeta_I - 1)^2(e^{is}\bar{\zeta}_I - 1)^2} \right| ds,$$

$$(7.9) \quad \text{Res}_{\zeta=\zeta_I} \frac{dz}{d\zeta} = 0.$$

These relations correspond with (5.4), (5.5), and (5.6) and are derived in the same manner.

In particular, to obtain (7.8), we have, by (3.1), (7.1), and (7.5),

$$(7.10) \quad \frac{dl}{ds} = \frac{M}{V} \exp [-T(e^{is})] \left| \frac{(e^{2is} - 1)(e^{i(s+s_0)} - 1)^2(e^{is} - \bar{\zeta}_D)^2(e^{is}\zeta_D - 1)^2}{(e^{is} - \zeta_I)^2(e^{is} - \bar{\zeta}_I)^2(e^{is}\zeta_I - 1)^2(e^{is}\bar{\zeta}_I - 1)^2} \right|.$$

Integration of (7.10) yields (7.8).

Also (7.9) is equivalent to

$$(7.11) \quad i\Omega'(\zeta_I) + \frac{2}{\zeta_I - e^{is_0}} - \frac{2}{\zeta_I - \bar{\zeta}_I} - \frac{2\bar{\zeta}_I}{\zeta_I \bar{\zeta}_I - 1} - \frac{1}{\zeta_I} + \frac{2}{\zeta_I - \bar{\zeta}_D} + \frac{2\zeta_D}{\zeta_I \zeta_D - 1} = 0.$$

The central equations of the reentrant jet problem. With $T(e^{is})$ defined by (7.6) in terms of $l(s)$, it is clear that (7.10) is a nonlinear integro-differential equation, satisfied by the boundary correspondence $l(s)$ and the constants, M , s_0 , ζ_I , and ζ_D . A system of equations satisfied by these quantities is attained by adjoining to (7.10) the equations (7.7), (7.8), and (7.11).

Conversely, suppose a set of quantities $l(s)$, M , s_0 , ζ_I , and ζ_D satisfies this system, and

$$0 < s_0 < \pi, \quad \zeta_I \in \Gamma, \quad \zeta_D \in \Gamma.$$

On the basis of this set, let us define $\omega(\zeta)$ and $w(\zeta)$ by the formulas of this section. Then the flow determined by $\omega(\zeta)$ and $w(\zeta)$ solves the reentrant jet problem for the obstacle defined by $\Psi(l)$. This is the basis for the existence theorem of §19.

The symmetric case. As in the Helmholtz problem, we consider only the function $l(s)$, $0 \leq s \leq \pi/2$, and define $l(s)$ throughout the interval $0 \leq s \leq \pi$ by (4.11).

When symmetry is assumed, we have $s_0 = \pi/2$, $\zeta_I = ih$, $\zeta_D = id$, $\Theta(i\eta) = \pi/2$, where h and d are real numbers, $0 < h < 1$, $0 < d < 1$. Then (7.7), (7.8), and (7.11) become

$$(7.7a) \quad \frac{U}{V} = \frac{(h-d)(1-hd)(1-h)}{(h+d)(1+hd)(1+h)} \exp [T(ih)];$$

$$(7.8a) \quad b = \frac{2M}{V} \int_0^\pi \exp [-T(e^{is})] \frac{(1+2d \sin s + d^2)^2(1+\sin s) \sin s \, ds}{(1+2h^2 \cos 2s + h^4)^2},$$

$$(7.11a) \quad 0 = \Omega'(ih) + \frac{2}{h^2-1} + \frac{2d(1-h^2)}{h(h+d)(1+hd)}.$$

The integral equation (7.10) becomes

$$(7.10a) \quad l(s) = \frac{4M}{V} \int_0^s \exp [-T(e^{is})] \frac{(1 + 2d \sin s + d^2)^2 (1 + \sin s) \sin s \, ds}{(1 + 2h^2 \cos 2s + h^4)^2}.$$

For $l(s) \in \mathcal{E}^*$ (§4) and real h and d , $0 < h < 1$, $0 < d < 1$, the expressions on the right sides of (7.6) and (7.8a) have meaning. For such $l(s)$, h , and d , therefore, $T(e^{is})$ and the ratio M/V may be determined by (7.6) and (7.8a). We may then consider the expression on the right side of (7.10a) as giving a functional transformation $\mathfrak{U}_J$ of $l(s)$, h , and d .

8. Continuity of $\mathfrak{U}$. The space $\mathcal{E}$ of real continuously differentiable functions $l(s)$, $0 \leq s \leq \pi/2$, is a Banach space if we put

$$\|l(s)\| = \text{Max } |l(s)| + \text{Max } |l'(s)|.$$

We wish to show that the functionals $\mathfrak{U}$ depend continuously on $l(s) \in \mathcal{E}^*$.

In order to prove this it is clearly sufficient to show that the function $\Omega(\zeta)$, $|\zeta| \leq 1$, depends continuously on $l(s)$. Now $\Theta(\zeta)$ is a harmonic function and consequently depends continuously on its boundary values. But the boundary values of $\Theta(\zeta)$ depend continuously on $l(s)$, for

$$|\Theta_1(e^{is}) - \Theta_2(e^{is})| = |\Psi[l_1(s)] - \Psi[l_2(s)]| < K \|l_1(s) - l_2(s)\|^\nu,$$

where K and ν are the Hölder constants for $\Psi(l)$.

It remains to show that $T(\zeta)$ depends continuously on $l(s)$. We have, for $\zeta = e^{is}$,

$$(8.1) \quad T(e^{is}) = \frac{1}{\pi} \int_0^\pi \{\Psi[l(s')] - \Psi[l(s)]\} \frac{\sin s}{\cos s' - \cos s} \, ds'.$$

Using (8.1) it is not difficult to show (by an appropriate partition of the range of integration) that, for $\|l_1(s) - l_2(s)\|$ sufficiently small,

$$|T_1(e^{is}) - T_2(e^{is})| < \epsilon$$

where ϵ is an arbitrary small positive number.

PART III. LIMITATION OF THE UNKNOWN

The flows solving the respective free boundary problems will be assumed henceforth to be symmetric about the x -axis. The general situation is indicated

by the "upper half" flows illustrated in figures 8a, 8b, and 8c. In this chapter we shall develop the properties of the solutions of the central equations which are basic to our later application of the Schauder-Leray theory.

9. The behavior of the free streamlines. A symmetric obstacle AOB will be called a *regular obstacle*, if there exists a number α , $\pi/2 \leq \alpha \leq \pi$, such that the inclination $\Psi(l)$ satisfies the inequality

$$(9.1) \quad \alpha - \pi \leq \Psi(l) \leq \alpha$$

along OB (that is, for $b \leq l \leq 2b$). A generalization of the concept of regular obstacles is discussed in §20.

The arc OB of a regular obstacle is cut in at most one point or along a segment by a straight line whose inclination is α . The following theorem proves that the free streamline Σ , detaching from B , enjoys the same property.

FIG. 8. Upper half flow regions for symmetric case

Theorem 9.1. *In a symmetric solution of either the Helmholtz problem or the finite cavity problem for a regular obstacle, the inclination θ of Σ satisfies the inequality $\alpha - \pi < \theta < \alpha$.*

Proof. The proof is the same in either case. The inclination of the velocity vector at a point $z(\zeta)$ is given (§3) by the real part $\theta(\zeta)$ of $\omega(\xi)$. Thus the inclination of the free streamline Σ is given by $\theta(\xi)$, $-1 < \xi < 0$.

Let Γ^* denote the left half unit circle in the ζ -plane ($\xi^2 + \eta^2 < 1$, $\xi < 0$). Because $\omega(\zeta)$ is analytic for $|\zeta| < 1$, $\theta(\zeta)$ is harmonic in Γ^* ; moreover

$$(9.2) \quad \theta(\zeta) = \theta(\bar{\zeta}).$$

Recalling that $\theta = \Psi$ on OB , we have by (9.1) and (9.2),

$$(9.3) \quad \alpha - \pi \leq \theta(e^{is}) \leq \alpha, \quad \pi/2 < s < 3\pi/2.$$

Since the flow is symmetric, the image of the segment $(0, i)$ is the central stream-

line, so that $\theta(i\eta) = 0$, $|\eta| < 1$. Now from the definition of regular obstacles, $\alpha - \pi \leq 0 < \alpha$; hence

$$(9.4) \quad \alpha - \pi \leq \theta(i\eta) < \alpha, \quad |\eta| < 1.$$

By (9.3), (9.4), and the well known maximum principle we obtain, for ζ in Γ^* ,

$$\alpha - \pi < \theta < \alpha.$$

This completes the proof. An analogous theorem holds for the reentrant jet problem.

Theorem 9.2. *In a symmetric solution of the reentrant jet problem for a regular obstacle AOB there is exactly one point P on the free streamline Σ where $\theta = \alpha - \pi$. Moreover,*

- 1) *on the finite arc BP of Σ (except at P), $\alpha - \pi < \theta < \alpha$;*
- 2) *on the infinite arc PE of Σ the inclination θ decreases monotonically from $\alpha - \pi$ to $-\pi$.*

FIG. 9.

Proof. We retain the terminology of Theorem 9.1. Consider the values that $\theta(\zeta)$ assumes on the boundary of Γ^* . We have

$$\begin{aligned} \alpha - \pi &\leq \theta(e^{is}) \leq \alpha, & \pi/2 < s < 3\pi/2, \\ \theta(i\eta) &= 0, & d < |\eta| < 1, \\ \theta(i\eta) &= -\pi, & |\eta| < d. \end{aligned}$$

From the definition of regular obstacles, $-\pi < \alpha - \pi \leq 0$. Therefore there is a level curve C in Γ^* , defined by

$$(9.5) \quad \theta(\zeta) = \text{const.} = \alpha - \pi,$$

which issues from $\zeta = id$. Since no level curve can be closed, C must reach the boundary of Γ^* . However, by the maximum principle C cannot terminate on $|\zeta| = 1$, and therefore must join $\zeta = id$ and $\zeta = -id$; see figure 9.

Let $\zeta = \xi_p$ be the point of intersection of C and the real axis. By (9.5), $\theta(\xi_p) = \alpha - \pi$. We shall prove that the image in the flow plane of the point ξ_p is the required point P . Surely it will be sufficient to show that properties 1) and 2) hold.

The segment $(-1, \xi_p)$ of the ξ -axis corresponds to the arc BP of Σ . Property 1) is obtained easily by applying the maximum principle in the shaded region.

The segment $(\xi_p, 0)$ corresponds to the arc PE of Σ . Let ξ_1 be any point on this segment. Applying the maximum principle to the region bounded by C and the segment $(-id, id)$ of the η -axis, we get $\alpha - \pi > \theta(\xi_1) > -\pi$. Therefore the level curve C_1 , $\theta(\zeta) = \text{const.} = \theta(\xi_1)$, passing through ξ_1 must join id and $-id$. A final use of the maximum principle gives $\alpha - \pi > \theta(\xi_2) > \theta(\xi_1)$, when $\xi_p > \xi_2 > \xi_1$, which proves property 2).

The importance of the preceding theorems cannot be underestimated; in many of the succeeding proofs more or less explicit applications will be made of their results.

It is perhaps worth noting here that Theorem 9.1 is intimately connected with the well-known first Brillouin condition which states that in the Helmholtz problem the free streamlines should not intersect themselves. Theorem 9.2 gives the corresponding generalization of this result to the reentrant jet problem.

10. The behavior of the free streamlines (continued). By a *flow* we shall continue to mean a steady irrotational motion of an ideal incompressible fluid. We suppose that the motion occurs in a closed simply connected region R naturally bounded by fixed walls and streamlines (the fluid is to fill up R entirely). Under these circumstances, the complex potential $w(z) = \varphi + i\psi$ is single valued and regular in the interior of R and maps the boundary of R into straight arcs.

The flows we consider in the remainder of Part III will not always solve a free boundary problem. *Without exception, however, each flow will occupy an infinite region and have a uniform speed U at infinity.* A flow of this type will be called an *admissible flow* if $w(z)$ maps R onto the upper half plane, $\psi \geq 0$. If in addition the inverse of $w(z)$ is single valued and regular for $\psi > 0$, then the flow will be called a *simple flow*. The speed at a point P in R of a flow in R will be denoted by $V(P, R)$; i.e. $V(P, R) = |w'(z_P)|$.¹⁰ Since only one flow will be considered corresponding to a given region R , this notation will cause no confusion.

(We note that the "upper half" of a flow solving the schlicht cavity problem is a simple flow, while a flow solving the reentrant jet problem is an admissible flow.)

We shall often be dealing with several flows simultaneously; although they are considered as occurring in the same plane (z -plane), they are independent of one another. Our results will depend on the following two theorems [21].

Theorem 10.1. *Let R_1 and R_2 be two closed simply connected regions over the complex z -plane, such that R_1 contains R_2 . Suppose that there exists an admissible*

¹⁰ If P is on the boundary of R , $w'(z_P)$ does not necessarily exist. However, by writing the symbol $V(P, R)$ we shall imply that the derivative in question does exist.

flow in R_1 and a simple flow in R_2 . If the streamlines $\psi = 0$ have a common point M , then

$$V(M, R_1) \geq V(M, R_2),$$

the equality holding if and only if the flows are identical.

Theorem 10.2. Let R_1 and R_2 be two simply connected closed regions over the complex z -plane. Suppose that there are admissible flows in R_1 and R_2 , and let S_1 and S_2 denote the respective streamlines $\psi = 0$. Let S_1 and S_2 have an arc MN in common such that the direction of each flow on MN is from M to N . Furthermore let an arc QM of S_1 and an arc NQ of S_2 bound, together with MN , a simply connected region $MNQ \equiv R_3$ which is contained in R_1 and R_2 . Then

$$(10.1) \quad V(M, R_2) < V(N, R_2)$$

implies

$$V(M, R_1) < V(N, R_1).$$

FIG. 10.

Figure 10 shows a particularly simple case of the theorem when Q is the point at infinity, R_1 and R_2 are both schlicht, and the flows are simple. We recall the following results obtained with the aid of the foregoing theorems [21]:

- i) In a symmetric solution of the finite cavity problem the cavity speed V is less than the speed at infinity U .
- ii) In a symmetric solution of the schlicht cavity problem for a regular obstacle the free streamline Σ stays entirely above the real axis.¹¹
- iii) There can be no more than one symmetric solution of the schlicht cavity problem for a regular obstacle AB . Assuming for the moment that the schlicht cavity problem actually has a solution, we can prove

Theorem 10.3. Let AB be a regular obstacle. Let R_c and R_J be, respectively, the region occupied by the "upper half" of the (unique) symmetric solution of the schlicht cavity problem for AB , and the region occupied by the "upper half" of any symmetric solution of the reentrant jet problem for AB . Then R_J contains R_c .

¹¹ Except, of course, at E if the flow solves the finite cavity problem. A result corresponding to ii) may be proved for the reentrant jet problem, but it is not necessary for the existence theorems.

Proof. Denote the corresponding free streamlines by Σ_c and Σ_J and let P be the point of Σ_J where $\theta = \alpha - \pi$.

Let us assume for the moment that the arc BP of Σ_J is below Σ_c . Then by Theorem 9.1 and property *ii*) above, Σ_c is in the shaded area of figure 11. But by Theorem 9.2 this shaded area is covered by R_J . Hence R_J contains R_c .

FIG. 11.

FIG. 12.

It remains to show that BP is below Σ_c . We consider separately case I, when the flow solving the schlicht cavity problem is a Helmholtz flow, and case II, when it is a finite cavity flow.

Case I. Suppose for contradiction that part of BP is above Σ_c . Then figure 12 illustrates the situation arising from a suitable translation of R_c parallel to the direction α . The arcs BP and Σ_c are tangent at N .¹² Now let R be the region in the upper half plane bounded by the curve consisting of the arcs $(-\infty, O)$, OB , BN , and $(N, +\infty)$. Then by Theorem 10.1,

$$V(N, R) > V(N, R_c) = U.$$

By another suitable translation of R_c it becomes apparent that there exists a point M of BN where

$$V(M, R) < V(M, R_c) = U.$$

¹² A rigorous justification of this geometrically obvious construction is indicated in the proof of Theorem 4 of [21].

Hence

$$(10.2) \quad V(M, R) < V(N, R).$$

Theorem 10.2 is evidently applicable to the regions R and R_J ; (compare figures 10 and 12). Since the hypothesis (10.1) is verified by (10.2), we have

$$(10.3) \quad V(M, R_J) < V(N, R_J).$$

On the other hand, since M and N are points of the free streamline Σ_J ,

$$V(M, R_J) = V_J = V(N, R_J),$$

where V_J is the assigned cavity speed. This contradicts (10.3).

Case II. Consider the region R_c (figure 8b). Let N be a point on the real axis to the right of E . Using Theorem 10.2 it can be shown [21] that

$$(10.4) \quad V(N, R_c) > V(E, R_c) = V_c,$$

where V_c is the cavity speed. With the help of (10.4), a duplication of the argument used in case I proves that BP is below Σ_c . We will omit the details.

Corollary 10.1. *Let AB be a regular obstacle. Let V_c be the cavity speed in the (unique) symmetric solution of the schlicht cavity problem for AB . Then there is no symmetric solution of the reentrant jet problem unless the assigned cavity speed V_J is greater than V_c .*

Proof. Consider any solution of the reentrant jet problem for AB . The region R_J contains the region R_c , while B is common to the bounding streamlines of each flow. Therefore by Theorem 10.1, $V_J > V_c$.

In view of this corollary we might expect a solution of the reentrant jet problem to exist if and only if the cavity speed $V_J > V_c$. This is indeed the case, as we shall see in §19. The following theorems are of fundamental importance in §13.

Theorem 10.4. *Let AB be a regular obstacle whose length is $2b$, and for which $\alpha \neq \pi$, and let W be the polygonal obstacle defined by*

$$\Psi(l) = \begin{cases} \alpha, & \text{for } b \csc \alpha < l \leq 2b \csc \alpha, \\ -\alpha, & \text{for } 0 \leq l < b \csc \alpha. \end{cases}$$

Then the region R_H occupied by the "upper half" of a symmetric solution of the Helmholtz problem for AB (if it exists) contains the region R_W occupied by the "upper half" of the symmetric solution of the Helmholtz problem for W .

Proof. First of all, we note that the flow past W is well known, and in fact may be explicitly computed [20].

Now suppose for contradiction that R_H does not contain R_W ; letting Σ_H and Σ_W be the corresponding free streamlines, it is evident that an arc of Σ_H must be above Σ_W . If Σ_W lies above Σ_H at infinity then by translating R_W a suitable distance to the left we can place R_W inside Σ_H but such that Σ_W and Σ_H have a common point. Theorem 10.1 then states that the speeds of the two

flows at this point are unequal, which contradicts the fact that both speeds are U .

Likewise, if Σ_w lies below Σ_H at infinity, then by translating R_w a suitable distance to the right a similar contradiction is obtained. This proves the theorem.

A similar theorem holds in case $\alpha = \pi$, the comparison obstacle W being formed by the three line segments joining, in order, the points $(-b - b^2/a, b)$, $(-b, b)$, $(-b, -b)$, $(-b - b^2/a, -b)$. The proof can be omitted.

Theorem 10.5. *In a symmetric solution of the finite cavity problem the free streamline Σ lies entirely below the straight line parallel to the real axis passing through the maximal point of the obstacle.*

Proof. Suppose the theorem is false. Then the straight line S parallel to the real axis and tangent to Σ at its maximal point M is entirely in the "upper half" flow region R_c . Hence, if R denotes the half plane above S , we have

$$(10.5) \quad V(M, R_c) > V(M, R),$$

by Theorem 10.1. But the flow in R is just a uniform flow so that $V(M, R) = U$. Also $V(M, R_c) = V$, where V is the cavity speed. Thus, from (10.5), $V > U$. But this is impossible, by property i) above.

Corollary 10.2. *The symmetric finite cavity problem has no solution if the end-point B is the maximal point of AB . If AB is a regular obstacle this will certainly occur if $\alpha = \pi$.*

11. Domination of h . Consider a family of regular obstacles defined by the functions $\Psi_k(l)$, $0 \leq l \leq 2b$, where k is some index. The family will be said to be bounded if the following four conditions hold.

1. There exists a number α , $\pi/2 \leq \alpha \leq \pi$, such that

$$\alpha - \pi \leq \Psi_k(l) \leq \alpha$$

holds along OB , uniformly in k .

2. $\Psi_k(l)$ satisfies the Hölder condition,

$$(11.1) \quad |\Psi_k(l') - \Psi_k(l)| < K |l' - l|^\nu,$$

uniformly in k .

3. The endpoints A and B of the obstacles are uniformly bounded away from the real axis. The minimum distance will be called a .

4. There exists a number ϵ , $0 < \epsilon \leq 1$, such that

$$(11.2) \quad \Psi_k(2b) - (\alpha - \pi) \leq \epsilon \pi,$$

holds uniformly in k .

In this section we shall be concerned with a question peculiar to the problems of the finite cavity and the reentrant jet. In particular, in a solution of the reentrant jet problem there appears a number ih corresponding to the image point ζ_I of I_∞ in the z -plane. If we consider a bounded family of regular obstacles, then should a solution exist for each member of the family, we obtain a set of

values of h . We wish, prior to showing that solutions even exist, to exhibit an upper bound H for possible values in this set (this bound must clearly depend only on $K, a, b, \epsilon, \alpha$ and ν), and to show that $H < 1$. (Geometrically, this assures us that ζ_I does not approach $\zeta = i$.) The procedure will be called the domination of h ; it obviously applies equally well to the finite cavity problem.

A similar logical process will be understood when we consider bounds for certain other quantities (i.e., $M, l(s)$, etc.), while the purpose of these bounds will become apparent when we consider the Schauder-Leray theory in the next chapter.

Lemma 11.1 *For a bounded family of regular obstacles there exists a number $H < 1$, independent of k , such that the number h appearing in a symmetric solution of the reentrant jet problem for any member of the family satisfies the inequality $h < H$.*

Proof. Consider an arbitrary obstacle AOB of the family, and let R be the region in the z -plane occupied by some flow which solves the reentrant jet problem for AOB . We shall construct two auxiliary regions R_1 and R_2 in the z -plane. Let L_1 be the half line issuing from B whose inclination is $-\pi$. Then R_1 is the two sheeted surface bounded by the curve consisting of L_1 , the obstacle, and the reflection L_1^* of L_1 in the real axis (figure 13a). Let I_∞ denote the interior point at infinity in the surface.

Let y_B be the ordinate of the point B , and let L_2 be the line perpendicular to the real axis which is a distance by_B/a to the left of B . Let L_3 be the line segment joining B and L_2 whose slope is α . Then R_2 is the subregion of R_1 whose boundary consists of L_1, L_3 , their reflections, L_1^*, L_3^* , in the real axis, and the segment of L_2 joining L_3 and its reflection L_3^* (figure 13b).

In virtue of Theorem 9.2, R_1 also contains the flow region R . Denote by $z_j(\zeta)$, $j = 1, 2$, the function which maps the upper half unit circle Γ onto R_j such that $z_j(-1) = z_B$, $z_j(+1) = z_A$, $z_j(0) = E_\infty$, $j = 1, 2$, where E_∞ is the boundary point of both R_1 and R_2 at infinity. Furthermore, suppose that $z_j(ih_j) = I_\infty$, $j = 1, 2$. We assert that h_2 depends only on α, a , and b . For the function $z_2(\zeta)$, which can only depend on α, a, b , and y_B , involves y_B merely as a multiplicative factor. Hence h_2 cannot depend on y_B and the assertion is proved.

Now the combination of the mappings $z_2(\zeta)$ and $z_1^{-1}(\zeta)$ produces a mapping $f(\zeta)$ of Γ into itself such that $f(\zeta)$ is real on the real axis and $f(0) = 0$. Hence, by Schwarz's reflection principle, we may define $f(\zeta)$ throughout $|\zeta| < 1$, and $|f(\zeta)| < 1$. Also

$$f(ih_2) = z_1^{-1}[z_2(ih_2)] = z_1^{-1}(I_\infty) = ih_1,$$

so that, by Schwarz's lemma,

$$(11.2) \quad h_2 > h_1.$$

Again, if $z(\zeta)$ maps Γ onto the interior of R (§3), the combination of the mappings $z(\zeta)$ and $z_1^{-1}(\zeta)$ produces a mapping $g(\zeta)$ of Γ into itself. Since the image

 FIG. 13(a) Two sheeted surface R_1 .

 FIG. 13(b). Two sheeted surface R_2 .

of the circumference Γ' under both $z(\zeta)$ and $z_1(\zeta)$ is the obstacle AB , it follows that $g(\zeta)$ takes Γ' into itself and $g(i) = i$. Now

$$g(ih) = z_1^{-1}[z(ih)] = z_1^{-1}(I_\infty) = ih_1.$$

A second application of the Schwarz reflection principle and the generalized Schwarz lemma leads to

$$(11.3) \quad h < h_1.$$

From (11.2) and (11.3) we obtain $h < h_2 = h_2(a, \alpha, b)$.

We may therefore take h_2 for the number H .

Lemma 11.2. *For a bounded family of regular obstacles there exists a number $H < 1$, independent of k , such that the number h appearing in a symmetric solution of the schlicht cavity problem for any member of the family satisfies the inequality $h < H$.*

Proof. Since the flow region of the schlicht cavity problem is interior to the flow region for the reentrant jet problem, we may, by another application of the generalized Schwarz lemma, take H to be the number $h_2(a, \alpha, b)$ of Lemma 11.1.

12. Domination of M . The method to be employed to bound M is a refinement of the method originated by Leray [8]. There are three cases to consider.

Case 1. The Helmholtz problem. From (4.10a) we obtain

$$\begin{aligned} (12.1) \quad M &= \frac{bU}{2} \left[\int_0^\pi \exp[-T(e^{is})](1 + \sin s) \sin s \, ds \right]^{-1} \\ &= \frac{bU}{2} \left[\int_0^\pi \exp\{-T(e^{is}) + \log[(1 + \sin s) \sin s]\} \, ds \right]^{-1} \\ &< \frac{bU}{2\pi} \exp \left[\frac{1}{\pi} \int_0^\pi \{T(e^{is}) - \log[(1 + \sin s) \sin s]\} \, ds \right].^{13} \end{aligned}$$

Now

$$\begin{aligned} (12.2) \quad & - \int_0^\pi \log(1 + \sin s) \sin s \, ds < - \int_0^\pi \log \sin s \, ds \\ & = \pi \log 2. \end{aligned}$$

¹³ In obtaining (12.1) we have used the inequality

$$\frac{1}{\pi} \int_0^\pi \exp[x(s)] \, ds > \exp \left[\frac{1}{\pi} \int_0^\pi x(s) \, ds \right],$$

which is a special case of Jensen's inequality, (see e.g. Pólya-Szegő, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, section II, problems 75 and 79; also problem 69).

Suppose that there exists a number μ such that

$$(12.3) \quad \int_0^\pi T(e^{is}) ds < \mu\pi;$$

from (12.1)–(12.3) we obtain a bound on M , namely,

$$M < (bU/\pi)e^\mu.$$

It remains only to prove (12.3).

Consider the analytic function

$$(12.4) \quad \begin{aligned} \Pi(\zeta) &= \int_1^\zeta \Omega(\zeta) \frac{d\zeta}{i\zeta}, \\ &= -i\Omega(0) \log \zeta + \Pi_1(\zeta), \\ &= -i2^{-1}\pi \log \zeta + \Pi_1(\zeta), \end{aligned}$$

where $\Pi_1(\zeta)$ is holomorphic for $|\zeta| < 1$. We have

$$(12.5) \quad \Pi(e^{is}) = \int_0^s [\Theta(e^{iu}) + iT(e^{iu})] ds.$$

Put

$$\rho(s) \equiv \operatorname{Re}[\Pi_1(e^{is})] = -2^{-1}\pi s + \int_0^s \Theta(e^{iu}) ds;$$

then, from the relations

$$\begin{aligned} \Theta(e^{-is}) &= \Theta(e^{is}) = \Psi[l(s)], & 0 \leq s \leq \pi, \\ \alpha - \pi &\leq \Psi[l(s)] \leq 2\pi - \alpha \end{aligned}$$

it follows that $|\rho'(s)| \leq \pi$. Thus $\rho(s)$ satisfies a Lipschitz condition; hence, in view of the Fatou-Privaloff theorem, there exists a number $\mu_1 > 0$ such that $|\Pi_1(\zeta)| < \mu_1$. Hence, by (12.5) and (12.4),

$$\int_0^\pi T(e^{is}) ds < |\Pi(-1)| < 2^{-1}\pi^2 + \mu_1 = \mu\pi.$$

Case 2. The finite cavity problem. We have for any h ,

$$\int_0^\pi \log(1 + 2h^2 \cos 2s + h^4) ds = 0,$$

(B. O. Pierce, *A Short Table of Integrals*, 3rd rev. ed., formula 523). Using this formula and the inequality $V < U$ we can derive the bound, $M < (bU/\pi)e^\mu$.

Case 3. The reentrant jet problem. If W is an upper bound for the preassigned cavity speeds, then $M < (bW/\pi)e^u$.

13. Domination of $l'(s)$. For a bounded family of regular obstacles there exists a number Λ , independent of k , such that the derivative $l'(s)$ appearing in a solution of the Helmholtz problem for any member of the family satisfies the inequality,

$$(13.1) \quad 0 \leq l'(s) < \Lambda.$$

Proof. Consider an arbitrary obstacle AOB of the family, and let $q(l) \equiv |w'(z_P)|$ denote the flow speed at a point P on OB ($b \leq l \leq 2b$) whose arc length is l . We put $\delta = \epsilon^{-1}$ for convenience in printing, where ϵ is defined by (11.2). The proof of (13.1) rests on the assertion

$$(13.2) \quad q(l) > \begin{cases} K_1(2b - l)^{\delta-1}, & (3b/2 \leq l \leq 2b), \\ K_1(l - b), & (b \leq l \leq 3b/2), \end{cases}$$

where $K_1 > 0$ is independent of K .

Let us suppose for the moment that (13.2) is true. Then, for $3b/2 \leq l \leq l' \leq 2b$,

$$(13.3) \quad \begin{aligned} \int_l^{l'} q(l) \, dl &> -\epsilon K_1(2b - l)^{\delta} \Big|_l^{l'} \\ &\geq \epsilon K_1 |l' - l|^{\delta}.^{14} \end{aligned}$$

On the other hand, putting $l = l(s)$, $l' = l(s')$,

$$(13.4) \quad \begin{aligned} \int_l^{l'} q(l) \, dl &= \int_s^{s'} \left| \frac{dw}{dz} \right| \left| \frac{dz}{ds} \right| ds = \int_s^{s'} \left| \frac{dw}{ds} \right| ds, \\ &= 2M \int_s^{s'} \sin 2s \, ds < 2M |s' - s|, \end{aligned}$$

¹⁴ In obtaining this result, we have employed the inequality

$$(A) \quad |x^{\delta} - y^{\delta}| \geq |x - y|^{\delta},$$

which holds for any two positive numbers x and y , and $\delta \geq 1$. To prove (A), consider the expression,

$$I(y) = (x^{\delta} - y^{\delta}) / (x - y)^{\delta},$$

for fixed x , and $0 \leq y < x$. Now,

$$I(0) = 1,$$

$$I'(y) = \delta x(x - y)^{-\delta-1}(x^{\delta-1} - y^{\delta-1}) \geq 0.$$

Hence,

$$I(y) \geq 1.$$

Since (A) is symmetric in x and y the proof is complete.

where $|dw/ds|$ is evaluated from equation (4.1a) with $s_0 = \pi/2$. From (13.3) and (13.4), we get

$$|l' - l| < \left(\frac{2M}{\epsilon K_1}\right)^\epsilon |s' - s|^\epsilon;$$

hence, using the result of §12, there exists a number K_2 , independent of k , such that

$$(13.5) \quad |l' - l| < K_2 |s' - s|^\epsilon, \quad 3b/2 \leq l \leq l' \leq 2b.$$

Similarly, there exists a number K_3 , independent of k , such that

$$(13.6) \quad |l' - l| < K_3 |s' - s|^\frac{1}{2}, \quad b \leq l \leq l' \leq 3b/2.$$

Let us put $\rho = \nu \min(\epsilon, 1/2)$. Then from (11.1), (13.5), and (13.6) we obtain

$$(13.7) \quad |\Psi[l(s')] - \Psi[l(s)]| < K(K_2 + K_3) |s' - s|^\rho,$$

when $\pi/2 \leq s \leq s' \leq \pi$; in virtue of symmetry (13.7) also holds for $0 \leq s \leq s' \leq \pi/2$. Hence it follows from the Fatou-Privaloff theorem that the function $\Omega(\zeta)$, defined by (4.6a), satisfies a Hölder condition, independent of k , for $|\zeta| \leq 1$. The function $l'(s)$, which is defined by

$$(4.9a) \quad l'(s) = 4MU^{-1} \exp[-T(e^{is})] (1 + \sin s) \sin s,$$

will likewise obey a Hölder condition,¹⁵ proving (13.1).

We turn now to the proof of (13.2). Our method is based on the possibility of constructing a *known* flow region R_2 which, with the application of Theorem 10.1, can be used to estimate $q(l)$.

We shall use the terminology of Theorem 10.4.¹⁶ Let L be the line segment joining B and Σ_w ; consider the simple region R_2 which is above the curve consisting of the negative real axis, the obstacle OB , L , and the infinite arc of Σ_w joining L to infinity (figure 14). There is an admissible flow in R_2 since any function $w = f(z)$ which maps R_2 onto $\psi \geq 0$ such that the points at infinity correspond has a derivative at infinity (by Lemma 1 of reference [21]).

Now in view of Theorems 9.1 and 10.4, R_2 is contained in R . Hence, by Theorem 10.1,

¹⁵ The Hölder exponent is ρ . This exponent may, however, be bettered: for, having proved that $l(s)$ has a bounded derivative, we could repeat the reasoning of this paragraph starting with a Lipschitz condition for $l(s)$. The result is that $l'(s)$ satisfies a Hölder condition with exponent ν .

¹⁶ The procedure to be given applies when $\alpha \neq \pi$. However, only minor changes in detail are necessary to handle the case when $\alpha = \pi$. In particular, the region R_w is different, corresponding to the remark at the end of the proof of Theorem 10.4.

Then there are two positive numbers μ_1 and μ_2 , depending only on ϵ , p and the constants of Warschawski's theorem, such that

$$\mu_1 |z - z_n|^{\delta n - 1} \leq |f'(z)| \leq \mu_2 |z - z_n|^{\delta n - 1}$$

holds in a neighborhood of each point z_n corresponding to a corner C_n , while

$$\mu_1 \leq |f'(z)| \leq \mu_2$$

holds elsewhere in R_2 .

Let Q be a fixed point in the interior of R_w and let $w = f(z)$ map the flow region R_2 onto the upper half w -plane such that $f(z_Q) = i$, $f(\infty) = \infty$. Now if R_2 and the upper half w -plane are considered on the Riemann sphere, it is clear from the nature of the conditions which must be satisfied by a member of a bounded family of regular obstacles that R_2 satisfies the hypotheses of the theorem above, with $p = 3$. At the corner O of S_2 , the interior angle is $\pi/2$, and at the corner B the interior angle is greater than $\epsilon\pi$. Hence

$$\begin{aligned} |f'(\infty)| &\leq \mu_2, \\ |f'(z_P)| &\geq \begin{cases} \mu_1 |z_P - z_B|^{\delta - 1}, & (3b/2 \leq l \leq 2b), \\ \mu_1 |z_P|, & (b \leq l \leq 3b/2). \end{cases} \end{aligned}$$

Moreover, if d is the constant of Warschawski's theorem,

$$\begin{aligned} |z_P - z_B| &\geq d |l - 2b|, \\ |z_P| &\geq d |l - b|. \end{aligned}$$

Thus,

$$\begin{aligned} (13.10) \quad V(P, R_2) &= \frac{U}{|f'(\infty)|} |f'(z_P)|, \\ &\geq \begin{cases} K_1 |2b - l|^{\delta - 1}, & (3b/2 \leq l \leq 2b), \\ K_1 |l - b|, & (b \leq l \leq 3b/2), \end{cases} \end{aligned}$$

where $K_1 = \mu_2^{-1} \mu_1 dU > 0$. The required inequality (13.2) follows from (13.8) and (13.10).

The schlicht cavity problem. For a bounded family of regular obstacles there exists a number Λ , independent of k , such that the derivative $l'(s)$ appearing in a solution of the schlicht cavity problem for any member of the family satisfies the inequality

$$0 \leq l'(s) < \Lambda.$$

Proof. Since the solution of this problem is either a solution of the Helmholtz problem or a solution of the finite cavity problem, we must consider each case separately. The former case, however, has been examined already, and we need only determine a bound for $l'(s)$ in the finite cavity problem.

The proof is an exact counterpart of the preceding one. With the help of an auxiliary region based on Theorem 10.5, and the application of Theorem 10.1 and Warschawski's theorem, we can show that

$$(13.2a) \quad q(l) \geq \begin{cases} K_1(2b - l)^{\delta-1}, & (3b/2 \leq l \leq 2b), \\ K_1(l - b), & (b \leq l \leq 3b/2); \end{cases}$$

thus, for $3b/2 \leq l \leq l' \leq 2b$,

$$(13.3a) \quad \int_l^{l'} q(l) dl > \epsilon K_1 |l' - l|^\delta.$$

Corresponding to (13.4) we have,

$$(13.4a) \quad \int_l^{l'} q(l) dl = 2M \int_s^{s'} (h^4 + 2h^2 \cos 2s + 1)^{-2} \sin 2s ds, \\ < 2M(1 - H^2)^{-4} |s' - s|,$$

where H is the constant of Lemma 11.2. From (13.3a) and (13.4a) we get

$$|l' - l| < \left[\frac{2M}{\epsilon K_1(1 - H^2)^4} \right]^\epsilon |s' - s|^\epsilon;$$

hence, using the result of §12, there exists a number K_2 , independent of k , such that

$$(13.5a) \quad |l' - l| < K_2 |s' - s|^\epsilon, \quad 3b/2 \leq l \leq l' \leq 2b.$$

Similarly

$$|l' - l| < K_3 |s' - s|^{\frac{1}{2}}, \quad b \leq l \leq l' \leq 3b/2.$$

The rest of the demonstration may be omitted.

The reentrant jet problem. For a bounded family of regular obstacles, and a bounded set of cavity speeds, there exists a number Λ , independent of k , such that the derivative $l'(s)$ appearing in the solution of the reentrant jet problem for any member of the family satisfies the inequality

$$0 \leq l'(s) < \Lambda.$$

With the exception that Theorem 10.3 is used to obtain the auxiliary comparison region, the proof is a duplication of the preceding two.

14. Domination of d . The lemma of this section depends on the existence of a solution of the schlicht cavity problem. Its application must therefore be deferred until a proof of existence is made.

Consider a bounded family of regular obstacles defined by the functions $\Psi_k(l)$. Let V_k be the preassigned cavity speed for the reentrant jet problem associ-

ated with the obstacle given by $\Psi_k(l)$, and let V_{sk} be the cavity speed in the (unique) symmetric solution of the schlicht cavity problem for the same obstacle.

Lemma 14.1. *Suppose there exist numbers W and δ such that the inequalities,*

$$V_k < W, \quad V_k - V_{sk} > \delta > 0,$$

hold uniformly in k . Then there exists a number Δ , independent of k , such that the number d appearing in a symmetric solution of the reentrant jet problem for any member of the family satisfies the inequality $0 < \Delta < d$.

Proof. We argue by contradiction. Upon supposing that no such Δ exists, it follows that there exists a sequence of solutions of the reentrant jet problem, which we denote by $\{l_n(s), h_n, d_n\}$, $n = 1, 2, \dots$, such that $\lim d_n = 0$. We can, without confusion, denote the inclination of the corresponding obstacles by $\Psi_n(l)$, and call the corresponding cavity speeds V_n . In addition, let $l_{sn}(s)$, h_{sn} , $n = 1, 2, \dots$, be the solution of the schlicht cavity problem corresponding to $\Psi_n(l)$.

Now it was shown in §13 that the functions $l_n(s)$, $l_{sn}(s)$, $n = 1, 2, \dots$, possess bounded equicontinuous derivatives. The functions $\Psi_n(l)$ are also bounded and equicontinuous, and

$$0 < h_n < H < 1, \quad 0 < h_{sn} < H < 1.$$

Hence, by the well-known theorem of Ascoli and the Weierstrass-Bolzano theorem, each of the sequences exhibited converges, provided that n approaches infinity through a suitable subsequence.

Having this in mind, and recalling that $\Omega(\zeta)$ is a continuous functional of $l(s)$ (§8), we may proceed to the limit n towards infinity in equations (6.2) and (7.10a) and (7.11a). Thus, if the limiting quantities are formed by dropping the subscript n , i.e. $\lim l_{sn}(s) = l_s(s)$, $\lim h_n = h$, etc. we obtain

$$(6.2a) \quad \begin{cases} l_s(s) = \mathcal{U}_c[l_s(s), h_s; \Psi(l)], \\ 0 = G[\Omega'_s(ih_s), h_s]; \end{cases}$$

$$(7.10b) \quad \begin{cases} l(s) = \mathcal{U}_J[l(s), h, 0; \Psi(l)], \\ 0 = \Omega'(ih) + 2(h^2 - 1)^{-1} + \kappa, \end{cases}$$

$$(7.11b)$$

where

$$(14.1) \quad \kappa = \lim \frac{2d_n(1 - h_n^2)}{h_n(h_n + d_n)(1 + h_nd_n)} \geq 0.$$

Let us show that

$$(14.2) \quad G[\Omega'(ih), h] = 0,$$

$$(14.3) \quad \lim \frac{d_n}{h_n} = 0.$$

Case 1: $h = 0$. From (7.11b) we obtain $\Omega'(0) = 2 - \kappa \leq 2$. It follows that the first factor in the expression (6.1) defining G is zero, proving (14.2). Moreover, by (14.1),

$$\lim \frac{d_n}{h_n} = \frac{\kappa}{2} \lim (h_n + d_n) = 0.$$

Case 2: $h > 0$. In this case (14.3) is obvious and $\kappa = 0$. Hence by (7.11b) the second factor in the expression defining G is zero, proving (14.2).

It is clear from the respective definitions of the functionals $\mathfrak{U}_J$ and $\mathfrak{U}_C$ that (7.10b) is equivalent to

$$(14.4) \quad l(s) = \mathfrak{U}_C[l(s), h; \Psi(l)].$$

Now (14.2) and (14.4) constitute the system of equations (6.2). It follows that $l(s)$ and h solve the schlicht cavity problem. Putting $V = \lim V_n$, and using (7.7a) and (14.3) we have,

$$\begin{aligned} V &= \lim U \frac{(1 + h_n)(h_n + d_n)(1 + h_n)}{(1 - h_n)(h_n - d_n)(1 - h_n)} \exp [-T_n(ih_n)] \\ &= U \frac{1 + h}{1 - h} \exp [-T(ih)]. \end{aligned}$$

By equation (6.4), V is the cavity speed of the flow defined by $l(s)$ and h .

In virtue of (6.2a), however, $l_s(s)$ and h_s also solve the schlicht cavity problem. Put

$$\begin{aligned} V_s &= \lim V_{s_n} \\ &= U \frac{1 + h_s}{1 - h_s} \exp [-T_s(ih_s)]; \end{aligned}$$

thus V_s is the cavity speed in this flow. Since the schlicht cavity problem possesses a unique solution, $V = V_s$. This is impossible, for

$$V - V_s = \lim (V_n - V_{s_n}) \geq \delta > 0.$$

PART IV. EXISTENCE THEOREMS

15. Preliminaries. The results of this chapter are based on a theory of functional equations developed by Leray & Schauder [7].

Consider an equation of the form

$$(15.1) \quad x = \mathfrak{F}(x),$$

where x is a point in a normed, linear and complete (Banach) space $\mathcal{E}$ and $\mathcal{F}(x)$ is a completely continuous functional transformation of $\mathcal{E}$ into itself. (A transformation is called completely continuous if it is continuous and in addition, transforms bounded sets into compact sets.) We associate with this equation the functional transformation

$$(15.2) \quad y = x - \mathcal{F}(x).$$

Let $\mathfrak{D}$ be a bounded domain in $\mathcal{E}$ whose boundary contains no solution of (15.1). The topological degree at the point 0 of the transformation (15.2) cut down to $\mathfrak{D}$ is called the total index of the solutions of (15.1) in $\mathfrak{D}$, or for brevity, the total index of (15.1). If $\mathcal{F}(x)$ is varied continuously, and if in this process no solution of (15.1) attains the boundary of $\mathfrak{D}$, then the total index remains constant.

It is therefore possible to infer the existence of a solution of (15.1) in $\mathfrak{D}$ by the following procedure. We alter the equation (15.1), continuously and in such a manner that no solution is possible on the boundary of $\mathfrak{D}$, to an equation which is sufficiently simple that its total index may be determined. If this total index is not zero, then the total index of (15.1), being the same, is not zero. It follows, then, from the fundamental properties of degree, that there is at least one solution of (15.1) in $\mathfrak{D}$.

In the remainder of the paper, as in Part II, $\mathcal{E}$ will denote the particular Banach space of real continuously differentiable functions $l(s)$, $0 \leq s \leq \pi/2$. We recall that $\mathcal{E}^*$ is the subset of $\mathcal{E}$ whose elements satisfy the inequality $0 \leq l(s) \leq b$. Finally the space of real numbers will be denoted by $\mathfrak{R}$.

16. The Helmholtz problem. *There is one and only one symmetric solution of the Helmholtz problem for a regular obstacle AB whose inclination $\Psi(l)$ satisfies*

- 1) a Hölder condition, and
- 2) $\Psi(2b) \neq \alpha - \pi$.

Proof. In another paper [21] we have shown that the Helmholtz problem possesses a unique solution.

The existence of a flow solving the Helmholtz problem has been shown equivalent to the existence of a solution of the functional equation

$$(4.9b) \quad l(s) = \mathcal{U}_H[l(s); \Psi(l)], \quad l(s) \in \mathcal{E}^*.$$

For $l(s) \in \mathcal{E}$ define

$$(16.1) \quad l_1(s) = \frac{b}{\lambda} [l(s) - \min l(s)],$$

where

$$\lambda = \text{Max} [b, \max l(s) - \min l(s)].$$

When $l(s) \in \mathcal{E}^*$ and $\min l(s) = 0$,

$$(16.2) \quad l_1(s) = l(s),$$

and in any case, $l_1(s) \in \mathcal{E}^*$. It is clear that the functional equation

$$(16.3) \quad l(s) = \mathcal{U}_H[l_1(s); \Psi(l)], \quad l(s) \in \mathcal{E},$$

is equivalent to (4.9b). Unlike (4.9b), however, the transformation on the right side of (16.3) is defined over the entire Banach space $\mathcal{E}$; this will be of importance later.

We shall now show that (16.3) has at least one solution, which will complete the proof. Let

$$(16.4) \quad \Psi_k(l) = k\Psi(l) + (1 - k)\pi/2, \quad (0 \leq k \leq 1).$$

It is easily verified that the functions $\Psi_k(l)$ define a bounded family of regular obstacles. Moreover, as k runs from 1 to 0, the obstacle given by (16.4) is continuously deformed from its initial configuration, which is just the given arc AB , to a straight segment having the same length, perpendicular to the current.

Consider the equation of the Helmholtz problem for the deformed obstacle $\Psi_k(l)$, namely,

$$(16.5) \quad l(s) = \mathcal{U}_H[l_1(s); \Psi_k(l)].^{18}$$

The result of §13 guarantees that any solution of (16.5) is interior to the sphere $\mathfrak{D}$ in $\mathcal{E}$ whose radius is $b + \Lambda$. Letting $i(k)$ be the total index of (16.5) in $\mathfrak{D}$, we have

$$(16.6) \quad \text{Total index of (16.3)} = i(1) = i(0).^{19}$$

But because $\mathcal{U}_H$ is independent of $l(s)$ when $k = 0$ we can conclude that $i(0) = 1$.

Q.E.D

¹⁸ The continuity properties required of $\mathcal{U}_H[l_1(s); \Psi_k(l)]$ for the application of the Schauder-Leray theory are easily verified. In particular, we must show that

- 1) $\mathcal{U}_H$ continuous in $l(s)$ for each k , $0 \leq k \leq 1$;
- 2) $\mathcal{U}_H$ continuous in k , uniformly for $l(s)$ in a bounded set $\mathfrak{D}$;
- 3) $\mathcal{U}_H$ transforms bounded sets into compact sets.

Proof. By §8, $\mathcal{U}_H$ is continuous in $l_1(s)$. But $l_1(s)$ is continuous in $l(s)$, proving 1).

If $\|l(s)\| < D$, then $\|l'_1(s)\| < D$. It is then apparent that the method given in §8 can equally well be applied to prove 2).

It remains to prove 3). If $\|l(s)\| < D$, then

$$\|\Psi[l_1(s)] - \Psi[l_1(s')]\| < KD \|s - s'\|^p.$$

Therefore, by the Fatou-Privaloff theorem, the functions $\Omega(\zeta)$ corresponding to a bounded set of functions $l(s)$ are equicontinuous. If $L(s)$ is the image of $l(s)$, we obtain out of (4.9a),

$$L(s) = \frac{4M}{U} \int_0^s \exp[-T(e^u)] (1 + \sin s) \sin s \, ds.$$

Thus the derivatives $L'(s)$ are bounded and equicontinuous. By Ascoli's theorem the set of functions $L(s)$ is compact in $\mathcal{E}$.

17. The schlicht cavity problem. This problem has been reduced to the system of equations

$$(6.2) \quad \begin{aligned} l(s) &= \mathcal{U}_c[l(s), h; \Psi(l)], \\ 0 &= G[\Omega'(ih), h], \end{aligned} \quad \begin{cases} l(s) \in \mathcal{E}^*, \\ 0 \leq h < 1. \end{cases}$$

For $-1 < h < 1$, and $l(s) \in \mathcal{E}^*$, define

$$\mathcal{G}[l(s), h; \Psi(l)] = \begin{cases} G[\Omega'(ih), h], & h \geq 0, \\ -h & h < 0, \Omega'(ih) \leq 2, \\ -h + [\Omega'(ih) - 2]^2, & h < 0, \Omega'(ih) > 2. \end{cases}$$

A short computation using (6.1) verifies that $\mathcal{G}$ is a continuous function of the arguments $\Omega'(ih)$, h . Hence, noting that $\Omega'(\zeta)$, $|\zeta| < 1$, depends continuously on $l(s)$ and ζ , we see that $\mathcal{G}$ is a continuous real valued functional.

We shall show that the system of equations (6.2) is equivalent to the system

$$(17.1) \quad \begin{aligned} l(s) &= \mathcal{U}_c[l_1(s), h; \Psi(l)], \\ 0 &= \mathcal{G}[l_1(s), h; \Psi(l)], \end{aligned} \quad \begin{cases} l(s) \in \mathcal{E}, \\ -1 < h < 1, \end{cases}$$

where $l_1(s)$ is given by (16.1). First of all, it is obvious that any solution of (6.2) is a solution of (17.1). Conversely, supposing that $l(s)$ and h are a solution of (17.1), we have

- i) $l(s) \in \mathcal{E}^*$ and $\min l(s) = 0$,
- ii) $l_1(s) = l(s)$,
- iii) $h \geq 0$.

i) holds since the range of $\mathcal{U}_c$ is $\mathcal{E}^*$, and ii) is then a consequence of (16.2); iii) follows from $\mathcal{G} = 0$. By virtue of properties i)–iii) and equations (17.1), the system (6.2) is satisfied by $l(s)$ and h .

To prove that the schlicht cavity problem has a solution, it is therefore sufficient to prove that (17.1) has a solution. For convenience we shall write (17.1) in the form

$$x = \mathcal{F}[x; \Psi(l)],^{20}$$

where $x = \{l(s), h\}$. The point x is confined to a subset $\mathcal{O}$ of the product space $\mathcal{E} \times \mathcal{R}$.

Proofs of the corresponding properties for the functional transformations which we obtain in later sections will be omitted.

¹⁹ Without the preliminary restatement of our problem in terms of equation (16.3) this result cannot be given. For if we had proceeded directly from (4.9b), we would be compelled to replace the set $\mathcal{D}$ by the intersection set $\mathcal{D} \cdot \mathcal{E}^*$. But solutions of (4.9b) are *all* on the boundary of $\mathcal{D} \cdot \mathcal{E}^*$ so that we could neither define total index nor infer (16.6). Seemingly this has not been noticed in the literature.

For the same reasons it is necessary to introduce the function $l_1(s)$ into the equations of the problems we consider in later sections.

²⁰ This is possible by adding h to both sides of the second equation of (17.1).

Consider the equations

$$(17.2) \quad x = \mathfrak{F}[x; \Psi_k(l)], \quad (0 \leq k \leq 1),$$

where $\Psi_k(l)$ is given by (16.4). From §13 and Lemma 11.2 we have $\|l(s)\| < b + \Lambda$, $|h| < H$, for any solution. Let $\mathfrak{D}$ be the subset of $\mathcal{O}$ defined by these inequalities, and let $i(k)$ be the total index of (17.2) in $\mathfrak{D}$. Then

$$\text{Total index of (17.1)} = i(1) = i(0).$$

Now when $k = 0$ we have $\Omega(\zeta) = \pi/2$, so that $\mathfrak{U}_c$ is independent of $l(s)$, and

$$\mathfrak{S} = \begin{cases} 2h(h^2 - 1)^{-1}, & h \geq 0, \\ -h, & h < 0. \end{cases}$$

It is thus evident by inspection that $i(0) = 1$. Therefore, (17.1) has at least one solution. In view of the uniqueness of solutions of the schlicht cavity problem (property iii, §10), we have proved the following result.

There is one and only one symmetric solution of the schlicht cavity problem for a regular obstacle AB whose inclination $\Psi(l)$ satisfies

- 1) a Hölder condition, and
- 2) $\Psi(2b) \neq \alpha - \pi$.

18. The schlicht cavity problem and the finite cavity problem. Detachment en proue. Suppose a regular obstacle AB is deformed in a continuous manner, such that the set of varied obstacles forms a bounded family of regular obstacles. (Throughout this section, we will allow the length of AB to vary, provided it stays bounded. Obviously this generalization will not affect any of the results of Part III concerning bounded families.) For convenience we shall let the deformation depend continuously on a real parameter k . Let $x_k = \{l_k(s), h_k\}$ be the solution of the schlicht cavity problem for the deformed obstacle corresponding to k . The functions $l_k(s)$ possess derivatives which are bounded and equicontinuous (§13), and (§12), $0 \leq h_k < H$.

The set of points x_k is therefore compact in the space $\mathcal{E} \times \mathcal{R}$. It is then an immediate consequence of the uniqueness of solutions and the continuity of $\mathfrak{F}[x, \Psi_k(l)]$, that x_k depends continuously on k . Since (§8) the analytic function $\Omega(\zeta)$ depends continuously on x we have proved

Lemma 18.1. *If ζ_k , $|\zeta_k| < 1$, depends continuously on k , then $\Omega_k(\zeta_k)$ depends continuously on k .*

Corollary 18.1. $\omega'_k(0)$ depends continuously on k .

Proof. We have

$$\omega'_k(0) = \Omega'_k(0) - 2.$$

Q.E.D.

Corollary 18.2. *The cavity speed V_k depends continuously on k .*

Proof. We have

$$V_k = U \frac{1 + h_k}{1 - h_k} \exp [-T_k(ih_k)].$$

Q.E.D.

Consider a simple closed curve T , symmetric about the x -axis. Let O and O' be the two points of intersection of T with the x -axis, with O to the left of O' . If B is any point on T , let B^* denote the conjugate point on T . Let T have the property that any symmetric arc BOB^* of T is a regular obstacle whose inclination $\Psi(l)$ satisfies

- 1) A Hölder condition, and
- 2) $\Psi(2b) \neq \alpha - \pi$.

We shall prove two theorems, which were originally stated only for strictly convex curves T .²¹

Theorem 18.1. *There is an arc BOB^* of T such that $\omega'(0) = 0$ in the symmetric solution of the schlicht cavity problem for BOB^* .*

Note. When $\omega'(0) = 0$ the free streamlines asymptotically approach each other; the interest in this flow is due to the fact that this flow, of all flows solving the Helmholtz problem for some arc of T , disturbs the uniform flow at infinity the least [15].

Proof. Consider the classical flow past T with uniform velocity U at infinity. Let $NO'N^*$ be the subarc of T , containing the stagnation point O' , along which the speed of this flow is less than U . Also let P be the point of the arc ON which has the maximum ordinate. Clearly $P \neq N$.²² Let k be arc length on PN , $p \leq k \leq n$, and denote the point of PN whose arc length is k by B_k .

Consider the bounded family of obstacles $B_kOB_k^*$, $p \leq k \leq n$; this family depends continuously on k . We have

$$(18.1) \quad \omega'_p(0) < 0,$$

$$(18.2) \quad \omega'_n(0) > 0.$$

To prove (18.1), we note by Corollary 10.2 that the solution of the schlicht cavity problem for the obstacle POP^* cannot solve the finite cavity problem; hence it must solve the Helmholtz problem and (§6) this implies (18.1).

The proof of (18.2) is equally easy. We are led to an immediate contradiction with Theorem 10.1 if we assume that the flow solving the schlicht cavity problem for NON^* solves the Helmholtz problem. But (18.2) holds for any solution of

²¹ The results are due to Pulliam [16] and Tiller [17]. Tiller's work, however, contains errors which invalidate his proof.

²² Remarkably, this is a consequence of Theorem 10.1. The details are obvious.

the finite cavity problem.²³ Having shown (18.1) and (18.2), the proof is completed by an application of Corollary 18.1.

Theorem 18.2. *If V is a fixed number, $0 < V \leq U$, then there is an arc BOB^* of T such that V is the cavity speed in the symmetric solution of the schlicht cavity problem for BOB^* .*

Proof. Let $N_1O'N_1^*$ be the subarc of T along which the speed of the classical flow considered in the last proof is less than V . Using the terminology of the preceding theorem we have

$$(18.3) \quad V_p = U,$$

$$(18.4) \quad V_{n_1} < V.$$

The proof of (18.4) parallels the proof of equation (18.2). Our proof is completed with the application of Corollary 18.2.

An important consequence of Theorem 18.2 is the *existence of flows solving the finite cavity problem*. A slight extension of Lemma 18.1 enables us to discuss the problem of detachment *en proue*.

Lemma 18.2. *If the inclination $\Psi_k(l)$ of each deformed obstacle possesses a continuous derivative which satisfies a Hölder condition independent of k , then $\Omega'_k(\zeta)$ depends continuously on k , $|\zeta| \leq 1$.*

Proof. Consider the function $\Pi(\zeta) = i\zeta\Omega'(\zeta)$. Since

$$\operatorname{Re} [\Pi(e^{is})] = \Psi'[l(s)]l'(s),$$

it follows that $\operatorname{Re} [\Pi(e^{is})]$ satisfies a Hölder condition; by the Fatou-Privaloff theorem, $\Pi(\zeta)$ satisfies a Hölder condition for $|\zeta| \leq 1$. Therefore the function $\Omega'_k(\zeta)$, $|\zeta| \leq 1$, formed for the solution x_k , obeys a Hölder condition uniformly in k .

The set of functions $\Omega'_k(\zeta)$ being compact, the continuous dependence of $\Omega_k(\zeta)$ on $l(s)$ implies the same of $\Omega'_k(\zeta)$. The proof may also be carried out by the method indicated in §8.

Without pausing to investigate the significance, we state the problem of detachment *en proue* as follows: determine an arc BOB^* of T such that in the symmetric solution of the schlicht cavity problem for BOB^* ,

$$\omega'(1) = 0,$$

while in the solution for any symmetric subarc of BOB^* ,

$$\omega'(1) < 0.$$

Clearly any solution of this problem is unique.

²³ In a solution of the finite cavity problem, the quantity $\omega'(0)$ may be interpreted geometrically as the curvature of Σ at E . Since Σ lies above the real axis it follows that $\omega'(0)$ is positive.

Theorem 18.3. *Let P be the point on T nearest O (measured along T) at which the ordinate of T attains a local maximum. If the curvature of T satisfies a Hölder condition, then there is a subarc BOB^* of POP^* which solves the problem of detachment *en proue* for T . Furthermore, the flow past BOB^* is a Helmholtz flow.*

Proof. Consider the flow past POP^* . We assert that

$$(18.5) \quad \omega'(1) > 0.$$

For, first of all, POP^* is a regular obstacle for the angle π . By Theorem 9.1 the free streamline Σ lies entirely above the horizontal straight line L through P . Leray [8; page 166] has shown that when $\omega'(1) \leq 0$ the curvature of Σ at P is such that in the neighborhood of P , Σ lies below L . Since Σ is above L , (18.5) must hold.

Suppose there is a sequence of subarcs $Q_n O Q_n^*$, $Q_n \rightarrow O$, such that

$$(18.6) \quad \omega'_n(1) \geq 0.$$

Since magnification of the flow plane leaves $\omega(\zeta)$ unchanged, we infer the existence of a sequence of obstacles $Q'_n O Q_n'^*$ such that

$$(18.7) \quad \omega'_n(1) \geq 0,$$

while the limit obstacle is just a straight segment of length $2b$, perpendicular to the current. Applying Lemma 18.2 and using (18.7) we see that the quantity $\omega'(1)$ corresponding to a straight segment must satisfy

$$\omega'(1) \geq 0.$$

However, calculating this directly from (4.5) we get $\omega'(1) = -1$. It follows that no sequence of obstacles $Q_n O Q_n^*$ exists for which (18.6) holds.

Consequently, using Lemma 18.2 and inequality (18.5), there is a first point B on OP where, in the solution of the schlicht cavity problem for BOB^* , $\omega'(1) = 0$. The obstacle BOB^* solves the problem of detachment *en proue*.

19. The reentrant jet problem. In order to prove that there is a symmetric flow solving the reentrant jet problem it is sufficient to prove that there exists at least one solution $x = \{l(s), h, d\}$,

$$l(s) \in \mathcal{E}, \quad 0 < h < 1, \quad 0 < d < 1,$$

of the system (§7),

$$(19.1) \quad \begin{cases} l(s) = \mathfrak{U}_T[l_1(s), h, d; \Psi(l)], \\ 0 = \Omega'(ih) + \frac{2}{h^2 - 1} + \frac{2d(1 - h^2)}{h(h + d)(1 + hd)}, \\ \frac{U}{V} = \frac{(h - d)(1 - hd)(1 - h)}{(h + d)(1 + hd)(1 + h)} \exp [T(ih)], \end{cases}$$

where $l_1(s)$ is given by (16.1). For convenience we will write this in the form

$$x = \mathfrak{F}[x; \Psi(l), V].$$

The point x is confined to a subset $\mathcal{O}$ of the product space $\mathcal{E} \times \mathcal{R} \times \mathcal{R}$.

We introduce the set of equations

$$(19.2) \quad x = \mathfrak{F}[x; \Psi_k(l), V_k], \quad (0 \leq k \leq 1),$$

where $\Psi_k(l)$ is given by (16.4), and

$$(19.3) \quad V_k = (V/V_s)V_{sk}.$$

Here V_{sk} is the cavity speed in the (unique) symmetric solution of the schlicht cavity problem for the obstacle given by $\Psi_k(l)$ and $V_s = V_{s1}$. Because of Corollary 10.1 we need only consider

$$(19.4) \quad V > V_s.$$

By Corollary 18.2, V_k depends continuously on k . The continuity properties required of $\mathfrak{F}$ are then easily verified.

Since $V_{sk} \leq U$ we have

$$V_k \leq UV/V_s;$$

since V_{sk} is a positive continuous function of k , there exists a number $v > 0$ such that $V_{sk} > v$. Hence

$$V_k - V_{sk} > \frac{V - V_s}{V_s} v > 0.$$

By virtue of these inequalities and Lemma 14.1 there exists a number $\Delta > 0$ such that

$$(19.5) \quad d > \Delta,$$

in any solution of (19.2). The third equation of (19.1) implies

$$(19.6) \quad d < h.$$

From §13, Lemma 11.1, and (19.5) and (19.6), we obtain finally,

$$\|l(s)\| < b + \Delta,$$

$$\Delta < d < H,$$

$$\Delta < h < H,$$

in any solution of (19.2). Let $\mathfrak{D}$ be the subset of $\mathcal{O}$ defined by these inequalities,

and let $i(k)$ be the total index of (19.2) in $\mathfrak{D}$. Then

$$\text{Total index of (19.1)} = i(1) = i(0).$$

We shall now prove that $i(0) = 1$. When $k = 0$ in (19.2), the transformation $\mathfrak{U}_J$ is independent of $l(s)$, and the remaining equations of (19.2) become

$$(19.7) \quad \begin{cases} 0 = \frac{2}{h^2 - 1} + \frac{2d(1 - h^2)}{h(h + d)(1 + hd)}, \\ \frac{U}{V_0} = \frac{(h - d)(1 - hd)(1 - h)}{(h + d)(1 + hd)(1 + h)}. \end{cases}$$

Putting

$$(19.8) \quad \begin{aligned} x &= -\frac{d^2 + 1}{2d}, & -\infty < x < -1, \\ y &= -\frac{h^2 + 1}{2h}, & -\infty < y < -1, \end{aligned}$$

equations (19.7) reduce to

$$(19.9) \quad \begin{cases} 0 = x + 2y^2 + y - 2 \\ \frac{U}{V_0} = \left[\frac{x - y}{x + y} \right] \left[\frac{y + 1}{y - 1} \right]^{\dagger}. \end{cases}$$

Let T and $\bar{T}$ be, respectively, the transformations defined by the right sides of (19.7) and (19.9), and let ν and $\bar{\nu}$ be, respectively, the topological degrees at $(0, U/V_0)$ of T and $\bar{T}$. Then it is evident that $i(0) = \nu$, and since (19.8) is a homeomorphism, $\nu = \bar{\nu}$.

To complete the proof it is sufficient to show that $\bar{\nu} = 1$. To accomplish this it will be shown i) that (19.9) possesses a unique solution (x_0, y_0) and ii) that the Jacobian of $\bar{T}$ is non zero at this point.

i) Eliminating x in (19.9) we obtain

$$(19.10) \quad \frac{U}{V_0} = \left[\frac{y^2 + y - 1}{y^2 - 1} \right] \left[\frac{y + 1}{y - 1} \right]^{\dagger} = g(y).$$

Figure 15 is a graph of $g(y)$ for $-\infty < y < -1$. By (19.3) and (19.4),

$$\frac{U}{V_0} = \frac{V_s}{V} < 1.$$

Hence there is a unique solution y_0 of (19.10), and consequently a unique solution (x_0, y_0) of (19.9). Furthermore,

$$y_0 - x_0 = 2(y_0^2 + y_0 - 1) > 0,$$

so that x_0 , as well as y_0 , is in the interval $(-\infty, -1)$.

ii) If J denotes the Jacobian of $\bar{T}$, we have by direct computation and the use of (19.9),

$$J(x_0, y_0) = \frac{U}{V_0} \frac{y_0(1 + 2y_0)}{(1 - y_0^2)(y_0^2 + y_0 - 1)} < 0.$$

Q.E.D

FIG. 15.

The following theorem summarizes the results of this section.

THEOREM. *Let AB be a regular obstacle whose inclination $\Psi(l)$ satisfies*

1) *a Hölder condition, and*

2) $\Psi(2b) \neq \alpha - \pi$.

Let V_s be the cavity speed in the (unique) symmetric solution of the schlicht cavity problem for AB . Then there exists a symmetric solution of the reentrant jet problem for AB , whose cavity speed is V , if and only if $V > V_s$.

20. Starlike obstacles. For simplicity we have heretofore restricted our considerations to the class of regular obstacles. In this section, however, we shall note a wider class of obstacles for which the same type of results are valid. A curve will be called *starlike* if

i) there exists a point σ in the fourth quadrant such that any straight line passing through σ intersects the curve in at most one point or along a segment, or if

ii) there exists a number α , $\pi/2 \leq \alpha \leq \pi$ such that any straight line whose inclination is α intersects the curve in at most one point or along a segment.

A symmetric obstacle AOB with a continuous tangent will be called a *starlike obstacle* if the curve consisting of the negative real axis together with OB is starlike. It is apparent that regular obstacles are also starlike obstacles.

Theorem 20.1. *In a symmetric solution of the schlicht problem for a starlike obstacle AOB , the streamline $\Psi = 0$ consisting of the negative real axis, OB , and Σ is a starlike curve.*

Proof. Let β be the inclination of the line σB . Then

$$\Psi(l) \geq \beta - \pi.$$

Hence, if θ is the inclination of Σ , we have, by a duplication of the proof of Theorem 9.1,

$$(20.1) \quad \theta > \beta - \pi.$$

Now suppose that Σ is not a starlike curve. It is clear then that there exists a line Λ passing through σ which has the properties: 1) Λ is tangent to Σ at a point M_1 and 2) the curve S consisting of the negative real axis, OB , and BM_1 is a starlike curve. If β^* is the inclination of Λ then $\theta_{M_1} = \beta^*$, for otherwise

$$\theta_{M_1} = \beta^* - \pi < \beta - \pi,$$

contradicting (20.1).

Let P be a variable point on S and let Λ_P be a straight cut on the Riemann surface R of the "upper half" flow which initiates at the point P , has inclination β^* , and either extends from P until meeting the infinite arc (M_1, ∞) of Σ or, if it does not meet this arc, extends from P to infinity. There is clearly some point P for which Λ_P is tangent to the arc (M_1, ∞) at some point N_1 and the extension of Λ_P past N_1 lies in R . Let R_2 be the simple admissible region bounded by the half line $(-\infty, M_1)$ of Λ , the arc M_1N_1 of Σ , and the half line (N_1, ∞) of Λ_P .

Now there exist two points M and N on the arc MN such that $V(M, R_2) < V(N, R_2)$; (see proof of Theorem 3, [21]). The hypotheses of Theorem 10.2 are satisfied by the regions R and R_2 . Hence $V(M, R) < V(N, R)$. But this is impossible since $V(M, R) = V(N, R) = V$. This completes the proof.

With the help of this result, which extends the validity of Theorem 9.1, and a similar result generalizing Theorem 9.2, we can verify that all the results of Part III for the schlicht cavity problem and the reentrant jet problem remain valid for starlike obstacles; (the properties of solutions of the schlicht cavity problem given in §10 have already been shown valid for starlike obstacles [21]). Hence we can replace, in the theorems of Part IV (except §16), the word "regular" with the word "starlike."

In conclusion we remark that the methods presented may be extended to apply to the existence problem for even more general obstacles. However, since this increase in generality is obtained only with a corresponding increase in the

length of the proofs, and since it is only for starlike obstacles that a uniqueness theory exists, we do not feel that a presentation of the existence theory in its most advanced state is necessary.

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